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ROUGH MAXIMAL OSCILLATORY SINGULAR INTEGRAL OPERATORS

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Abstract

In this paper, we establish the L^p boundedness of certain maximal oscillatory singular integral operators with rough kernels belonging to certain block spaces. Our L^p boundedness result improves previously known results.

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1. Introduction and statement of results

Let $\mathbf{R}^d$ $(d = n, m \geq 2)$ be the d-dimensional Euclidean space and $\mathbf{S}^{d-1}$ be the unit sphere in $\mathbf{R}^d$ equipped with the normalized Lebesgue measure $d\sigma$. For nonzero $y \in \mathbf{R}^d$, we let $y' = |y|^{-1} y$. Let $\Omega \in L^1(\mathbf{S}^{d-1})$ be a homogeneous function of degree zero on $\mathbf{R}^n$ which satisfies the cancelation property

$$\int_{\mathbf{S}^{d-1}} \Omega(y')d\sigma(y') = 0. \tag{1.1}$$

For a suitable mapping $P : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, consider the oscillatory singular integral operator defined by

$$T_{P,\Omega}(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} f(x-y)\Omega(y') \, |y|^{-n} \, dy, \tag{1.2}$$

and the corresponding maximal operator defined by

$$T^*_{P,\Omega}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} e^{iP(x,y)} f(x-y)\Omega(y')\,|y|^{-n}\,dy \right|. \tag{1.3}$$

The operators in (1.2) and (1.3) have been extensively studied by many authors. For their significance, we refer the reader to consult ([1], [10], [17], [18], [19], among others). In this paper, we are interested in studying maximal operators of the form (1.3). Clearly, if $P = 0$, then the operator $T^*_{P,\Omega}$ is the classical maximal singular integral operator of Calderón-Zygmund type ([4], [5]). When P is a polynomial mapping and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$, Lu-Zhang ([12]) showed that $T^*_{P,\Omega}$ is bounded on L^p for all $1 < p < \infty$. Subsequently, Lu-Wu in ([13]) proved that Lu-Zhang's result still holds under a weaker condition on Ω. In fact, they showed that the operator $T^*_{P,\Omega}$ is bounded on L^p for all $1 < p < \infty$ provided that the function Ω belongs to certain block space $B_q^{(0,0)}(\mathbf{S}^{n-1}), q > 1$. The same result was obtained, as a consequence of a more general result, by A-Salman in ([1]). It should be pointed out here that block spaces were introduced by Jiang and Lu (definition of block spaces will be recalled in Section 2). For background information about block spaces and their use in harmonic analysis, see ([14], [15]).

Motivated by the work of Fan-Pan on singular integrals along subvarieties ([7]), Fan-Yang ([9]) studied L^p estimates of maximal oscillatory singular integral operators of the form (1.3) with singularities spread over sets more general than the diagonal $\{y = x\}$. More precisely, Fan-Yang considered the following maximal oscillatory singular integral operator

$$T^*_{\mathcal{P},\Phi,\Omega}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} e^{i\Phi(y)} f(x-\mathcal{P}(y))\Omega(y')\,|y|^{-n}\,dy \right|, \tag{1.4}$$

where $\mathcal{P}(y) = (P_1, \ldots, P_d)$ is a polynomial mapping from $\mathbf{R}^n$ into $\mathbf{R}^d$, and $\Phi : \mathbf{R}^n \to \mathbf{R}$ is a homogeneous function that satisfies

$$\Phi(ty') = t^{\beta}\Phi(y') \quad \text{for} \quad t > 0, \tag{1.5}$$

$$\Phi(y') \in L^\infty(\mathbf{S}^{n-1}), \text{ and } \int_{\mathbf{S}^{n-1}} \left|\Phi(y')\right|^{-\delta} d\sigma(y') < \infty, \tag{1.6}$$

for some $\delta > 0$ and for some $\beta \neq 0$.

Fan-Yang proved the following result:

THEOREM A. ([9]) *Suppose that* Ω *is a homogeneous function of degree zero on* $\mathbf{R}^n$ *that satisfies (1.1) and that* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q > 1$. *Suppose also that* $\mathcal{P}(y) = (P_1, \ldots, P_d)$ *is a polynomial mapping. If* Φ *is a homogeneous function that satisfies (1.5)-(1.6) with either the index* $\beta \neq 0$ *is not a positive integer or* β *is a positive integer larger than* $\max\{\deg(P_j) : 1 \leq j \leq d\}$, *then the operator* $T^*_{\mathcal{P},\Phi,\Omega}$ *is bounded on* L^p *for all* $1 < p < \infty$. *Moreover, the operator norm is independent of the coefficients of the polynomial mappings* $\{P_j : 1 \leq j \leq d\}$.

In this paper, we are interested in weakening the assumption $\Omega \in L^q(\mathbf{S}^{n-1})$ in Theorem A. In order to state our main result, we cite the following related remarks:

(i) It can be easily shown that if $\Phi : \mathbf{R}^n \to \mathbf{R}$ is a homogeneous function of degree $\beta \neq 0$ which is real analytic on $\mathbf{S}^{n-1}$ (i.e. $\Phi\mid_{\mathbf{S}^{n-1}}$ is real analytic), then the assumptions (1.6) hold. In fact, if $\Phi_1, ..., \Phi_l$ are linearly independent real analytic functions on $\mathbf{S}^{n-1}$and that each Φ_j is homogeneous of degree $\beta \neq 0$, then there exist positive constants $\delta = \delta(\Phi, \mathbf{S}^{n-1}, \mathbf{S}^l)$ and $A = A(\Phi, \mathbf{S}^{n-1}, \mathbf{S}^l)$ such that

$$\sup_{\eta' \in \mathbf{S}^l} \int_{\mathbf{S}^{n-1}} \left|\eta' \cdot (\Phi_1(y'), ..., \Phi_l(y'))\right|^{-\delta} d\sigma(y') < A. \tag{1.7}$$

Detailed proof of (1.7) can be obtained following a similar argument as in the proof of Lemma 2.6 in ([6], see also [1]).

(ii) In a recent paper ([3]), Al-Qassem, Al-Salman, and Pan showed that the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1}), q > 1$ is an optimal size condition for the L^p boundedness of the classical Calderón-Zygmund singular integral operator $T_{0,\Omega}$ to hold. In fact, they proved that if Ω is assumed to be in $B_q^{(0,\varepsilon)}(\mathbf{S}^{n-1}) \backslash B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $\varepsilon < 0$, then the L^p boundedness of the operator $\mathbf{T}_{0,\Omega}$ may fail for any $1 < p < \infty$.

(iii) Also, by a result obtained by the authors of ([3]), it is known that if $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $\Phi : \mathbf{R}^n \to \mathbf{R}$ is a homogeneous function of degree $\beta \neq 0$ such that $\Phi\mid_{\mathbf{S}^{n-1}}$ is real analytic, then the operator

$$\mathbf{T}_{\Phi,\Omega} f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\Phi(y)} f(x-y) \left|y\right|^{-n} \Omega y) dy,$$

is bounded on L^p for all $1 < p < \infty$.

(iv) By Fatou's lemma, and a well known limiting argument it can be shown that if the operator $T^*_{\mathcal{P},\Phi,\Omega}$ is bounded on L^p for some $1 < p < \infty$, then the oscillatory singular integral operator

$$T_{\mathcal{P},\Phi,\Omega}(f)(x)=\int_{\mathbf{R}^n} e^{i\Phi(y)} f(x-\mathcal{P}(y))\Omega(y')\,|y|^{-n}\,dy,$$

is also bounded on L^p.

(v) It is known that the space $B_s^{(0,0)}(\mathbf{S}^{n-1})$, $s>1$ contains $\bigcup_{q>1} L^q(\mathbf{S}^{n-1})$ properly ([11], [14]).

In the light of the above remarks, it is natural to ask if the result in Theorem A still holds under the weaker and more natural condition $\Omega \in B_s^{(0,0)}(\mathbf{S}^{n-1})$. In the following result, we answer this question affirmatively:

Theorem B. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1) and that $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1}), q>1$. Suppose also that $\mathcal{P}(y)=(P_1,\dots,P_d)$ is a polynomial mapping. If Φ is a homogeneous function that satisfies (1.5)-(1.6) with either the index $\beta \neq 0$ is not a positive integer or β is a positive integer larger than* $\max\{\deg(P_j): 1\le j\le d\}$*, then the operator $T^*_{\mathcal{P},\Phi,\Omega}$ is bounded on L^p for all $1<p<\infty$. Moreover, the operator norm is independent of the coefficients of the polynomial mappings $\{P_j : 1\le j\le d\}$.*

By remark (i) above and a careful review of the proof of Theorem B in Section 4 in this paper, it can be easily seen that if the function Φ is assumed to be real analytic on $\mathbf{S}^{n-1}$, then the index of homogeneity β can be allowed to equal $\max\{\deg(P_j): 1\le j\le d\}$. By this, Theorem B, and remark (iv), we immediately obtain the following improvement of the L^p boundedness result in remark (iii) above:

Corollary C. *Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1) and that $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1}), q>1$. Suppose also that $\mathcal{P}(y)=(P_1,\dots,P_d)$ is a polynomial mapping. If Φ is a homogeneous function of degree $\beta\neq 0$ such that $\Phi\,|_{\mathbf{S}^{n-1}}$ is real analytic with either the index $\beta\neq 0$ is not a positive integer or β is a positive integer larger than or equal* $\max\{\deg(P_j): 1\le j\le d\}$*, then the operator $T_{\mathcal{P},\Phi,\Omega}$ is bounded on L^p for all $1<p<\infty$. Moreover, the operator norm is independent of the coefficients of the polynomial mappings $\{P_j : 1\le j\le d\}$.*

It should be remarked here that the requirement $\beta\neq 0$ in Theorem A and Theorem B can not be removed even for smooth functions Ω. This can be easily seen by using Proposition 6.1 in ([1]) and remark (iv) above.

Throughout this paper the letter C denotes a constant that may vary at each occurrence, but it is independent of the essential variables. Finally, for a set A, we let $\chi_{_A}$ denote the characteristic function of A.

2. Definition of block spaces

In this section we recall the definition of block spaces introduced by Jiang and Lu.

By a cap on $\mathbf{S}^{n-1}$, we mean a subset $I \subset \mathbf{S}^{n-1}$ of the form $I = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \alpha\}$ for some α and $x_0' \in \mathbf{S}^{n-1}$.

Definition 2.1. For $1 < q \leq \infty$, we say that a measurable function $b(x')$ on $\mathbf{S}^{n-1}$ is a $q-$block, if there exists some cap I on $\mathbf{S}^{n-1}$ such that $supp(b) \subset I$ and $\|b\|_{L^q} \leq |I|^{-\frac{1}{q'}}$, where $1/q + 1/q' = 1$.

The block functions are defined in terms of $q-$block functions. In fact, the following definition takes place.

Definition 2.2. Let $1 < q \leq \infty$ and $\nu > -1$. The class $B_q^{0,\nu}(\mathbf{S}^{n-1})$ consists of all functions $\Omega \in L^1(\mathbf{S}^{n-1})$ of the form $\Omega = \sum_{\mu=1}^{\infty} c_{\mu} b_{\mu}$, where each c_{μ} is a complex number; each b_{μ} is a $q-$block supported on a cap I_{μ} on $\mathbf{S}^{n-1}$; and

$$M_q^{0,\nu}(\{c_{\mu}\}) = \sum\nolimits_{\mu=1}^{\infty} \left|c_{\mu}\right| (1 + (\log |I_{\mu}|^{-1})^{\upsilon+1}) < \infty. \tag{2.1}$$

The block functions enjoy many properties ([11], [14]). The following are closely related to our work:

(i) $B_q^{0,\upsilon} \subset B_q^{0,0} \quad (q > 1), \nu > 0$;
(ii) $B_{q_2}^{0,\upsilon} \subset B_{q_1}^{0,\upsilon}$ $(1 < q_1 < q_2)$; $L^q(\mathbf{S}^{n-1}) \subseteq B_q^{0,\upsilon}(\mathbf{S}^{n-1}) \quad (\upsilon > -1)$;
(iii) $\bigcup\limits_{q>1} B_q^{0,\upsilon}(\mathbf{S}^{n-1}) \not\subseteq \bigcup\limits_{p>1} L^p(\mathbf{S}^{n-1})$, $\upsilon > -1$.

3. Preparation

We start by recalling the following result from ([8]) which is a simple consequence of a theorem due to Stein and Wainger, [19].

Lemma 2.1. ([8]) *Let $\mathcal{P} = (P_1, ..., P_d)$ be a polynomial mapping from $\mathbf{R}^n$ into $\mathbf{R}^d$. Suppose $\Omega \in L^1(\mathbf{S}^{n-1})$ and*

$$\mu_{\Omega,\mathcal{P}} f(x) = \sup_{j \in \mathbf{Z}} \int_{2^j \leq |y| < 2^{j+1}} |f(x - \mathcal{P}(y))| \, |y|^{-n} \left|\Omega(y')\right| dy.$$

Then for $1 < p \leq \infty$ *there exists a constant* $C_p > 0$ *independent of* Ω, *and the coefficients of* $P_1, ..., P_d$ *such that* $\|\mu_{\Omega,\mathcal{P}} f\|_p \leq C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_p$ *for every* $f \in L^p(\mathbf{R}^d)$.

Following similar arguments as in the proof of Theorem 1.1 in ([2]), we obtain the following:

LEMMA 2.2. *Suppose that* $h \in L^\infty(\mathbf{R}^+)$ *and* $\mathcal{P} = (P_1, ..., P_d)$ *is a polynomial mapping from* $\mathbf{R}^n$ *into* $\mathbf{R}^d$. *Suppose also that* $\Omega \in L^\infty(\mathbf{S}^{n-1})$ *is a homogeneous function of degree zero on* $\mathbf{R}^n$ *that satisfies (1.1) with* $\|\Omega\|_{L^1} \leq 1$ *and* $\|\Omega\|_{L^\infty} \leq 2^\kappa$ *for some* $\kappa \geq 1$. *Then the operator*

$$S^*_{\mathcal{P},\Omega,h}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} f(x - \mathcal{P}(y))\Omega(y')h(|y|)\, |y|^{-n}\, dy \right| \tag{3.1}$$

satisfies $\left\| S^*_{\mathcal{P},\Omega,h}(f) \right\|_p \leq \kappa \|h\|_\infty C_p \|f\|_p$ *for all* $1 < p < \infty$ *with constant* C_p *independent of* κ, h, Ω, *and the coefficients of* $P_1, ..., P_d$.

Suppose that $a \geq 2$. For a homogeneous function $\Omega \in L^1(\mathbf{S}^{n-1})$ of degree zero on $\mathbf{R}^n$, suitable mappings $\mathcal{P}(y) : \mathbf{R}^n \to \mathbf{R}^d$ and $\Phi : \mathbf{R}^n \to \mathbf{R}$, and a suitable sequence $\{\psi_{k,a} : k \in \mathbf{Z}\}$ of non-negative real valued functions defined on $\mathbf{R}^+$, define the sequence of measures $\{\sigma_{a,\Omega,k} : k \in \mathbf{Z}\}$ on $\mathbf{R}^n$ by

$$\int f d\sigma_{a,\Omega,k} = \int e^{i\Phi(y)}\, |y|^{-n}\, \Omega(y')\psi_{k,a}(|y|) f(\mathcal{P}(y)) dy. \tag{3.2}$$

Then, we prove the following:

LEMMA 2.3. *Suppose that* $\|\Omega\|_1 \leq 1$ *and* $\|\Omega\|_q \leq 2^a$, *where* $q > 1$ *and* $1/q + 1/q' = 1$. *Suppose also that* $\mathcal{P}(y) = (P_1, \ldots, P_d)$ *is a polynomial mapping and is a homogeneous function that satisfies (1.5)-(1.6) with a negative index* $\beta \neq 0$. *Let* $\{\sigma_{a,\Omega,k} : k \in \mathbf{Z}\}$ *be the sequence of measures given by (3.2). Suppose also that* $0 \leq \psi_{k,a} \leq 1$, *supp*$(\psi_{k,a}) \subseteq [2^{-a(k+1)}, 2^{-a(k-1)}]$, *and* $\left| \frac{d\psi_{k,a}}{du}(u) \right| \leq Cu^{-1}$ *with constant* C *independent of* a *and* k. *Let* $G_{\Omega,a}$ *be the maximal function given by*

$$G_{\Omega,a}(f)(x) = \sup_{j<1} \left| \sum\nolimits_{k=0}^{1-j} \sigma_{a,\Omega,k} * f(x) \right|. \tag{3.3}$$

Then

$$\|G_{\Omega,a}(f)\|_p \leq aC \|f\|_p \tag{3.4}$$

for all $1 < p < \infty$ *with constant* C *independent of* a.

P r o o f. We start by observing that

$$G_{\Omega,a}(f)(x) \leq \sum\nolimits_{k=0}^{\infty} M_{a,k}(f)(x), \tag{3.5}$$

where $M_{a,k}(f)$ is the maximal function given by

$$M_{a,k}(f)(x) = \sup_{j<1} \left|\sigma_{a,\Omega,k-j} * f(x)\right|. \tag{3.6}$$

Thus, to prove (3.4) we only need to estimate $\left\|M_{a,k}(f)\right\|_p$.

First, by the observation that $M_{a,k}(f)(x) \leq 2a\mu_{\Omega,\mathcal{P}}f(x)$ and Lemma 2.1, the following crude estimate

$$\left\|M_{a,k}(f)\right\|_p \leq aC \left\|f\right\|_p \tag{3.7}$$

holds for all $1 < p < \infty$ with constant C independent of a.

Next, we seek a good $\left\|M_{a,k}(f)\right\|_2$. By Plancherel's theorem, we have

$$\|M_{a,k}(f)\|_2 \leq \sum\nolimits_{j=-\infty}^{1} \|\sigma_{a,\Omega,k-j} * f\|_2 \leq \|f\|_2 \sum\nolimits_{j=-\infty}^{1} \|\hat{\sigma}_{a,\Omega,k-j}\|_\infty . \tag{3.8}$$

Now, by Hölder's inequality and the fact that $\left\|\Omega\right\|_q \leq 2^a$, we have

$$\|\hat{\sigma}_{a,\Omega,k-j}\|_\infty \leq 2^a \sup_{\xi\in\mathbf{R}^n} (\int_{\mathbf{S}^{n-1}} |I_{k-j}(\Phi,\xi,\Omega,a)|^{q'} \, d\sigma)^{\frac{1}{q'}}, \tag{3.9}$$

where

$$I_{k-j}(\Phi,\xi,\Omega,a) = \left|\int_1^{2^{2a}} e^{iE_{j,\beta,a}(\Phi,\mathcal{P},y',\xi,t)}\psi_{k,m}(2^{-a(k-j+1)}t)t^{-1}dt\right| \tag{3.10}$$

and $E_{j,\beta,a}(\Phi,\mathcal{P},y',\xi,t) = 2^{-a\beta(k-j+1)}\Phi(y')t^{\beta} - \mathcal{P}(2^{-a(k-j+1)}y't)\cdot\xi$. Thus, by van der Corput lemma ([18]), we immediately obtain

$$I_{k-j}(\Phi,\xi,\Omega,a) \leq C2^{\gamma a} \left|2^{-a\beta(k-j+1)}\Phi(y')\right|^{-\frac{1}{l}}, \tag{3.11}$$

for some $\gamma > 0$. When interpolated with the trivial estimate $I_{k-j}(\Phi,\xi,\Omega,a) \leq 2a\ln 2$, this implies that

$$I_{k-j}(\Phi,\xi,\Omega,a) \leq aC \left|2^{-a\beta(k-j+1)}\Phi(y')\right|^{-\frac{\delta}{lq'}}. \tag{3.12}$$

Therefore, by (3.9), (3.12), and (1.6), we obtain

$$\left\|\hat{\sigma}_{a,\Omega,k-j}\right\|_{\infty} \leq 2^{a} a C \left|2^{-a\beta(k-j+1)}\right|^{-\frac{\delta}{lq'}}. \tag{3.13}$$

By interpolation between (3.13) and the estimate $\left\|\hat{\sigma}_{a,\Omega,k-j}\right\|_{\infty} \leq 2a$ we get

$$\left\|\hat{\sigma}_{a,\Omega,k-j}\right\|_{\infty} \leq aC2^{\frac{\delta\beta}{lq'}(k-j+1)}. \tag{3.14}$$

By (3.8) and (3.14), we immediately get

$$\left\|M_{a,k}(f)\right\|_{2} \leq aC2^{\frac{\delta\beta}{lq'}k} \left\|f\right\|_{2}. \tag{3.15}$$

Therefore, by interpolation between (3.7) and (3.15), we get

$$\left\|M_{a,k}(f)\right\|_{p} \leq a2^{\alpha\beta k} \left\|f\right\|_{p} \tag{3.16}$$

for all $1 < p < \infty$ with constant C independent of a. Hence the proof is complete by (3.5) and (3.16). ■

By a similar argument as in the proof of Lemma 2.3, we can easily obtain the following:

LEMMA 2.4. *Suppose that* $\left\|\Omega\right\|_{1} \leq 1$ *and* $\left\|\Omega\right\|_{q} \leq 2^{a}$, *where* $q > 1, 1/q + 1/q' = 1$. *Suppose also that* $\mathcal{P}(y) = (P_1, \ldots, P_d)$ *is a polynomial mapping and* Φ *is a homogeneous function that satisfies (1.5)-(1.6) with a positive index* β *which is either not an integer, or is a positive integer larger than* $\max\{\deg(P_j) : 1 \leq j \leq d\}$. *Let* $\{\sigma_{a,\Omega,k} : k \in \mathbf{Z}\}$ *be the sequence of measures given by (3.2). Suppose also that* $0 \leq \psi_{k,a} \leq 1$, *supp*$(\psi_{k,a}) \subseteq [2^{-a(k+1)}, 2^{-a(k-1)}]$, *and* $\left|\frac{d\psi_{k,a}}{du}(u)\right| \leq Cu^{-1}$ *with constant* C *independent of* a *and* k. *Let* $G_{\Omega,a}$ *be the maximal function given by*

$$G_{\Omega,a}(f)(x) = \sup_{j\geq 0}\left|\sum\nolimits_{k=-j-1}^{0} \sigma_{a,\Omega,k} * f(x)\right|. \tag{3.17}$$

Then

$$\left\|G_{\Omega,a}(f)\right\|_{p} \leq aC\left\|f\right\|_{p} \tag{3.18}$$

for all $1 < p < \infty$ *with constant* C *independent of* a.

We now prove the following lemma:

LEMMA 2.5. *Let* $a \geq 2$ *and let* $\mathcal{P} = (P_1, ..., P_d)$ *be a polynomial mapping from* $\mathbf{R}^n$ *into* $\mathbf{R}^d$. *Let* Ω *be a homogeneous function of degree zero on* $\mathbf{R}^n$ *that satisfy (1.1) with* $\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$ *and let* $\eta_a : \mathbf{R}_+ \to \mathbf{R}$ *be a smooth function that satisfies* $0 \leq \eta_a \leq 1$, *supp*$(\eta_a) \subset [1, \infty)$, *and* $\eta_a(t) = 1$ *for* $t \geq 2^{2a}$. *Let* $K_a(y) = \Omega(y')\eta_a(|y|)$. *For a function* Φ *that satisfies (1.5)-(1.6) with index* $\beta < 0$, *let* $T^*_{\mathcal{P},\Phi,K_a}$ *be the operator given by (1.4) with* Ω *replaced by* K_a. *Then*

$$\left\|T^*_{\mathcal{P},\Phi,K_a}(f)\right\|_p \leq aC\,\|f\|_p \tag{3.19}$$

for all $1 < p < \infty$.

P r o o f. By the assumptions, the factor $e^{i\Phi(y)}\,|y|^{-n}\,K_a(y)$ can be written as

$$\begin{aligned} e^{i\Phi(y)}\,|y|^{-n}\,K_a(y) &= |y|^{-n}\,\Omega(y')\chi_{\{|y|>2^{2a}\}} + (e^{i\Phi(y)} - 1)\,|y|^{-n}\,\Omega(y') \\ &\times \chi_{\{|y|>2^{2a}\}} + e^{i\Phi(y)}\,|y|^{-n}\,K_a(y)\chi_{\{1\leq|y|<2^{2a}\}}. \end{aligned} \tag{3.20}$$

Therefore, we have

$$T^*_{\mathcal{P},\Phi,K_a(y)}(f)(x) \leq S^*_{\mathcal{P},\Omega,h_a}(f)(x) + M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)(x) + M^{(2)}_{\mathcal{P},\Phi,K_a}(f)(x), \tag{3.21}$$

where

$$M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)(x) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon}(e^{i\Phi(y)} - 1)f(x - \mathcal{P}(y))\,|y|^{-n}\,\Omega(y^{'})\chi_{\{|y|>2^{2a}\}}\right|dy, \tag{3.22}$$

$$M^{(2)}_{\mathcal{P},\Phi,K_a}(f)(x) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} e^{i\Phi(y)}f(x - \mathcal{P}(y))\,|y|^{-n}\,K_a(y)\chi_{\{1\leq|y|<2^{2a}\}}\right|dy, \tag{3.23}$$

and $S^*_{\mathcal{P},\Omega,h_a}$ is the operator given by (3.1) with h replaced by $h_a = \chi_{\{|y|>2^{2a}\}}$.

Now,

$$\begin{aligned} M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)(x) &\leq \|\Phi\|_\infty \sum\nolimits_{j=2}^{\infty}\{2^{a\beta j}\int_{2^{aj}<|y|<2^{a(j+1)}}\left|\Omega(y^{'})\right|\,|y|^{-n}\,| \\ &\times f(x - \mathcal{P}(y))|\,dy\} \leq Ca\mu_{\Omega,\mathcal{P}}f(x); \end{aligned}$$

when combined with Lemma 2.1, implies that

$$\left\|M^{(1)}_{\mathcal{P},\Phi,\Omega}(f)\right\|_p \leq aC\,\|f\|_p \tag{3.24}$$

for all $1 < p < \infty$.

On the other hand, by the observation that $M^{(2)}_{\mathcal{P},\Phi,K_a}(f)(x) \leq 2a\mu_{\Omega,\mathcal{P}} f(x)$ and Lemma 2.1, we have

$$\left\| M^{(2)}_{\mathcal{P},\Phi,K_a}(f) \right\|_p \leq aC \|f\|_p \tag{3.25}$$

for all $1 < p < \infty$. Hence (3.19) follows by Minkowsky's inequality, Lemma 2.2, (3.21), (3.24), and (3.25). This completes the proof. ∎

By a similar argument as in the proof of Lemma 2.5, we can easily obtain the following:

LEMMA 2.6. *Let $a \geq 2$ and let $\mathcal{P} = (P_1, ..., P_d)$ be a polynomial mapping from $\mathbf{R}^n$ into $\mathbf{R}^d$. Let Ω be a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies (1.1) with $\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$ and let $\eta_a : \mathbf{R}_+ \to \mathbf{R}$ be a smooth function that satisfies $0 \leq \eta_a \leq 1$, supp$(\eta_a) \subset [0,1]$, and $\eta_a(t) = 1$ for $t \leq 2^{-a}$. Let $K_a(y) = \Omega(y')\eta_a(|y|)$. For a function Φ that satisfies (1.5)-(1.6) with a positive index β which is either not an integer or is a positive integer larger than $\max\{\deg(P_j) : 1 \leq j \leq d\}$, let $T^*_{\mathcal{P},\Phi,K_a}$ be the operator given by (1.4) with Ω replaced by K_a. Then $\left\| T^*_{\mathcal{P},\Phi,K_a}(f) \right\|_p \leq aC \|f\|_p$ for all $1 < p < \infty$.*

4. Proof of main result

Proof of Theorem B. Assume that $\Omega \in B^{0,0}_q(\mathbf{S}^{n-1})$, $q > 1$. Then $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$, where each b_μ is a $q-$block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and $\{c_\mu\}$ is a sequence of complex numbers that satisfies

$$M^{0,0}_q(\{c_\mu\}, \{I_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu| \left(1 + \log(|I_\mu|^{-1})\right) < \infty. \tag{4.1}$$

For each μ, we define the function $\bar{b}_\mu$ by $\bar{b}_\mu(x) = \bar{b}_\mu(x) - \int_{\mathbf{S}^{n-1}} \bar{b}_\mu(u) d\sigma(u)$.

Then, it is easy to see that $\bar{b}_\mu$ satisfies the cancelation property (1.1). Moreover, the following hold

$$\begin{aligned} \|\bar{b}_\mu\|_{L^q} &\leq C |I_\mu|^{-\frac{1}{q'}}, \ \|\bar{b}_\mu\|_{L^1} \leq C, &(4.2)\\ \Omega &= \sum_{\mu=1}^{\infty} c_\mu \bar{b}_\mu. &(4.3)\end{aligned}$$

By (4.3), we have

$$T^*_{\mathcal{P},\Phi,\Omega}f(x) \le \sum\nolimits_{\mu=1}^{\infty} \left|c_\mu\right| T^*_{\mathcal{P},\Phi,\bar{b}_\mu} f(x), \tag{4.4}$$

where $T^*_{\mathcal{P},\Phi,\bar{b}_\mu}$ is given by (1.4) with Ω replaced by $\bar{b}_\mu$.

To prove Theorem B, it suffices by (4.1) and (4.4) to prove that

$$\left\|T^*_{\mathcal{P},\Phi,\bar{b}_\mu} f\right\|_p \le (1+\log(\left|I_\mu\right|^{-1})C \left\|f\right\|_p \tag{4.5}$$

for all $1 < p < \infty$. To prove (4.5), we argue as follows:

Given $\bar{b}_\mu$. Let $a = 2$ if $\left|I_\mu\right| \ge 2^{q'}e^{-2q'}$ and $a = \log 2 \left|I_\mu\right|^{-\frac{1}{q'}}$ if $\left|I_\mu\right| < 2^{q'}e^{-2q'}$. Choose a collection of $\mathcal{C}^\infty$ functions $\{\psi_{k,a}\}_{k\in\mathbf{Z}}$ on $(0,\infty)$ that satisfy $supp(\psi_{k,a}) \subseteq [2^{-a(k+1)}, 2^{-a(k-1)}], 0 \le \psi_{k,a} \le 1, \sum\limits_{k\in\mathbf{Z}} \psi_{k,a}(u) = 1$, and $\left|\frac{d^s\psi_{k,a}}{du^s}(u)\right| \le C_s u^{-s}$ with constants C_s independent of a (see [2] for more details).

Now, as in ([9]), we have two cases:

Case 1. $\beta < 0$. Let

$$\eta(y) = \sum\nolimits_{k=-\infty}^{-1} \psi_{k,a}(|y|);$$

$$K_{a,\infty}(y) = \bar{b}_\mu(y')\eta(y);\ K_{a,0}(y) = \sum\nolimits_{k=0}^{\infty} \bar{b}_\mu(y')\psi_{k,a}(|y|).$$

Then, it is clear that

$$\begin{aligned} supp(K_{a,\infty}) &\subset \{y \in \mathbf{R}^n : |y| \ge 1\}; && (4.6)\\ K_{a,\infty}(y) &= \bar{b}_\mu(y') \text{ for all } |y| > 2^{2a}; && (4.7)\\ supp(K_{a,0}) &\subset \{y \in \mathbf{R}^n : |y| \le 2^a\}. && (4.8)\end{aligned}$$

Therefore, we have

$$T^*_{\mathcal{P},\Phi,\bar{b}_\mu} f(x) \le T^*_{\mathcal{P},\Phi,K_{a,\infty}}(f)(x) + T^*_{\mathcal{P},\Phi,K_{a,0}}(f)(x). \tag{4.9}$$

Therefore, by Lemma 2.5, we have

$$\left\|T^*_{\mathcal{P},\Phi,K_{a,\infty}}(f)\right\|_p \le aC \left\|f\right\|_p \tag{4.10}$$

for all $1 < p < \infty$.

Now, we show that

$$\left\| T^{*}_{\mathcal{P},\Phi,K_{a,0}}(f) \right\|_{p} \leq aC \left\| f \right\|_{p} \tag{4.11}$$

for all $1 < p < \infty$. To prove (4.11), we argue as follows:

By (4.8), we observe that

$$T^{*}_{\mathcal{P},\Phi,K_{a,0}}(f)(x) = \sup_{0<\varepsilon<2^{a}} \left| \sum\nolimits_{k=0}^{\infty} \int_{|y|>\varepsilon} L(a,k,\mu,\Phi,n)(y) f(x-\mathcal{P}(y))dy \right|, \tag{4.12}$$

where $L(a,k,\mu,\Phi,n)(y) = e^{i\Phi(y)} |y|^{-n} \bar{b}_{\mu}(y')\psi_{k,a}(|y|)$. For any $0 < \varepsilon < 2^{a}$, choose $j \leq 1$ such that $2^{a(j-1)} \leq \varepsilon < 2^{aj}$. Let I_1 and I_2 be the operators given by

$$I_1(f)(x) = \left| \sum\nolimits_{k=0}^{\infty} \int_{2^{aj} \leq |y| < 2^{a}} L(a,k,\mu,\Phi,n)(y) f(x-\mathcal{P}(y))dy \right|; \tag{4.13}$$

$$I_2(f)(x) = \left| \sum\nolimits_{k=0}^{\infty} \int_{\varepsilon < |y| < 2^{aj}} L(a,k,\mu,\Phi,n)(y) f(x-\mathcal{P}(y))dy \right|. \tag{4.14}$$

Therefore,

$$\left| \sum\nolimits_{k=0}^{\infty} \int_{|y|>\varepsilon} L(a,k,\mu,\Phi,n)(y) f(x-\mathcal{P}(y))dy \right| \leq I_1(f)(x) + I_2(f)(x), \tag{4.15}$$

Now, it can be easily seen that

$$I_2(f)(x) \leq 3a\mu_{\bar{b}_{\mu},\mathcal{P}} f(x), \tag{4.16}$$

where $\mu_{\bar{b}_{\mu},\mathcal{P}} f$ is the operator given in Lemma 2.1 with Ω replaced by $\bar{b}_{\mu}$.

On the other hand, by the support property of $\psi_{k,a}$, we have

$$I_1(f)(x) \leq \left| \sum\nolimits_{k=0}^{1-j} \int_{\mathbf{R}^{n}} L(a,k,\mu,\Phi,n)(y) f(x-\mathcal{P}(y))dy \right| + 2a\mu_{\bar{b}_{\mu},\mathcal{P}} f(x). \tag{4.17}$$

Therefore by (4.12), (4.15)-(4.17), we have

$$T^{*}_{\mathcal{P},\Phi,K_{a,0}} f(x) \leq G_{\bar{b}_{\mu},a}(f)(x) + 5a\mu_{\bar{b}_{\mu},\mathcal{P}} f(x), \tag{4.18}$$

where $G_{\bar{b}_\mu,a}$ is given by (3.3) with Ω replaced by $\bar{b}_\mu$. Thus, by (4.18), Lemma 2.3, and Lemma 2.1, we obtain (4.11). Hence, the proof of Case 1 is complete by (4.9)-(4.11).

Case 2. $\beta > 0$ is not a positive integer or β is a positive integer larger than $\max\{\deg(P_j) : 1 \le j \le d\}$. The proof of this case follows by a similar argument as in Case 1. In fact, by taking $\eta(y) = \sum_{k=1}^{\infty} \psi_{k,a}(|y|)$, $K_{a,0}(y) = \bar{b}_\mu(y')\eta(y)$, and $K_{a,\infty}(y) = \sum_{k=-\infty}^{0} \bar{b}_\mu(y')\psi_{k,a}(|y|)$, it follows that $supp(K_{a,0}) \subset \{y \in \mathbf{R}^n : |y| \le 1\}$, $K_{a,0}(y) = \bar{b}_\mu(y')$ for all $|y| < 2^{-2a}$, and $supp(K_{a,\infty}) \subset \{y \in \mathbf{R}^n : |y| \ge 2^{-a}\}$. Thus a proof of Case 2 follows by repeating the same argument as in the proof of Case 1 using Lemmas 2.4 and 2.6 in place of Lemmas 2.3 and 2.5 at this time. This completes the proof. ∎

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