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FRACTIONAL POWERS OF ALMOST NON-NEGATIVE OPERATORS [1]

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Dedicated to Acad. Bogoljub Stanković,
on the occasion of his 80-th birthday

Abstract

In this paper, we extend the theory of complex powers of operators to a class of operators in Banach spaces whose spectrum lies in $\mathbb{C} \setminus]-\infty, 0[$ and whose resolvent satisfies an estimate $\left\| (\lambda + A)^{-1} \right\| \leq \left(\lambda^{-1} + \lambda^{m} \right) M$ for all $\lambda > 0$ and for some constants $M > 0$ and $m \in \mathbb{R}$. This class of operators strictly contains the class of the non negative operators and the one of operators with polynomially bounded resolvent. We also prove that this theory may be extended to sequentially complete locally convex spaces.

Mathematics Subject Classification: Primary 47A60, 47D06

Key Words and Phrases: fractional powers, non-negative operators, almost sectorial operators, functional calculus, semigroups of operators

1. Introduction

The theory of fractional powers of linear operators in Banach spaces has been developed during the second half of the last century, both in abstract settings and for their applications to partial differential equations. The first

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class of operators under consideration was the one of *non-negative* operators introduced by A. Balakrishnan in [1]. An operator A in this class satisfies that its resolvent set, $\rho(A)$, contains the interval $]-\infty, 0[$ and the estimate

$$\left\|(\lambda + A)^{-1}\right\| \leq \frac{M}{\lambda} \tag{1}$$

holds for all $\lambda > 0$ and some $M > 0$. It is proved (see [12, Prop. 1.2.1, p. 7]) that (1) may always be extended to some sector $\mathbb{C} \backslash \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \omega\}$, for an appropriate $\omega \in\,]-\pi, \pi]$, so that these operators are called *sectorial* as well.

During the last two decades, the theory of fractional powers has been extended to a class of operators whose spectrum is contained in a certain region of the complex plane and whose resolvent is uniformly bounded by a polynomial (see [15], [17], [2]-[4], [6], [14] and [16]). This class of operators is usually called the class of *operators with polynomially bounded resolvent.*

The aim of this paper is to construct a theory of complex powers for a class of operators which strictly contains the class of the non-negative operators and the class of operators with polynomially bounded resolvent. Injective operators of this type have been recently studied by R. DeLaubenfels and J. Pastor [5].

Our construction does not require the condition of injectivity and it is based on the ideas underlying the theory of complex powers of non-negative operators defined in a sequentially complete locally convex space (see [12, Section 1.4 and 5.6]), and on the properties of a semigroup that it is associated with the base operator A. This paper is organized as follows. In Section 2, we introduce the class of almost non-negative operators and give some examples. Next, we study the spectrum of these operators and present more theoretical examples. Section 3 is devoted to the construction of the complex powers. In Section 4, we study some properties such as additivity and multiplicativity and we prove that the spectral mapping theorem, in general, does not hold. Finally, in Section 5 we prove that this theory may be extended to a sequentially complete locally convex space.

2. Almost non-negative operators

Throughout this paper, with the exception of Section 5, the operators which are considered are linear and defined in a complex Banach space $(X, \|\cdot\|)$. As usual, $D(A)$ and $R(A)$ stand for the domain and range of A, respectively, $\rho(A)$ is the resolvent set of A, $\sigma(A)$ its spectrum, and $A\,|_Z$ denotes the restriction of A to a vector subspace Z.

DEFINITION 2.1. The operator $A : D(A) \subseteq X \to X$ is said to be *almost non-negative* if $\rho(A) \supset]-\infty, 0[$ and if there exists $m \in \mathbb{R}$ and $M > 0$ such that

$$\left\| (\lambda + A)^{-1} \right\| \leq \left(\lambda^{-1} + \lambda^{m} \right) M \qquad \text{for all } \lambda > 0. \tag{2}$$

We denote this class of operators by $\mathcal{M}(m)$.

REMARK 2.1. Note that the class of non-negative operators coincides with $\mathcal{M}(-1)$. If $m > -1$, then the norm of the resolvent of the operator has a polynomial behavior when $\lambda \to +\infty$, but if $m < -1$, then we have a polynomial behavior for $\lambda \to 0$.

REMARK 2.2. In the case $m < -1$, we only consider operators A with $0 \in \sigma(A)$, since if $0 \in \rho(A)$, then $(\lambda + A)^{-1}$ is bounded for $\lambda \to 0$ and consequently A is non-negative.

REMARK 2.3. Obviously, if $m > m_0 \geq -1$, then $\mathcal{M}(m_0) \subset \mathcal{M}(m)$, and if $m < m_0 \leq -1$, then $\mathcal{M}(m_0) \subset \mathcal{M}(m)$. Consequently, for $m \in \mathbb{R}$, $\mathcal{M}(-1) \subset \mathcal{M}(m)$, and if $m_1 < -1$ and $m_2 > -1$, then $\mathcal{M}(-1) = \mathcal{M}(m_1) \cap \mathcal{M}(m_2)$.

Next, we give some examples of operators in the class $\mathcal{M}(m)$.

EXAMPLE 2.1. The operators with polynomially bounded resolvent considered in [15], [17], [2]-[4], [6], [14] and [16] are examples of almost non-negative operators with $0 \in \rho(A)$. In particular, the realization of an elliptic differential operator in a Hölder space is not sectorial, but has polynomially bounded resolvent. An example of this class of operators which belongs to $\mathcal{M}(m)$ with $-1 < m < 0$, can be found in [18, Th. 1].

EXAMPLE 2.2. Consider the Banach space $X = C([0,1])$ of complex-valued continuous functions on $[0,1]$ endowed with the supremum norm. The operator $A = -i\dfrac{d}{dt}$, with domain

$$D(A) = \left\{ \phi \in X : \phi' \in X,\, \phi(0) = 0 \right\},$$

is an example of an operator which is not sectorial but its resolvent is uniformly bounded (see [12, Example 1.4.4, p. 28] for the details). So, this operator belongs to $\mathcal{M}(m)$, $m \geq 0$, its domain in not dense and its spectrum is empty.

Remark 2.4. If A is an injective operator and $m \in \mathbb{R}$,then A belongs to $\mathcal{M}(m)$ if and only if A^{-1} belongs to $\mathcal{M}(-m-2)$.

It is a direct consequence from the relation

$$\left(\lambda + A^{-1}\right)^{-1} = \lambda^{-1} A \left(\lambda^{-1} + A\right)^{-1} = \lambda^{-1} - \lambda^{-2}\left(\lambda^{-1} + A\right)^{-1}.$$

In fact, if $A \in \mathcal{M}(m)$ and $m \in \mathbb{R}$, then for $\lambda > 0$

$$\left\|\left(\lambda + A^{-1}\right)^{-1}\right\| \leq \lambda^{-1} + \lambda^{-2}\left(\lambda + \lambda^{-m}\right) M \leq \left(\lambda^{-1} + \lambda^{-m-2}\right)(M+1)$$

and $A^{-1} \in \mathcal{M}(-m-2)$. Conversely, if $A^{-1} \in \mathcal{M}(-m-2)$, we can reason for A^{-1} and their inverse A as before.

Our next goal in this section is to locate the spectrum of an almost non-negative operator. This will allow us to generate operators of any class $\mathcal{M}(m)$, to separate each one of these classes and to prove that the classes of sectorial operators and of operators with polynomially bounded resolvent are strictly contained in $\cup_{m\in\mathbb{R}} \mathcal{M}(m)$.

We will use the following notations:

$$S_\omega = \{z \in \mathbb{C}^* : |\arg z| < \omega\}, \;\; 0 < \omega < \frac{\pi}{2}$$

$$H^+_{m,a,\infty} = \left\{z \in \mathbb{C} : \operatorname{Re} z \geq 1, \;\; |\operatorname{Im} z| < a (\operatorname{Re} z)^{-m}\right\}, \;\; m \geq -1, \;\; a > 0$$

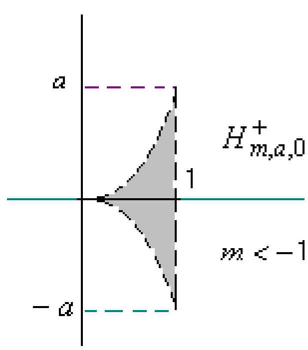
$$H^+_{m,a,0} = \left\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, \;\; |\operatorname{Im} z| < a (\operatorname{Re} z)^{-m}\right\}, \;\; m \leq -1, \;\; a > 0$$

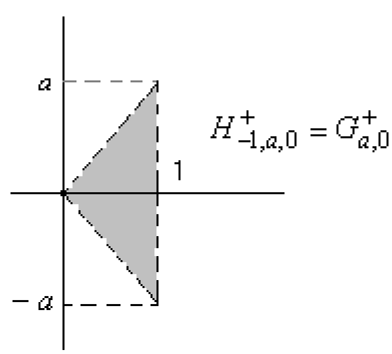
$$G^+_{a,\infty} = \{z \in S_{\arctan a} : \operatorname{Re} z \geq 1\}, \;\; a > 0$$

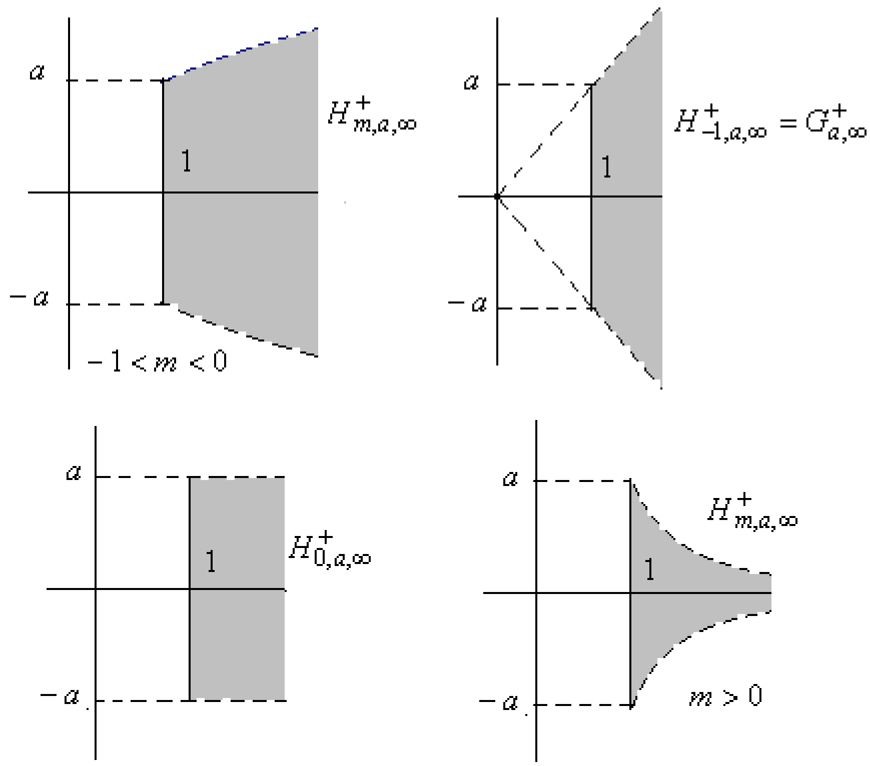
$$G^+_{a,0} = \{z \in S_{\arctan a} : 0 < \operatorname{Re} z < 1\}, \;\; a > 0$$

$$H^-_{m,a,\infty} = \left\{z \in \mathbb{C} : -z \in H^+_{m,a,\infty}\right\} \;\; \text{and} \;\; H^-_{m,a,0} = \left\{z \in \mathbb{C} : -z \in H^+_{m,a,0}\right\}$$

$$G^-_{a,\infty} = \left\{z \in \mathbb{C} : -z \in G^+_{a,\infty}\right\} \;\; \text{and} \;\; G^-_{a,0} = \left\{z \in \mathbb{C} : -z \in G^+_{a,0}\right\}.$$

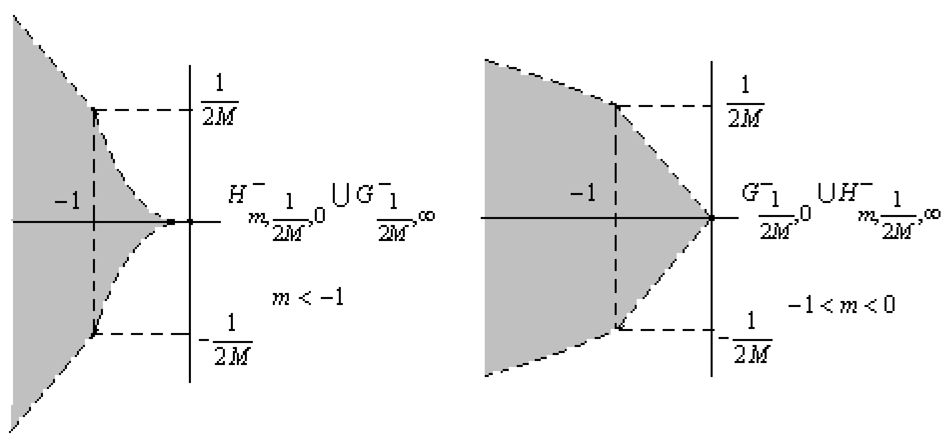


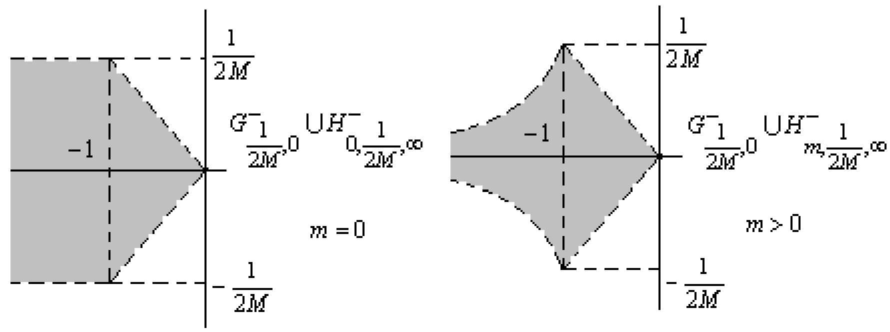




From now on, by M we will denote the constant of (2).

In the next theorem we obtain that the resolvent set of almost non-negative operators with $m \neq -1$ contains one of the following regions:





THEOREM 2.1. *If $A \in \mathcal{M}(m)$, with $m \geq -1$, then the resolvent set $\rho(A)$ contains the open region $G^-_{\frac{1}{2M},0} \cup H^-_{m,\frac{1}{2M},\infty}$. Further, if $0 < \tau < \frac{1}{2M}$, then the function χ defined on $\rho(A)$ by*

$$\eta \longmapsto \chi(\eta) = \left(|\eta|^{-1} + |\eta|^m\right)^{-1} (\eta + A)^{-1} \tag{3}$$

is uniformly bounded on $G^+_{(\frac{1}{2M}-\tau),0} \cup H^+_{m,(\frac{1}{2M}-\tau),\infty}$.

If $A \in \mathcal{M}(m)$, for $m \leq -1$, then the region $H^-_{m,\frac{1}{2M},0} \cup G^-_{\frac{1}{2M},\infty}$ is contained in $\rho(A)$, and if in addition $0 < \tau < \frac{1}{2M}$, then the function (3) is uniformly bounded on $H^+_{m,(\frac{1}{2M}-\tau),0} \cup G^+_{(\frac{1}{2M}-\tau),\infty}$.

Consequently, in particular case $m = -1$ (non-negative operator) the resolvent set $\rho(A)$ contains the sector

$$G^-_{\frac{1}{2M},0} \cup H^-_{-1,\frac{1}{2M},\infty} = H^-_{-1,\frac{1}{2M},0} \cup G^-_{\frac{1}{2M},\infty}.$$

P r o o f. Suppose that $A \in \mathcal{M}(m)$ with $m \geq -1$, and consider the bijective mapping

$$\begin{array}{rcll} \zeta : & S_{\arctan\left(\frac{1}{2M}\right)} & \rightarrow & G^+_{\frac{1}{2M},0} \cup H^+_{m,\frac{1}{2M},\infty} \\ & z & \mapsto & \eta = \zeta(z) = \operatorname{Re} z - (\operatorname{Re} z - z)(\operatorname{Re} z)^{-k-1}, \end{array} \tag{4}$$

where $k = -1$ whenever $z \in G^+_{\frac{1}{2M},0}$, and $k = m$ for $z \in G^+_{\frac{1}{2M},\infty}$.

Given $z \in S_{\arctan\left(\frac{1}{2M}\right)}$, the operator $B_z = (\operatorname{Re} z - z)(\operatorname{Re} z)^{-k-1}$ $(\operatorname{Re} z + A)^{-1}$ is bounded and $\|B_z\| < 1$. So,

$$[I - B_z]^{-1} = \sum_{n=0}^{\infty} \left[(\operatorname{Re} z - z)(\operatorname{Re} z)^{-k-1}\right]^n \left[(\operatorname{Re} z + A)^{-1}\right]^n$$

is bounded as well and consequently the same holds for the inverse of $\eta + A = (\operatorname{Re} z + A)[I - B_z]$. Therefore $G^{-}_{\frac{1}{2M},0} \cup H^{-}_{m,\frac{1}{2M},\infty} \subset \rho(A)$.

Suppose $m \geq -1$. Given $0 < \tau < \frac{1}{2M}$ let us prove that (3) is bounded. Assume first that $\eta \in G^{+}_{(\frac{1}{2M}-\tau),0}$. In this case $\eta = \zeta(\eta)$ and denoting $\theta = \arg(\eta)$, we get $|\tan\theta| < \frac{1}{2M} - \tau$, and since $0 < \operatorname{Re} z < 1$,

$$\|\chi(\eta)\| = |\eta| \left\|(\eta + A)^{-1}\right\| \leq \frac{|\eta|}{\operatorname{Re}\eta}\tau^{-1} \leq (1 + |\tan\theta|)\,\tau^{-1} \leq \frac{2M+1}{2M\tau}.$$

If we assume that $\eta \in H^{+}_{m,(\frac{1}{2M}-\tau),\infty}$, by (4) there is a unique $z = \lambda e^{i\theta} \in G^{+}_{\frac{1}{2M},\infty}$ such that $\eta = \zeta(z)$, $|\tan\theta| < \frac{1}{2M} - \tau$, $\operatorname{Re} z \geq 1$, and we obtain

$$\|\chi(\eta)\| = |\eta|^{-m} \left\|(\eta + A)^{-1}\right\| \leq |\eta|^{-m} (\operatorname{Re} z)^m \tau^{-1}.$$

Note that if $m \geq 0$, then

$$|\eta|^{-m} (\operatorname{Re} z)^m = \left[1 + \tan^2\theta\,(\lambda\cos\theta)^{-2m-2}\right]^{-\frac{m}{2}} \leq 1,$$

and that if $-1 \leq m < 0$, we can obtain a similar estimate with a constant that only depends on M and m. This completes the first part of the proof.

If $A \in \mathcal{M}(m)$ and $m \leq -1$, then we consider the mapping

$$\begin{array}{rccl} \zeta: & S_{\arctan(\frac{1}{2M})} & \rightarrow & H^{+}_{m,\frac{1}{2M},0} \cup G^{+}_{\frac{1}{2M},\infty} \\ & z & \mapsto & \eta = \zeta(z) = \operatorname{Re} z - (\operatorname{Re} z - z)(\operatorname{Re} z)^{-k-1} \end{array}$$

where if $z \in G^{+}_{\frac{1}{2M},0}$ we take $k = m$; and if $z \in G^{+}_{\frac{1}{2M},\infty}$, then $k = -1$. The proof then follows in the same way as in the preceding case. ∎

In Theorem 2.1 we have proved that the spectrum of an almost non-negative operator is a closed set which is contained in the complement of an open region that depends on m and M. Next, we will see that given a closed set F, which is contained in the complement of any region with these characteristics, it is possible to find an almost non-negative operator A whose spectrum coincides with F.

Let us first consider the open region $G^{-}_{a,0} \cup H^{-}_{m,a,\infty}$ with $a > 0$, $m \geq -1$, and a closed set $F \subset \mathbb{C} \setminus \left(G^{-}_{a,0} \cup H^{-}_{m,a,\infty}\right)$. Assume that $F \neq \varnothing$. Otherwise, it is sufficient to take a non-negative operator A. As F is separable, there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $\overline{\{z_n : n \in \mathbb{N}\}} = F$.

As usual, we denote by l^p, $1 \leq p < \infty$, the Banach space of complex sequences $\phi = \{\phi_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |\phi_n|^p < \infty$, endowed with the norm $\|\phi\|_{l^p} = (\sum_{n=1}^{\infty} |\phi_n|^p)^{\frac{1}{p}}$. By l^∞ we denote the Banach space of bounded complex sequences endowed with the norm $\|\phi\|_{l^\infty} = \sup_{n\in\mathbb{N}} |\phi_n|$.

Now consider in l^p, $1 \leq p \leq \infty$ the operator

$$A\phi = A(\phi_1, \phi_2, \ldots) = (z_1\phi_1, z_2\phi_2, \ldots),$$

defined on $D(A) = \left\{\phi = \{\phi_n\}_{n\in\mathbb{N}} \in l^p : A\phi \in l^p\right\}$. This operator is bounded if and only if F is bounded. If F is unbounded, then $D(A)$ is dense if and only if $1 \leq p < \infty$. Moreover $\sigma(A) = F$ and for $\phi \in l^p$, $1 \leq p \leq \infty$, we have

$$(\lambda + A)^{-1}\phi = \left(\phi_1 (\lambda + z_1)^{-1}, \phi_2 (\lambda + z_2)^{-1}, \ldots\right) \qquad (\lambda > 0).$$

Since $-\lambda \notin F$ and $\{z_n\}_{n\in\mathbb{N}} \subset F$, the distance from $-\lambda$ to F, say $d(-\lambda, F)$, is greater than zero, happening for $\phi \in l^p$, $1 \leq p \leq \infty$,

$$d(-\lambda, F) \left\|(\lambda + A)^{-1}\phi\right\|_{l^p} \leq \|\phi\|_{l^p}.$$

Elementary geometrical arguments imply that the expression $\left(\lambda^{-1} + \lambda^m\right)$ $d(-\lambda, F)$ is low bounded, uniformly for all $\lambda > 0$. This yields that the operator generated holds (2) for $m \geq -1$. Thus $A \in \mathcal{M}(m)$.

Choosing F equal to the boundary of $G^-_{a,0} \cup H^-_{m,a,\infty}$, we obtain operators whose resolvent set cannot contain a sector $\mathbb{C} \setminus \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \omega\}$, $\omega \in\ ]-\pi, \pi]$. So, these operators are not sectorial and different from the ones considered in [15], [17], [2]-[4], [6], [14] and [16].

Analogously, given $m \leq -1$ and $a > 0$, we find an operator in $\mathcal{M}(m)$ whose spectrum coincides with any closed set contained in $\mathbb{C} \setminus \left(H^-_{m,a,0} \cup G^-_{a,\infty}\right)$. In the same manner, if we consider the boundary of this set, then the resulting operator is strictly almost non-negative.

3. Definition of the fractional powers

In this section we define the fractional powers for the operators considered in the preceding section. The key to construct these fractional powers is Proposition 3.6, where we prove that for $m \geq -1$ the subspace $D(A^\infty) = \bigcap_{n\in\mathbb{N}} D(A^n)$ endowed with a certain system of seminorms is a Fréchet space and that the restriction of A to this space is non-negative. So if $m \geq -1$ it makes sense to consider the fractional powers of A on $D(A^\infty)$. In Proposition 3.7 we prove that these fractional powers are closable on the initial Banach space and from this we define the fractional powers of the operator A for $m \geq -1$. We begin by studying a semigroup associated with the operator A. This semigroup has two properties proved in Propositions 3.3 and 3.5 which will be very useful later on. For that we first consider the function

$$\begin{aligned} f(\lambda) &= -\frac{1}{\pi} e^{-t\lambda^\gamma \cos \pi\gamma} \sin\left(t\lambda^\gamma \sin \pi\gamma\right) \\ &= \frac{1}{2\pi i}\left(e^{-t\lambda^\gamma e^{-i\pi\gamma}} - e^{-t\lambda^\gamma e^{i\pi\gamma}}\right) \qquad (\lambda > 0) \end{aligned} \tag{5}$$

defined for $t > 0$ and $0 < \gamma < 1/2$. Writing $\xi = t\cos\pi\gamma > 0$, it is clear that

$$|f(\lambda)| \leq \frac{1}{\pi} e^{-\lambda^\gamma \xi}. \tag{6}$$

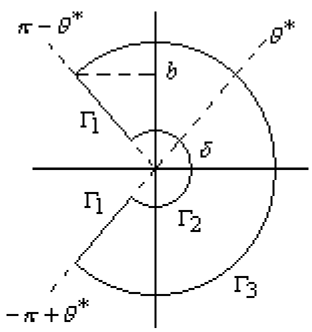
In general, for $0 < r < \frac{\pi}{2}$, $s > 0$, $0 < \varepsilon < \frac{\pi}{2}$ and $|\tau| \leq \frac{\pi}{2} - \varepsilon$, we have

$$\left| e^{-se^{i(r+\tau)}} - e^{-se^{-i(r-\tau)}} \right| \leq 2rse^{-s\sin\varepsilon}.$$

So, in particular, if $s = t\lambda^\gamma$, $r = \pi\gamma$, $\tau = \arg t$, and $0 < \varepsilon < \frac{\pi}{2}$, then

$$|f(\lambda)| \leq \gamma t \lambda^\gamma e^{-t\lambda^\gamma \sin\varepsilon}. \tag{7}$$

Now let $b > 0$, $\delta > 0$, $0 < \theta^* < \frac{\pi}{2}$, and consider the paths



$$\Gamma_1(\theta^*, b, \delta) = \left\{ z \in \mathbb{C} : \delta \leq |z| \leq \tfrac{b}{\sin\theta^*},\ |\arg z| = \pi - \theta^* \right\}$$

$$\Gamma_2(\theta^*, \delta) = \left\{ z \in \mathbb{C} : |z| = \delta,\ |\arg z| \leq \pi - \theta^* \right\}$$

$$\Gamma_3(\theta^*, b) = \left\{ z \in \mathbb{C} : |z| = \tfrac{b}{\sin\theta^*},\ |\arg z| \leq \pi - \theta^* \right\}$$

and finally,

$$\Gamma(\theta^*, b, \delta) = \Gamma_1(\theta^*, b, \delta) \cup \Gamma_2(\theta^*, \delta) \cup \Gamma_3(\theta^*, b) \tag{8}$$

with a positive orientation.

Lemma 3.1. *Let $f(\cdot)$ be the function defined in (5). Then*

$$\int_0^\infty \lambda^n f(\lambda)\, d\lambda = 0, \qquad n = 0, 1, 2, \ldots .$$

P r o o f. The function $F(z) = e^{-tz^\gamma}$, $0 < \gamma < \frac{1}{2}$, $t > 0$ is analytic on $\mathbb{C} \setminus]-\infty, 0]$ and satisfies

$$|F(z)| \le e^{-|z|^\gamma \xi}, \text{ with } \xi = t \cos \pi\gamma.$$

By Cauchy's theorem

$$\int_{\Gamma(\theta^*, b, \delta)} z^n F(z)\, dz = 0, \quad n = 0, 1, 2, \ldots,$$

and letting $\theta^* \to 0$, by the Dominated Convergence theorem, we obtain the result. ■

Lemma 3.2. *Let $A \in \mathcal{M}(m)$, with $m \ge -1$ and $f(\cdot)$ be as in (5). Then the integral*

$$\int_0^\infty \lambda^n f(\lambda)(\lambda + A)^{-1}\, d\lambda, , \quad n = 0, 1, 2, \ldots$$

is absolutely convergent and defines a bounded linear operator.

P r o o f. From (2), (7) and (6) it follows that

$$\int_0^\infty \left\| \lambda^n f(\lambda)(\lambda + A)^{-1} \right\| d\lambda \le \frac{2M}{\sin \varepsilon} + \frac{2M}{\pi\gamma} \xi^{-\left(\frac{n+1}{\gamma}\right)} \Gamma\left(\frac{n + m + 1}{\gamma}\right),$$

for $0 < \varepsilon < \frac{\pi}{2}$, and where $\Gamma(\cdot)$ denotes the gamma function. This completes the proof. ■

Now let $A \in \mathcal{M}(m)$, with $m \ge -1$, $t > 0$ and $0 < \gamma < \frac{1}{2}$. Lemma 3.2 shows that the operator

$$S_\gamma(t) = \frac{1}{2\pi i} \int_0^\infty \left(e^{-t\lambda^\gamma e^{-i\pi\gamma}} - e^{-t\lambda^\gamma e^{i\pi\gamma}} \right) (\lambda + A)^{-1}\, d\lambda \tag{9}$$

is well-defined and bounded.

Proposition 3.3. *Let $A \in \mathcal{M}(m)$, with $m \ge -1$, $t > 0$ and $0 < \gamma < \frac{1}{2}$. Then*

$$R(S_\gamma(t)) \subset D(A^\infty) = \bigcap\nolimits_{n \in \mathbb{N}} D(A^n). \tag{10}$$

P r o o f. By Lemmas 3.1 and 3.2,

$$\int_0^\infty \lambda^{n-1} f(\lambda)\, d\lambda = \int_0^\infty \lambda^n f(\lambda)(\lambda + A)^{-1}\, d\lambda$$

$$+ \int_0^\infty \lambda^{n-1} f(\lambda)\, A(\lambda + A)^{-1}\, d\lambda = 0, \quad n \in \mathbb{N},$$

and the integral $\int_0^\infty \lambda^{n-1} f(\lambda)(\lambda + A)^{-1}\, d\lambda$ is convergent. On the other hand, as the operator A is closed,

$$A \int_0^\infty \lambda^{n-1} f(\lambda)(\lambda + A)^{-1}\, d\lambda = -\int_0^\infty \lambda^n f(\lambda)(\lambda + A)^{-1}\, d\lambda, \qquad n \in \mathbb{N}.$$

We can now proceed by induction on n to obtain

$$\int_0^\infty \lambda^n f(\lambda)(\lambda + A)^{-1}\, d\lambda = (-1)^{n+1} A^n S_\gamma(t),$$

which shows that $R(S_\gamma(t)) \subset D(A^n)$, $n \in \mathbb{N}$. ■

LEMMA 3.4. *Let* $F(z) = e^{-tz^\gamma}$, $0 < \gamma < \frac{1}{2}$, $t > 0$, $z_0 \in \mathbb{C}$ *with* $Re\, z_0 > 0$ *and* $f(\lambda)$ *be as in (5). Then*

$$\int_0^\infty \frac{f(\lambda)}{(\lambda + z_0)^{n+1}} d\lambda = \frac{(-1)^{n+1}}{n!} F^{(n)}(z_0), \qquad n = 0, 1, 2, \ldots \tag{11}$$

where $F^{(n)}$ *denotes the* $n-th$ *derivative of* F, $n \geq 1$, *and* $F^{(0)} = F$.

P r o o f. Consider the path (8), with $0 < \delta < |z_0|$, $b > 0$ and θ_0^* sufficiently small so that $\frac{b}{\sin\theta_0^*} > |z_0|$. With these assumptions, for $0 < \theta^* \leq \theta_0^*$, by Cauchy's theorem we have

$$\frac{1}{2\pi i} \int_{\Gamma(\theta^*, b, \delta)} \frac{F(z)}{(z - z_0)^{n+1}} dz = \frac{1}{n!} F^{(n)}(z_0), \qquad n = 0, 1, 2, \ldots.$$

Note that with this choice of θ_0^* and taking $0 < \theta^* < \theta_0^*$, if $z \in \Gamma_3(\theta^*, b)$, then

$$|z_0| < \frac{b}{\sin\theta^*} \quad \text{and} \quad |z - z_0| \geq \frac{b}{\sin\theta^*} - |z_0| \geq \frac{b}{\sin\theta_0^*} - |z_0|,$$

and hence, by dominated convergence, equality (11) is easily obtained. ■

From now on, the notation $[x]$ means the integer part of $x \in \mathbb{R}$.

Proposition 3.5. *If $A \in \mathcal{M}(m)$, $m \geq -1$ and $p = [m+2]$, then*

$$D(A^p) \subset \left\{ \phi \in X : \lim_{t \to 0} S_\gamma(t)\phi = \phi \right\}. \tag{12}$$

P r o o f. Let us first note that the derivatives of the function $F(z) = e^{-tz^\gamma}$, $t > 0$, $0 < \gamma < \frac{1}{2}$, can be written as $F^{(n)}(z) = tH_n(z)$, $n \in \mathbb{N}$, for certain functions $H_n(z)$ which are analytic in $\mathbb{C} \setminus]-\infty, 0]$.

Suppose that $m \geq 0$, $\phi \in D(A^p)$, $\lambda > 0$ and $z_0 \in \mathbb{C}$ with $\operatorname{Re} z_0 > 0$. Then from the identity

$$(\lambda + A)^{-1}\phi = \left(\sum_{k=1}^{p} \frac{1}{(\lambda+z_0)^k}(z_0 - A)^{k-1}\phi \right) + \frac{1}{(\lambda+z_0)^p}(\lambda+A)^{-1}(z_0 - A)^p \phi$$

and (11), it follows that

$$\begin{aligned} S_\gamma(t)\phi &= e^{-tz_0^\gamma}\phi + t\sum_{k=2}^{p} \frac{(-1)^{k+1}}{(k-1)!} H_{k-1}(z_0)(z_0 - A)^{k-1}\phi \\ &\quad - \int_0^{+\infty} \frac{f(\lambda)}{(\lambda+z_0)^p}(\lambda+A)^{-1}(z_0-A)^p \phi d\lambda \end{aligned} \tag{13}$$

for all $t > 0$. Now letting $t \to 0$, we find that

$$\lim_{t \to 0} \left[e^{-tz_0^\gamma}\phi + t\sum_{k=2}^{p} \frac{(-1)^{k+1}}{(k-1)!} H_{k-1}(z_0)(z_0 - A)^{k-1}\phi \right] = \phi.$$

On the other hand, for a fixed $\lambda > 0$, $\lim_{t \to 0} f(\lambda) = 0$. Taking into account (2), (6), (7) and the fact that $m + 1 < p$, and applying dominated convergence to the integral in (13), we obtain that

$$\lim_{t \to 0} \int_0^{+\infty} \frac{f(\lambda)}{(\lambda+z_0)^p}(\lambda+A)^{-1}(z_0-A)^p \phi d\lambda = 0$$

and therefore $\lim_{t \to 0} S_\gamma(t)\phi = \phi$.

If $-1 < m < 0$, then in (13) not would appear the second term but we would reason in the same way. In the particular case $m = -1$ (non-negative operator) we know that the continuity with respect to t of operator $S_\gamma(t)$ holds in X. ■

In the sequel, we let p stand for $[m+2]$.

Proposition 3.6. *Let $A \in \mathcal{M}(m)$ with $m \geq -1$. Then the following properties hold.*

(i) The subspace $D(A^{\infty}) = \bigcap_{n\in\mathbb{N}} D(A^n) \neq \emptyset$*, endowed with the system of seminorms*

$$\|\phi\|_q = \max_{o\leq k\leq q}\left\{\left\|A^k\phi\right\|\right\}, \quad \phi \in D(A^{\infty}), \quad q = 0,1,2,\ldots \tag{14}$$

is a Fréchet space. We will denote this space by $Y = \left\{D(A^{\infty}), \|\cdot\|_q\right\}$.

(ii) The operator $B = A\mid_{D(A^{\infty})}: D(A^{\infty}) \longrightarrow D(A^{\infty})$ *is continuous and* $\rho(B) \supset \rho(A)$.

(iii) If $\phi \in D(A^{\infty})$, $\lambda > 0$ *and* $q = 0,1,2,\ldots$, *then*

$$\left\|\lambda(\lambda + B)^{-1}\phi\right\|_q \leq (1 + c(m,M))\|\phi\|_{q+[m+2]}, \tag{15}$$

where $c(m,M) > 0$ *depends only on* m *and* M*. Consequently, the operator* B *is non-negative in the Fréchet space* Y.

P r o o f. Note first that since $D(A^{\infty}) \supset R(S_{\gamma}(t))$, $D(A^{\infty}) \neq \{0\}$. On the other hand, as the operators A^r, with $r \in \mathbb{N}$, are closed, Y is a Fréchet space. Moreover, from the inequality $\|B\phi\|_q \leq \|\phi\|_{q+1}$, with $\phi \in D(A^{\infty})$, it follows that B is continuous. The inclusion $\rho(B) \supset \rho(A)$ is evident and so $]-\infty, 0[\subset \rho(B)$.

The non-negative character of B, for $0 < \lambda < 1$, is a straightforward consequence of (2). Now let $m \geq 0$, $\lambda \geq 1$ and $\phi \in D(A^{\infty})$. Writing $n = [m+1]$ and using the relation

$$(\lambda+B)^{-1}\phi = \lambda^{-n}(\lambda+B)^{-1}B^n\phi + \sum_{t=0}^{n-1}\binom{n}{t}\sum_{s=0}^{n-t-1}\binom{n-t-1}{s}\lambda^{s-n}B^{n-1-s}\phi$$

on deduces the estimate

$$\left\|(\lambda + B)^{-1}\phi\right\|_q \leq c(m,M)\|\phi\|_{q+[m+1]}, \quad q = 0,1,2,\ldots, \tag{16}$$

where

$$c(m,M) = 2M + \sum_{t=0}^{n-1}\binom{n}{t}\sum_{s=0}^{n-t-1}\binom{n-t-1}{s}.$$

Further, for $-1 \leq m < 0$ and $\lambda \geq 1$, the estimate (16) is evident. Thus, for $\lambda > 0$, $\phi \in D(A^{\infty})$ and $q = 0,1,2,\ldots$ we obtain

$$\left\|\lambda(\lambda + B)^{-1}\phi\right\|_q \leq \|\phi\|_q + \left\|(\lambda + B)^{-1}B\phi\right\|_q \leq (1 + c(m,M))\|\phi\|_{q+[m+2]}.$$

This proves that the operator B in non-negative in Y. ∎

The theory of fractional powers of non-negative operators defined in sequentially complete complex spaces ([9]-[10], [11]-[13]) allows us to define the power B^{α}, with $\alpha \in \mathbb{C}$, as the closure in Y of the operator J_B^{α} which was introduced by Balakrishnan and Komatsu (see [1] and [12, Prop.3.1.3]), ([7]-[8] and [12, Def. 7.2.1 y 7.2.2]).

Definition 3.1. Let $A \in \mathcal{M}(m)$, with $m \geq -1$, $\alpha \in \mathbb{C}$, $\tau \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $B = A\,|_{D(A^{\infty})}$. If $\operatorname{Re}\alpha > 0$ and $n \in \mathbb{N}$ such that $0 < \operatorname{Re}\alpha < n$, then we define the operator B^{α}, with domain $D(B^{\alpha}) = D(A^{\infty})$, as

$$B^{\alpha}\phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^{\infty} \lambda^{\alpha-1} \left[B(\lambda+B)^{-1}\right]^n \phi d\lambda, \quad \phi \in D(B^{\alpha}). \tag{17}$$

If in addition the operator A is injective, we define:

– for $\phi \in D(B^{-\alpha}) = D(A^{\infty}) \cap R(B^n)$,

$$B^{-\alpha}\phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^{\infty} \lambda^{n-\alpha-1} \left[(\lambda+B)^{-1}\right]^n \phi d\lambda \tag{18}$$

– and for $\phi \in D\left(B^{i\tau}\right) = D(A^{\infty}) \cap R(B)$,

$$B^{i\tau}\phi = \frac{\sinh \pi\tau}{\pi\tau} \int_0^{\infty} \lambda^{i\tau} (\lambda+B)^{-2} B\phi d\lambda. \tag{19}$$

Proposition 3.7. *If $A \in \mathcal{M}(m)$, $m \geq -1$, $\alpha \in \mathbb{C}$ and $B = A\,|_{D(A^{\infty})}$, then the operator B^{α} is closable in X.*

P r o o f. Let $\{\phi_n\}_{n\in\mathbb{N}} \subset D(B^{\alpha}) \subset D(A^{\infty})$ be a sequence such that

$$\phi_n \to 0 \quad \text{and} \quad B^{\alpha}\phi_n \to \phi \quad \text{for the topology of } X.$$

The inclusion (10) and the closedness of the operators A^k, $k = 0, 1, 2, \ldots$, imply that the operator $A^k S_{\gamma}(t)(1+A)^{-1}$ is bounded and hence the sequences

$$\left\{S_{\gamma}(t)(1+A)^{-p}\phi_n\right\}_{n\in\mathbb{N}} \quad \text{and} \quad \left\{S_{\gamma}(t)(1+A)^{-p}B^{\alpha}\phi_n\right\}_{n\in\mathbb{N}}$$

converge to zero and $S_{\gamma}(t)(1+A)^{-p}\phi$, respectively, for the topology of Y. Taking into account that $S_{\gamma}(t)(1+A)^{-p}$ is a bounded operator which commutes with B^{α}, and since B^{α} is closed, $S_{\gamma}(t)(1+A)^{-p}\phi = 0$. Now letting $t \to 0$, from (12) it follows that $(1+A)^{-p}\phi = 0$. Since $(1+A)^{-p}$ is injective, $\phi = 0$. This proves that B^{α} is closable in X. ∎

Given a closed linear operator A and a linear subspace $Z \subset X$, by $A \parallel_Z$ we denote the part of A in Z, that is, the operator defined as $(A \parallel_Z)\,\phi = A\phi$, with domain

$$D(A) = \{\phi \in D(A) \cap Z : A\phi \in Z\}.$$

PROPOSITION 3.8. *Let $A \in \mathcal{M}(m)$, $m \geq -1$ and $B = A\,|_{D(A^\infty)}$. Then the following properties hold:*

(i) If $\alpha, \beta \in \mathbb{C}$, with $Re\,\alpha, Re\,\beta > 0$, then

$$\overline{B^{\alpha+\beta}} \subset \overline{B^\alpha B^\beta} \subset \overline{\overline{B^\alpha}\ \overline{B^\beta}} \tag{20}$$

and

$$\overline{B^\alpha}\ \overline{B^\beta} \parallel_{D(A^p)} \subset \overline{B^{\alpha+\beta}}. \tag{21}$$

Moreover, if A is injective, then (20) and (21) hold for $\alpha, \beta \in \mathbb{C}$.

(ii) If A is injective and $\alpha \in \mathbb{C}$, then

$$\overline{B^{-\alpha}}\ \overline{B^\alpha} = I\,|_{D\left(\overline{B^\alpha}\right)} \quad \text{and} \quad \overline{B^\alpha}\ \overline{B^{-\alpha}} = I\,|_{D\left(\overline{B^{-\alpha}}\right)}. \tag{22}$$

Hence $\overline{B^\alpha}$ is injective and $\left(\overline{B^\alpha}\right)^{-1} = \overline{B^{-\alpha}}$.

P r o o f. The second extension of (20) is trivial and the first one is a straightforward consequence of the additivity of the fractional powers of B. Now let $\phi \in D\left(\overline{B^\alpha}\ \overline{B^\beta} \parallel_{D(A^p)}\right)$. Since $S_\gamma(t)$ commutes with $\overline{B^\alpha}$ and $\overline{B^\beta}$, from (12) one deduces that the nets $\{S_\gamma(t)\,\phi\}_{t>0}$ and $\left\{B^{\alpha+\beta} S_\gamma(t)\,\phi\right\}_{t>0}$ converge to ϕ and $\overline{B^\alpha}\ \overline{B^\beta}\phi$, respectively as $t \to 0$. This proves (21).

The statement (ii) is a direct consequence of the non-negativity of B in Y, since B is injective and the equality

$$\phi = B^{-\alpha} B^\alpha \phi = B^\alpha B^{-\alpha} \phi$$

holds for all $\phi \in D(A^\infty)$. ∎

PROPOSITION 3.9. *Let $A \in \mathcal{M}(m)$, $m \geq -1$, $\alpha \in \mathbb{C}$, with $Re\,\alpha > 0$ and $n \in \mathbb{N}$ be such that $0 < Re\,\alpha < n$. Then*

$$D\left(A^{n(p+1)+p}\right) \subset D\left(\overline{B^\alpha}\right). \tag{23}$$

If in addition A is injective and $\tau \in \mathbb{R}$, then

$$D\left(A^{n(p+1)+p}\right) \cap R\left(A^{n}\right) \subset D\left(\overline{B^{-\alpha}}\right) \tag{24}$$

and

$$D\left(A^{3p+2}\right) \cap R\left(A\right) \subset D\left(\overline{B^{i\tau}}\right). \tag{25}$$

P r o o f. Since A is non-negative in zero, from (17) and (15) it follows that, $\varphi \in D\left(A^{\infty}\right)$,

$$\left\| \frac{\Gamma\left(\alpha\right)\Gamma\left(n-\alpha\right)}{\Gamma\left(n\right)} B^{\alpha}\varphi \right\| \leq \frac{\left(1+2M\right)^{n}}{\operatorname{Re}\alpha}\left\|\varphi\right\| + \frac{\left(1+c\left(m,M\right)\right)^{n}}{n-\operatorname{Re}\alpha}\left\|\varphi\right\|_{n(p+1)}.$$

That is, there exists $K_1 = K_1\left(m, M, n, \alpha\right) > 0$ such that

$$\left\|B^{\alpha}\varphi\right\| \leq K_1 \left\|\varphi\right\|_{n(p+1)}, \quad \varphi \in D\left(A^{\infty}\right).$$

Now let $\phi \in D\left(A^{n(p+1)+p}\right) \subset D\left(A^{2n+p}\right) \subset D\left(A^{p}\right)$ and $0 < \gamma < 1/2$. By (12), the net $\left\{S_{\gamma}\left(t\right)\phi\right\}_{t>0}$ converges to ϕ as $t \to 0$, and the nets $\left\{A^{k}S_{\gamma}\left(t\right)\phi\right\}_{t>0}$, with $k = 0, 1, \cdots, 2n$, are convergent as well.

From the preceding estimate we obtain that

$$\left\|B^{\alpha}S_{\gamma}\left(t\right)\phi - B^{\alpha}S_{\gamma}\left(t'\right)\phi\right\| \leq K_1 \left\|S_{\gamma}\left(t\right)\phi - S_{\gamma}\left(t'\right)\phi\right\|_{n(p+1)}, \quad t, t' > 0,$$

and therefore $\left\{B^{\alpha}S_{\gamma}\left(t\right)\phi\right\}_{t>0}$ is a Cauchy net in X. Hence, $\phi \in D\left(\overline{B^{\alpha}}\right)$.

Next, we assume that A is injective. Let $\varphi \in D\left(A^{\infty}\right) \cap R\left(B^{n}\right)$. From (18) it follows that there exists $\eta \in D\left(B^{n}\right)$, with $\varphi = B^{n}\eta$, such that

$$\left\| \frac{\Gamma\left(\alpha\right)\Gamma\left(n-\alpha\right)}{\Gamma\left(n\right)} B^{-\alpha}\varphi \right\| \leq \frac{\left(1+2M\right)^{n}}{n-\operatorname{Re}\alpha}\left\|\eta\right\| + \frac{\left(1+c\left(m,M\right)\right)^{n}}{\operatorname{Re}\alpha}\left\|\eta\right\|_{n(p+1)}.$$

So, there exists $K_2 > 0$ such that

$$\left\|B^{-\alpha}\varphi\right\| \leq K_2 \left\|B^{-n}\varphi\right\|_{n(p+1)}, \quad \varphi \in D\left(A^{\infty}\right) \cap R\left(B^{n}\right).$$

A similar reasoning as before leads to (24).

Finally, if $\varphi \in D\left(A^{\infty}\right) \cap R\left(B\right)$, then there exists $\eta \in D\left(B\right)$, with $\varphi = B\eta$, such that

$$\left\| \frac{\pi\tau}{\sinh \pi\tau} B^{i\tau}\varphi \right\| \leq \left(1+2M\right)^{2}\left\|\eta\right\| + \left(1+c\left(m,M\right)\right)^{2}\left\|\eta\right\|_{2(p+1)}, \quad \tau \in \mathbb{R}\backslash\left\{0\right\}.$$

This estimate is an immediate consequence of (19). As before, there exists $K_3 > 0$ such that

$$\left\|B^{i\tau}\varphi\right\| \leq K_3 \left\|B^{-1}\varphi\right\|_{2(p+1)}.$$

This proves (25) for the case $\tau \neq 0$. The case $\tau = 0$ is obvious since $\left\|B^{i\tau}\varphi\right\| \leq \|\varphi\|$. ∎

Theses assertions and Remark 2.4 allow us to give the following definitions.

Definition 3.2. Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $A \in \mathcal{M}(m)$, $m \geq -1$ and $B = A\,|_{D(A^\infty)}$. Given $n \in \mathbb{N}$ such that $0 < \operatorname{Re}\alpha < n$, we define

$$A^\alpha = (1+A)^{n(p+1)+p}\overline{B^\alpha}(1+A)^{-n(p+1)-p} \tag{26}$$

with domain

$$D\left(A^\alpha\right) = \left\{\phi \in X : \overline{B^\alpha}(1+A)^{-n(p+1)-p}\phi \in D\left(A^{n(p+1)+p}\right)\right\}.$$

Remark 3.1. The operator A^α is well-defined by (23) and it is closed since $\overline{B^\alpha}(1+A)^{-n(p+1)-p}$ is bounded and $(1+A)^{n(p+1)+p}$ is closed. Note that the operator $\overline{B^\alpha}$ commutes with the resolvent and that the property of additivity of the powers of integer exponents of the operator $(1+A)$ implies that given an integer $r_0 > n$, writing $k = r_0 - n > 0$, we have that

$$\begin{aligned} A^\alpha &= (1+A)^{r_0(p+1)+p}(1+A)^{k(p+1)}\overline{B^\alpha}(1+A)^{-n(p+1)-p} \\ &= (1+A)^{r_0(p+1)+p}\overline{B^\alpha}(1+A)^{-r_0(p+1)-p}. \end{aligned}$$

So, the preceding definition does not depend on the particular choice of n. A similar argument shows that the definition does not depend on $m \geq -1$.

Definition 3.3. Let $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$, $A \in \mathcal{M}(m)$ be injective, $m \geq -1$ and $B = A\,|_{D(A^\infty)}$. Given $n \in \mathbb{N}$ such that $0 < \operatorname{Re}\alpha < n$, we define

$$A^{-\alpha} = (1+A)^{n(p+2)+p}A^{-n}\overline{B^{-\alpha}}A^n(1+A)^{-n(p+2)-p} \tag{27}$$

with domain

$$D\left(A^{-\alpha}\right) = \left\{\phi \in X : \overline{B^{-\alpha}}A^n(1+A)^{-n(p+2)-p}\phi \in D\left(A^{n(p+1)+p}\right) \cap R\left(A^n\right)\right\}.$$

As above, from (24) it follows that the operator $A^{-\alpha}$ is well-defined, and it is closed since $\overline{B^{-\alpha}}A^n(1+A)^{-n(p+2)-p}$ is bounded and $(1+A)^{n(p+2)+p}A^{-n}$ is closed.

Given $r_0 > n$, writing $k = r_0 - n > 0$, the bounded linear operator $A^k (1+A)^{-(p+2)k}$ commutes with A^{-n} and $\overline{B^{-\alpha}}$. Moreover, $(1+A)^{-(p+2)k}$ and $A^n(1+A)^{-n(p+2)-p}$ also commute. Hence,

$$\begin{aligned} A^{-\alpha} &= (1+A)^{r_0(p+2)+p} A^{-k} A^k (1+A)^{-(p+2)k} A^{-n} \overline{B^{-\alpha}} A^n (1+A)^{-n(p+2)-p} \\ &= (1+A)^{r_0(p+2)+p} A^{-r_0} \overline{B^{-\alpha}} A^{r_0} (1+A)^{-r_0(p+2)-p} \end{aligned}$$

which proves that this definition does not depend on n. The same holds with respect to m.

Definition 3.4. Let $\tau \in \mathbb{R}$, $A \in \mathcal{M}(m)$ be injective, with $m \geq -1$ and $B = A\,|_{D(A^\infty)}$. We define

$$A^{i\tau} = (1+A)^{3p+3} A^{-1} \overline{B^{i\tau}} A (1+A)^{-3p-3} \tag{28}$$

with domain

$$D\left(A^{i\tau}\right) = \left\{ \phi \in X : \overline{B^{i\tau}} A (1+A)^{-3p-3} \phi \in D\left(A^{3p+2}\right) \cap R(A) \right\}.$$

The inclusion (25) implies that this operator is well-defined and it is closed. This definition is also independent of m.

Remark 3.2. Note that the three preceding definitions can be unified, for all $\alpha \in \mathbb{C}$, in the form of the operator

$$A^{\alpha} = (1+A)^{r_0(p+2)+p} A^{-r_0} \overline{B^{\alpha}} A^{r_0} (1+A)^{-r_0(p+2)-p},$$

where r_0 is an integer which satisfies $r_0 > \max\{|\mathrm{Re}\,\alpha|, 3\}$. Moreover, taking into account that $\overline{B^{\alpha}}$ commutes on its domain with the bounded linear operator $A^{r_0}(1+A)^{-r_0(p+2)-p}$, the operator A^{α} is an extension of $\overline{B^{\alpha}}$.

Definition 3.5. Let $A \in \mathcal{M}(m)$ be injective, with $m \leq -1$ and $\alpha \in \mathbb{C}$. we define

$$A^{\alpha} = \left(A^{-1}\right)^{-\alpha}. \tag{29}$$

Remark 3.3. Note that by Remark 2.4, A^{-1} belongs to $\mathcal{M}(-m-2)$, with $-m-2 \geq -1$ and therefore the power $\left(A^{-1}\right)^{-\alpha}$ is given by means of (26), (27) and (28). In particular, if A is a non-negative operator ($m = -1$), then the power can be defined through the operators constructed in (26), (27) and (28), or in (29), which coincide with ones of the usual definition ([12, Th. 5.2.1, Def. 7.1.1 and 7.1.2]). The coherence of the definitions is evident for $\mathrm{Re}\,\alpha > 0$, since $\overline{B^{\alpha}}$ is a restriction of the closure of the Balakrishnan operator $\overline{J^{\alpha}}$ and (23) holds. For the remaining exponents the coherence is also immediate since our power satisfies $A^{-\alpha} = (A^{\alpha})^{-1}$ and the additivity property, as we will see later on.

4. Basic properties of fractional powers

PROPOSITION 4.1. *Let $A \in \mathcal{M}(m)$.*

(i) If A is injective, then $A^0 = I$.

(ii) $A^n = A \overset{n-times}{\cdots} A$, for all $n \in \mathbb{N}$.

(iii) If A is injective and $\alpha \in \mathbb{C}$, then A^α is injective as well. Moreover,

$$A^\alpha A^{-\alpha} = I\mid_{D(A^{-\alpha})}, A^{-\alpha}A^\alpha = I\mid_{D(A^\alpha)} \text{ and } A^{-\alpha} = (A^\alpha)^{-1} = \left(A^{-1}\right)^\alpha.$$

P r o o f. In order to prove (i) in the case $m \geq -1$, we fix $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$. Taking into account (20) and (22), we find that

$$\overline{B^0} = \overline{B^{\alpha+(-\alpha)}} \subset \overline{\overline{B^\alpha}\ \overline{B^{-\alpha}}} \subset I.$$

So, $A^0 \subset I$. To see that $I \subset A^0$, take $\phi \in X$ and an integer r_0, large enough. By (22) and (21),

$$\phi = (1+A)^{r_0(p+2)+p} A^{-r_0} \overline{B^{\alpha+(-\alpha)}} A^{r_0} (1+A)^{-r_0(p+2)-p} \phi.$$

Suppose $m \geq -1$ in the statement (ii). Since the operators $\overline{B^n}$ and $S_\gamma(t)$ commute, and as A is closed, by using (12) is not difficult to prove that $\overline{B^n}\parallel_{D\left(A^{[m+2]}\right)}$ is a restriction of $A \overset{n-\text{times}}{\cdots} A$. From this one easily obtains (ii). From Remark 2.4, it follows that the preceding results are also true for $m \leq -1$.

Finally, if A is injective, by (22), then $\overline{B^\alpha}$ is also injective, and A^α is always a composition of injective operators. So, A^α is injective.

If $m \geq -1$, in the definition of A^α we take $s_0 = r_0(p+2) + p$, with $r_0 > 0$ large enough, and from (22) we obtain

$$(A^\alpha)^{-1} = (1+A)^{s_0}\left(\overline{B^\alpha}\right)^{-1}(1+A)^{-s_0} = (1+A)^{s_0}\left(\overline{B^{-\alpha}}\right)(1+A)^{-s_0} = A^{-\alpha}.$$

Now let $m \leq -1$. For the operator A^{-1} the preceding equality holds, and then

$$(A^\alpha)^{-1} = \left[\left(A^{-1}\right)^{-\alpha}\right]^{-1} = \left(A^{-1}\right)^\alpha = A^{-\alpha}.$$

Both operators $(A^\alpha)^{-1}$ and $A^{-\alpha}$ coincide with $\left(A^{-1}\right)^\alpha$ in accordance with Remark 2.4. ∎

Proposition 4.2. (Additivity) *Let* $A \in \mathcal{M}(m)$ *with* $m \geq -1$ *and* $\alpha, \beta \in \mathbb{C}$, *with* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$. *Then the following properties hold:*

(i) $A^{\alpha} A^{\beta} \subset A^{\alpha+\beta}$.

(ii) If $\phi \in D\left(A^{\alpha+\beta}\right) \cap D\left(A^{\beta}\right)$, *then*

$$A^{\beta}\phi \in D\left(A^{\alpha}\right) \quad \text{and} \quad A^{\alpha+\beta}\phi = A^{\alpha}A^{\beta}\phi.$$

In particular, if $D\left(A^{\alpha+\beta}\right) \subset D\left(A^{\beta}\right)$, *then* $A^{\alpha+\beta} = A^{\alpha}A^{\beta}$.

Furthermore, if A is injective, then the preceding assertions remain true for all $\alpha, \beta \in \mathbb{C}$ *and* $m \leq -1$.

P r o o f. Consider $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$ and $m \geq -1$. Fix $r_0 > \operatorname{Re}\alpha + \operatorname{Re}\beta$ and take $s_0 = r_0(p+2) + p$ for the expressions of A^{α} and A^{β}. Hence, for $\phi \in D\left(A^{\alpha}A^{\beta}\right)$ we get

$$\begin{aligned} A^{\alpha}A^{\beta}\phi &= (1+A)^{s_0}\,\overline{B^{\alpha}}\;\overline{B^{\beta}}\,\|_{D(A^{p+1})}\,(1+A)^{-s_0}\,\phi \\ &= (1+A)^{s_0}\,\overline{B^{\alpha+\beta}}\,(1+A)^{-s_0}\,\phi \end{aligned}$$

and since $r_0 > \operatorname{Re}(\alpha+\beta)$, $\phi \in D\left(A^{\alpha+\beta}\right)$ and $A^{\alpha}A^{\beta}\phi = A^{\alpha+\beta}\phi$.

To prove (ii), take s_0 and r_0 as before. Then for $\phi \in D\left(A^{\alpha+\beta}\right)$ we get $(1+A)^{-s_0}\,\phi \in D\left(A^{p}\right)$ and $A^{\alpha+\beta}\phi = (1+A)^{s_0}\,\overline{B^{\alpha+\beta}}\,(1+A)^{-s_0}\,\phi$.

On the other hand, given $\phi \in D\left(A^{\beta}\right)$ and the preceding choice of r_0, we find that

$$(1+A)^{-s_0}\,(1+A)^{s_0}\,\overline{B^{\beta}}\,(1+A)^{s_0}\,\phi \in D\left(A^{s_0}\right) \subset D\left(\overline{B^{\alpha}}\right).$$

Thus, $\overline{B^{\alpha}}\,(1+A)^{-s_0}\,A^{\beta}\phi \in D\left(A^{s_0}\right)$ and $A^{\beta}\phi \in D\left(A^{\alpha}\right)$, and taking into account (21) we have that $A^{\alpha+\beta}\phi = A^{\alpha}A^{\beta}\phi$.

For the rest of exponents we can proceed in a similar way, since we can express the operators A^{α}, A^{β} and $A^{\alpha+\beta}$ in the a unified form with a large enough integer r_0.

If A is injective and $m \leq -1$, by Remark 2.4, the operator A^{-1} satisfies the above properties, and hence the rest of the proof runs as before. ∎

The next example makes evident that the additivity is not complete.

Example 4.1. Let $X = C_{\infty}\left(\mathbb{R}; \mathbb{C}^2\right)$ be the Banach space of the continuous functions which vanish at ∞, $f = (f_1, f_2) : \mathbb{R} \to \mathbb{C}^2$, with the norm

$$\|f\|_{\infty} = \max_{i=1,2}\left\{\|f_i\|_{\infty}\right\}, \;\; \|f_i\|_{\infty} = \sup_{x\in\mathbb{R}}\left\{|f_i(x)|\right\} \quad (i = 1, 2).$$

The multiplication operator A given by

$$Af(x) = \begin{pmatrix} x + i\sqrt{x^2+1} & x^2 \\ 0 & x - i\sqrt{x^2+1} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \quad , \quad (x \in \mathbb{R})$$

defined on $D(A) = \{f \in X : Af \in X\}$, is not sectorial but $(\lambda + A)^{-1}$ is uniformly bounded with respect to $\lambda > 0$. Therefore $A \in \mathcal{M}(0)$.

It can be proved that for all $\alpha \in \mathbb{C}$, the power A^α is the multiplication operator by the matrix

$$\begin{pmatrix} \left(x + i\sqrt{x^2+1}\right)^\alpha & k_{12}(x) \\ 0 & \left(x - i\sqrt{x^2+1}\right)^\alpha \end{pmatrix} \quad , \quad (x \in \mathbb{R})$$

with

$$k_{12}(x) = \frac{x^2}{2i\sqrt{x^2+1}} \left[\left(x + i\sqrt{x^2+1}\right)^\alpha - \left(x - i\sqrt{x^2+1}\right)^\alpha \right].$$

Fix $-\frac{1}{2} \leq \alpha, \beta < 0$, and p such that $\alpha + \beta + 1 < p < \beta + 1$. Let us consider the function

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)) = \left(0, \left(x + i\sqrt{x^2+1}\right)^{-p}\right) \in C_\infty\left(\mathbb{R}\,; \mathbb{C}^2\right),$$

then

$$A^{\alpha+\beta}\varphi(x) = \begin{pmatrix} k_{\alpha+\beta}(x)\left(x + i\sqrt{x^2+1}\right)^{-p} \\ \left(x - i\sqrt{x^2+1}\right)^{\alpha+\beta}\left(x + i\sqrt{x^2+1}\right)^{-p} \end{pmatrix}.$$

Since $\alpha + \beta + 1 - p < 0$ and $\alpha + \beta - p < 0$, then

$$\left| k_{\alpha+\beta}(x)\left(x + i\sqrt{x^2+1}\right)^{-p} \right| \leq \frac{x^2}{\sqrt{1+x^2}} \left(\sqrt{2x^2+1}\right)^{\alpha+\beta-p} \overset{x\to\infty}{\longrightarrow} 0$$

$$\left| \left(x - i\sqrt{x^2+1}\right)^{\alpha+\beta}\left(x + i\sqrt{x^2+1}\right)^{-p} \right| = \left(\sqrt{2x^2+1}\right)^{\alpha+\beta-p} \overset{x\to\infty}{\longrightarrow} 0.$$

So, $\varphi \in D\left(A^{\alpha+\beta}\right)$. As $p < \beta + 1$, then

$$\left| k_\beta(x)\left(x + i\sqrt{x^2+1}\right)^{-p} \right| \geq C_1 \frac{x^2}{\sqrt{1+x^2}} \left(\sqrt{2x^2+1}\right)^{\beta-p} \overset{x\to\infty}{\longrightarrow} \infty,$$

where $C_1 = \min\{\sin\alpha\varepsilon, \sin\alpha\omega\} > 0$. Hence $\varphi \notin D\left(A^{\beta}\right)$. It follows that $A^{\alpha}A^{\beta}$ cannot be an extension of $A^{\alpha+\beta}$.

The statement (iii) of Proposition 4.1 implies that for this operator, the complete additivity neither holds for $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$.

Remark 4.1. Obviously, if $A^{-\alpha}$ is bounded, then $A^{\alpha+\beta} = A^{\alpha}A^{\beta}$ for $\beta \in \mathbb{C}$. In fact, given $\phi \in D\left(A^{\alpha+\beta}\right)$, by Proposition 4.2 (i), $\phi \in D\left(A^{-\alpha}A^{\alpha+\beta}\right) \subset D\left(A^{\beta}\right)$.

Proposition 4.3. *If $A \in \mathcal{M}(m)$, with $-1 \le m < 0$, $0 \in \rho(A)$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > m+1$, then $A^{-\alpha}$ is a bounded operator.*

P r o o f. Firstly note that as A^{-1} is bounded, the operator $A^{-\alpha}$ can be expressed in the form $(1+A)^{s_0}\,\overline{B^{-\alpha}}\,(1+A)^{-s_0}$, with $s_0 = r_0(p+1)+p$ and $r_0 \in \mathbb{N}$ such that $0 < m+1 < \operatorname{Re}\alpha < r_0$.

Now suppose that $0 \le m+1 < \operatorname{Re}\alpha < 1$. In this case, from (18) it follows that

$$B^{-\alpha}\phi = \frac{\Gamma(2)}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^{\infty}\lambda^{1-\alpha}(\lambda+A)^{-2}\phi d\lambda \quad , \quad \phi \in D\left(A^{\infty}\right).$$

Denote by

$$L^{-\alpha} = \frac{\Gamma(2)}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^{\infty}\lambda^{1-\alpha}(\lambda+A)^{-2}d\lambda. \tag{30}$$

As $0 \in \rho(A)$, the function $\lambda \to (\lambda+A)^{-2}$ is continuous in $[0,1]$. So, the integral of (30) is absolutely convergent in zero. Taking into account that $m+1-\operatorname{Re}\alpha < 0$ and integrating by parts it is not difficult to prove that the mentioned integral is also absolutely convergent in infinity. Hence, $L^{-\alpha}$ is a bounded extension of $B^{-\alpha}$. This shows that $\overline{B^{-\alpha}} \subset L^{-\alpha}$ and

$$\overline{B^{-\alpha}}(1+A)^{-r_0}\phi = L^{-\alpha}(1+A)^{-r_0}\phi = (1+A)^{-r_0}L^{-\alpha}\phi, \quad \text{for all } \phi \in X.$$

So, $\overline{B^{-\alpha}}(1+A)^{-r_0}\phi$ belongs to $D\left(A^{r_0}\right)$, that is, $\phi \in D\left(A^{-\alpha}\right)$.

If $\operatorname{Re}\alpha \ge 1$, then we take $k \in \mathbb{N}$ such that $m+1 < k \le \operatorname{Re}\alpha < k+1$. We proceed as before to show that

$$B^{-\alpha}\phi = \frac{\Gamma(k+1)}{\Gamma(\alpha)\Gamma(k+1-\alpha)}\int_0^{\infty}\lambda^{k-\alpha}\left[(\lambda+A)^{-1}\right]^{k+1}\phi d\lambda \quad , \quad \phi \in D\left(A^{\infty}\right)$$

and that the operator

$$L^{-\alpha} = \frac{\Gamma(k+1)}{\Gamma(\alpha)\Gamma(k+1-\alpha)}\int_0^{\infty}\lambda^{k-\alpha}\left[(\lambda+A)^{-1}\right]^{k+1}d\lambda$$

is bounded. This completes the proof. ∎

REMARK 4.2. An immediate consequence of the previous proposition and Remark 4.1, is that for an operator $A \in \mathcal{M}(m)$, with $-1 \leq m < 0$ and $0 \in \rho(A)$, the complete additivity holds for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha < -m-1$ and $\beta \in \mathbb{C}$.

The following corollary is an immediate consequence of Propositions 4.3 and 4.1, part (iii).

COROLLARY 4.4. *If $A \in \mathcal{M}(m)$, with $-1 \leq m < 0$, $0 \in \rho(A)$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > m+1$, then $0 \in \rho(A^\alpha)$.*

In general, the spectral mapping theorem, say

$$\sigma(A^\alpha) = \{z^\alpha : z \in \sigma(A)\},$$

does not hold for the class of operators we are considering in this work. In fact, there exist operators A in the class $\mathcal{M}(m)$ with $\sigma(A^\alpha) = \mathbb{C}$ if $-|m+1| < \operatorname{Re}\alpha < |m+1|$. For an example we refer to [16, Example 2.4], where it is considered in the Banach space $X = C_\infty(\mathbb{R}\,;\mathbb{C}^2)$ of continuous functions which vanish at infinity, the multiplication operator

$$\begin{array}{llll} A_q : D(A_q) & \subset X & \to & X \\ & f & \mapsto & A_q f = qf \end{array}$$

with domain $D(A_q) = \{f \in X : qf \in X\}$, where q is the matrix-valued function given by

$$q(x) = \begin{pmatrix} 1+x^2+i\left(x^2+1\right) & x^{4+2m} \\ 0 & 1+x^2-i\left(x^2+1\right) \end{pmatrix}, \quad x \in \mathbb{R}.$$

This operator satisfies $0 \in \rho(A)$ and its spectrum is contained in the closure of some sector S_ω, $0 \leq \omega < \pi$. Moreover, for $\omega < \mu < \pi$ there exists a constant C_μ such that

$$\left\|(z-A)^{-1}\right\| \leq C_\mu |z|^m \quad , \quad z \notin \overline{S_\mu} \quad , \quad -1 < m < 0.$$

Therefore, $A_q \in \mathcal{M}(m)$, with $-1 < m < 0$. The inverse A_q^{-1} of this operator is in the class $\mathcal{M}(m_0)$, $m_0 = -m-2$, with $-2 < m_0 < -1$, and its powers $\left(A_q^{-1}\right)^\alpha$ with $m_0 + 1 < \operatorname{Re}\alpha < -m_0 - 1$, also have spectrum equal to $\mathbb{C}$. It is sufficient to note that for these exponents α, we have $-m-1 < -\operatorname{Re}\alpha < m+1$ and $\left(A_q^{-1}\right)^\alpha = \left[\left(A_q^{-1}\right)^{-1}\right]^{-\alpha} = A_q^{-\alpha}$.

If an almost non-negative operator A is not sectorial, then the powers A^α with $\alpha > 1$, may not belong to $\cup_{m\in\mathbb{R}}\mathcal{M}(m)$. So, in this case, it has no sense to consider $(A^{\alpha})^{\beta}$. For instance, it is sufficient to generate, with the method of Section 2, an operator with spectrum equal to the boundary of the region

$$\left\{z+1 : z \in G_{a,0}^{-} \cup H_{m,a,\infty}^{-}\right\}, \quad a>0, \quad m>-1.$$

It is evident that if $\alpha > 1$, then $\sigma\left(A^{\alpha}\right) \cap \left]-\infty, 0\right[\neq \varnothing$, and therefore A^{α} cannot be almost non-negative.

So, we can only expect it to make sense and it fulfills the property of additivity $(A^{\alpha})^{\beta} = A^{\alpha\beta}$ for $|m+1| \leq |\alpha| < 1$, and these exponents can only exist whether $-2 < m < 0$. Before proving such as multiplicativity result let us prove the following lemma.

Lemma 4.5. *Let $A \in \mathcal{M}\,(m)$, with $-1 \leq m < 0$ and $0 \in \rho\,(A)$. If $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, then*

$$D\left[(A^{\alpha})^{\infty}\right] = D\left(A^{\infty}\right).$$

P r o o f. Given $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$ and $q \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all integer r with $0 \leq r \leq q$, we have $D(A^{s}) \subset D(A^{r\alpha})$. We proceed by induction on r. We know that $D(A^{s})$ is contained in $D(A^{\alpha})$. Assume that $D(A^{s}) \subset D\left[(A^{\alpha})^{r-1}\right]$ for $1 \leq r \leq q$, and take $\phi \in D(A^{s})$. Taking into account that

$$\phi \in D(A^{r\alpha}) \cap D\left[(A^{\alpha})^{r-1}\right] \subset D(A^{r\alpha}) \cap D\left[\left(A^{(r-1)\alpha}\right)\right],$$

by additivity, we get $A^{(r-1)\alpha}\phi = (A^{\alpha})^{r-1}\,\phi \in D(A^{\alpha})$, and thus $\phi \in D\left[(A^{\alpha})^{r}\right]$. Therefore, $D\left(A^{\infty}\right) \subset D\left[(A^{\alpha})^{q}\right]$ and reasoning for all q we obtain the inclusion $D\left(A^{\infty}\right) \subset D\left[(A^{\alpha})^{\infty}\right]$.

In the same way, given $s \in \mathbb{N}$ and taking $r \in \mathbb{N}$ such that $r\alpha - s > 1+m$, due to Proposition 4.8, the operator $A^{s-r\alpha}$ is bounded. By additivity, if $\phi \in D\left(A^{r\alpha}\right)$, then $\phi \in D\left(A^{s-r\alpha}A^{r\alpha}\right) \subset D\left(A^{s}\right)$. Hence, $D((A^{\alpha})^{r}) \subset D\left(A^{r\alpha}\right) \subset D(A^{s})$, and so $D\left[(A^{\alpha})^{\infty}\right] \subset D\left(A^{\infty}\right)$. ■

Theorem 4.6. (Multiplicativity) *Let $A \in \mathcal{M}\,(m)$, with $-1 \leq m < 0$ and $0 \in \rho\,(A)$. If $\alpha \in \mathbb{C}$ with $1+m < |\alpha| < 1$, then $A^{\alpha} \in \mathcal{M}\left(-1+\frac{m+1}{\alpha}\right)$ and*

$$A^{\alpha\beta} = (A^{\alpha})^{\beta}, \quad \textit{for all } \beta \in \mathbb{C}.$$

If $A \in \mathcal{M}\,(m)$, with $-2 < m \leq -1$, is injective and bounded, then for all α such that $|1+m| < |\alpha| < 1$, we have $A^{\alpha} \in \mathcal{M}\left(-1+\frac{m+1}{\alpha}\right)$ and

$$A^{\alpha\beta} = (A^{\alpha})^{\beta}, \quad \textit{for all } \beta \in \mathbb{C}.$$

P r o o f. The multiplicativity for non-negative operators is already well-known and in this case the hypothesis of injectivity and boundedness of the operators A and A^{-1} are not necessary.

First consider the case $-1 \leq m < 0$, $0 \in \rho\,(A)$ and $1 + m < \alpha < 1$. For $\mu > 0$, consider the operator

$$R_\mu = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{\lambda^\alpha}{\lambda^{2\alpha} + 2\lambda^\alpha \mu \cos \alpha\pi + \mu^2} (\lambda + A)^{-1} d\lambda. \tag{31}$$

Taking into account the expression

$$R_\mu = \frac{\sin \alpha\pi}{\alpha\pi} \int_0^\infty \frac{\mu^{-1}\mu^{1/\alpha}t^{1/\alpha}}{t^2 + 2t\cos\alpha\pi + 1} (\mu^{1/\alpha}t^{1/\alpha} + A)^{-1} dt,$$

which is obtained after changing $\lambda^\alpha = \mu t$ and decomposing the integral given into $\int_0^{\mu^{-1}} + \int_{\mu^{-1}}^\infty$ we find that the integral of the second member of (31) is absolutely convergent and moreover there exists a constant $M_1 > 0$ such that $\|R_\mu\| \leq M_1(\mu^{-2} + \mu^{-1+\frac{m+1}{\alpha}})$.

Since B is non-negative in the Fréchet space Y, we know that the operator R_μ, restricted to the domain $D\,(A^\infty)$, coincides with the operator $(\mu + B^\alpha)^{-1}$ (see [12, (5.24)]). So, given $\phi \in X$, $t > 0$ and $0 < \gamma < \frac{1}{2}$ and taking into account (12) and the inclusion $D\left(A^3\right) \subset D\left(\overline{B^\alpha}\right)$, we have

$$\begin{aligned} & S_\gamma(t)(\mu + \overline{B^\alpha})(1 + A)^{-4} R_\mu \phi = (\mu + \overline{B^\alpha}) R_\mu S_\gamma(t)(1 + A)^{-4}\phi \\ = \ & (\mu + B^\alpha)\,(\mu + B^\alpha)^{-1}\, S_\gamma(t)(1 + A)^{-4}\phi = S_\gamma(t)(1 + A)^{-4}\phi. \end{aligned}$$

Now letting $t \to 0$, $(\mu + \overline{B^\alpha})(1 + A)^{-4} R_\mu \phi = (1 + A)^{-4}\phi$, which proves that $R_\mu \phi \in D\,(A^\alpha)$ and $(\mu + A^\alpha) R_\mu \phi = \phi$.

A similar reasoning shows that for $\phi \in D\,(A^\alpha)$, $R_\mu(\mu + A^\alpha)\phi = \phi$. So $-\mu \in \rho\,(A^\alpha)$ and $(\mu + A^\alpha)^{-1} = R_\mu$.

Hence, if $\mu \geq 1$, then

$$\left\|(\mu + A^\alpha)^{-1}\right\| \leq 2M_1 \mu^{-1+\frac{m+1}{\alpha}}.$$

On the other hand, in Corollary 4.4 we have proved that for those exponents, $0 \in \rho(A^\alpha)$. Hence, due to the continuity of the mapping $\mu \to \mu\,(\mu + A^\alpha)^{-1}$ in the interval $[0, 1]$, there exists a constant $M_2 > 0$, such that, for all $\mu < 1$ we have

$$\left\|(\mu + A^\alpha)^{-1}\right\| \leq M_2 \mu^{-1}.$$

In order to prove the inclusion $A^{\alpha\beta} \subset (A^\alpha)^\beta$, note that since $(A^\alpha)^{-1}$ is bounded, we can consider a sufficiently large $m_0 \in \mathbb{N}$, so that $D\left[(A^\alpha)^{m_0}\right]$ is contained in $D\left[(A^\alpha)^\beta\right]$. Let $S_{A^\alpha}(t)$ stand for the operator associated with A^α by (9). If $\phi \in D(A^{\alpha\beta})$, then

$$\begin{aligned} & \left(A^\alpha|_{D[(A^\alpha)^\infty]}\right)^\beta S_{A^\alpha}(t)(1+A^\alpha)^{-m_0}\phi = (B^\alpha)^\beta S_{A^\alpha}(t)(1+A^\alpha)^{-m_0}\phi \\ = \; & A^{\alpha\beta} S_{A^\alpha}(t)(1+A^\alpha)^{-m_0}\phi = S_{A^\alpha}(t)(1+A^\alpha)^{-m_0}A^{\alpha\beta}\phi, \end{aligned}$$

where the first equality follows from Lemma 4.5 and the second one from the multiplicativity of the powers of non-negative operators in locally convex spaces. Hence, we have a net which converges to $(1+A^\alpha)^{-m_0}A^{\alpha\beta}\phi$, as $t \to 0$.

On the other hand, the net $\{S_{A^\alpha}(t)(1+A^\alpha)^{-m_0}\phi\}_{t>0}$ converges to $(1+A^\alpha)^{-m_0}\phi$ when t tends to zero. It follows that

$$\overline{\left(A^\alpha|_{D[(A^\alpha)^\infty]}\right)^\beta}(1+A^\alpha)^{-m_0}\phi = (1+A^\alpha)^{-m_0}A^{\alpha\beta}\phi$$

which implies that $\phi \in D(A^{\alpha\beta})$ and $A^{\alpha\beta}\phi = (A^\alpha)^\beta \phi$. The opposite inclusion can be shown in a similar way.

The case $-2 < m \leq -1$, with A bounded and injective, and $-1 < \alpha < m+1$, is reduced to the previous one making use of Remark 2.4, Proposition 4.1 (*iii*) and (29).

In fact, in this case we have $A^{-1} \in \mathcal{M}(-m-2)$ with $-1 \leq -m-2 < 0$ and $0 \in \rho\left(A^{-1}\right)$. Hence

$$A^\alpha = \left(A^{-1}\right)^{-\alpha} \in \mathcal{M}\left(-1 + \frac{(-m-2)+1}{-\alpha}\right) = \mathcal{M}\left(-1 + \frac{m+1}{\alpha}\right)$$

and

$$(A^\alpha)^\beta = \left[\left(A^{-1}\right)^{-\alpha}\right]^\beta = \left(A^{-1}\right)^{(-\alpha)\beta} = \left(A^{-1}\right)^{-\alpha\beta} = A^{\alpha\beta}.$$

If $A \in \mathcal{M}(m)$ with $-1 \leq m < 0$, $0 \in \rho(A)$ and $-1 < \alpha < -m-1 < 0$, then $0 \leq m+1 < |\alpha| < 1$, and $A^{|\alpha|} \in \mathcal{M}\left(-1 + \frac{m+1}{|\alpha|}\right)$. Therefore it follows by Proposition 4.1 (*iii*) and Remark 2.4,

$$A^\alpha = \left(A^{|\alpha|}\right)^{-1} \in \mathcal{M}\left(-\left(-1 + \frac{m+1}{|\alpha|}\right) - 2\right) = \mathcal{M}\left(-1 + \frac{m+1}{\alpha}\right)$$

and consequently

$$\left(A^{-\alpha}\right)^{-\beta} = \left(A^{|\alpha|}\right)^{-\beta} = A^{|\alpha|(-\beta)} = A^{\alpha\beta}.$$

On the other hand, using the Proposition 4.1 (*iii*), we have

$$\left(A^{-\alpha}\right)^{-\beta} = \left[\left(A^{-\alpha}\right)^{-1}\right]^{\beta} = \left(A^{\alpha}\right)^{\beta}.$$

Hence $A^{\alpha\beta} = \left(A^{\alpha}\right)^{\beta}$ holds.

In the remaining case we can proceed in a similar way. ■

5. Powers in locally convex spaces

Throughout this section we assume that X is a locally convex space, endowed with a directed family of seminorms $\left\{\|\cdot\|_q : q \in P\right\}$. Definition 2.1 can be generalized in this way.

Definition 5.1. A closed linear operator $A : D(A) \subseteq X \to X$, is said to be almost non-negative if $\rho(A) \supset]-\infty, 0[$ and if there exists $m \in \mathbb{R}$ satisfying that for each seminorm $q \in P$ there exists another seminorm $s(q) \in P$ and a constant $M_q \geq 0$, such that

$$\left\|\frac{\lambda}{1+\lambda^{m+1}}(\lambda + A)^{-1}\phi\right\|_q \leq M_q \|\phi\|_{s(q)} \quad , \quad \phi \in X,\ \lambda > 0. \tag{32}$$

For these operators we cannot reasoning as in Theorem 2.1 and therefore we cannot guarantee that the region mentioned there, exists. However, we can define the operator $S_\gamma(t)$ in the same manner as in (9). The results of Propositions 3.3 and 3.5 remain true and the same holds for its respective Lemmas. It is due to the fact that all the integrals which appear can be considered in the sense of a Riemann improper integral of a continuous function defined on $]0, +\infty[$.

Proposition 3.6 was the key to construct our definition of power in a Banach space. We see now that we can also generalize this process. In fact, if $m \geq -1$, then the set $D(A^\infty)$, with the family of seminorms

$$\|\phi\| = \max_{o \leq j,k \leq w} \left\{\left\|A^k \phi\right\|_j\right\}, \quad \phi \in D(A^\infty), \quad w = 0, 1, 2, \ldots,$$

is a Fréchet space. Denote by $Y = \{D(A^\infty), |||\cdot|||_w\}$. In this space, the operator $B = A|_{D(A^\infty)}$ satisfies $|||B\phi|||_w \leq |||\phi|||_{w+1}$ for all w and $\phi \in D(A^\infty)$. Hence B is continuous in Y. Moreover, $\rho(B) \supset \rho(A)$.

If $0 < \lambda < 1$, from (32) it follows that for $w = 0, 1, 2, \ldots$ and $\phi \in D(A^\infty)$ we have

$$\left|\left|\left|\lambda(\lambda + B)^{-1}\phi\right|\right|\right|_w \leq M(w)\, |||\phi|||_{w+r(w)} \quad , \quad 0 < \lambda < 1,$$

where $M(w) = \max_{0\leq j\leq w}\{2M_j\}$ and $r(w)$ satisfies $\|\cdot\|_{r(w)} \geq \max_{0\leq j\leq w}\left\{\|\cdot\|_{s(j)}\right\}$, M_j and $s(j)$, $0 \leq j \leq w$, being the constants and seminorms given by (32) for its respective seminorms $\|\cdot\|_j$, $0 \leq j \leq w$.

On the other hand, if $\lambda \geq 1$ and $-1 < m < 0$, then for $w = 0, 1, 2, \ldots$ and $\phi \in D(A^\infty)$ we have

$$\left|\left|\left|(\lambda + B)^{-1}\phi\right|\right|\right|_w \leq M(w)\,|||\phi|||_{w+r(w)} \quad , \quad \lambda \geq 1.$$

For $m \geq 0$ and $\phi \in D(A^\infty)$, by using the same relation of Proposition 3.6 we obtain, for $\lambda \geq 1$,

$$\left|\left|\left|(\lambda + B)^{-1}\phi\right|\right|\right|_w \leq C_w(m, M)\,|||\phi|||_{[m+1]+w+r(w)}$$

where

$$C_w(m, M) = M(w) + \sum_{t=0}^{[m]}\binom{[m+1]}{t}\left[\sum_{s=0}^{[m]-t}\binom{[m]-t}{s}\right].$$

Since for $w = 0, 1, 2, \ldots$, $\phi \in D(A^\infty)$ and $\lambda > 0$, we have

$$\left|\left|\left|\lambda(\lambda + B)^{-1}\phi\right|\right|\right|_w \leq (1 + C_w(m, M))\,|||\phi|||_{w+r(w)+[m+2]}$$

which shows that B is non-negative in Y. From here we can construct a theory of fractional powers which is similar to the one developed for the Banach space case.

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