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ON A SINGULAR VALUE PROBLEM FOR THE FRACTIONAL LAPLACIAN ON THE EXTERIOR OF THE UNIT BALL

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Abstract

We study a singular value problem and the boundary Harnack principle for the fractional Laplacian on the exterior of the unit ball.

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1. Introduction

The potential theory of α-stable processes, $0 < \alpha \leq 2$, was studied intensively in the recent years. In [7] and [8], the boundary Harnack principle for bounded Lipschitz domains of $\mathbb{R}^d$ was proved for α-harmonic functions using probabilistic proof. In [3], for $\alpha = 2$ Bachar, Maagli and Zeddini treated the following non linear singular elliptic problem

$$\begin{cases} \triangle u + f(.,u) = 0, \text{ in } D, \\ u = \phi, \text{ on } \partial D, \\ \lim\limits_{|x| \to +\infty} u(x) = 0, \end{cases}$$

where D is an unbounded regular domain in $\mathbb{R}^d$, $(d \geq 3)$, with compact boundary, and f is a nonnegative Borel function in $D \times (0, \infty)$, that belongs to a convex cone which contains, in particular, all functions $f(x,t) = q(x)t^{-\gamma}$, $\gamma > 0$ with q is in a certain Kato class $K(D)$.

In [11], the authors considered the following problem

$$\begin{cases} \triangle u + F(x,u) = -g(x), \text{ in } D, \\ u = \phi, \text{ on } \partial D, \\ \lim\limits_{|x| \to +\infty} u(x) = \beta, \text{when } D \text{ is unbounded,} \end{cases}$$

where D is a domain in $\mathbb{R}^d$, $(d \geq 3)$, F is a measurable function defined on $D \times (0, b)$ for some $b \in (0, \infty]$ and $-U(x)f(x) \leq F(x,u) \leq V(x)f(u)$, where U and V are Green tight functions on D such that $\sup\limits_{0<y<\varepsilon} \dfrac{f(y)}{y} < \dfrac{1}{C\|V\|_D}$. The authors used the implicit probabilistic representation for solutions of Dirichlet boundary value problem combined with Schauder's fixed point theorem.

For the fractional Laplacian with $\alpha \in (0, 2]$ Belhaj Rhouma and Bezzarga in [4], considered the following problem

$$\begin{cases} -(-\triangle)^{\frac{\alpha}{2}} u = f(.,u), \text{ in } D, \\ u = \phi, \text{ on } D^c, \end{cases}$$

where $\phi \in C(D^c)$, D is a bounded $C^{1,1}$-domain in $\mathbb{R}^d$,$(d \geq 3)$ and f is assumed to be a measurable function in $D \times (0, \infty)$ that belongs to a convex cone which contains, in particular, all functions $f(x,t) = q(x)t^{-\gamma}$, $\gamma > 0$, with Borel function q is in some class of functions.

The main goal of this paper is to obtain criteria for the existence and uniqueness of positive solutions, bounded below by a positive α-harmonic function, of a class of semilinear elliptic problems

$$\begin{cases} -(-\triangle)^{\frac{\alpha}{2}} u = f(.,u), \text{ in } \overline{B}^c, \\ u = \phi, \text{ on } \overline{B}, \\ \lim\limits_{|x| \to +\infty} |x|^{d-\alpha} u(x) = \lambda > 0, \end{cases} \tag{1.1}$$

where $\overline{B}^c$ is the exterior of the unit ball of $\mathbb{R}^d$. By a solution of (1.1), we mean a continuous function u which satisfies the equivalent integral equation

$$u(x) = h(x) - \int_{\overline{B}^c} G_{\overline{B}^c}(x,y) f(y, u(y)) dy, \quad x \in \mathbb{R}^d, \tag{1.2}$$

where $G_{\overline{B}^c}$ is the Green function of $(-\triangle)^{\frac{\alpha}{2}}$ on $\overline{B}^c$ and h is the α-harmonic extension of ϕ. The function f is assumed to be a measurable function on $\overline{B}^c \times (0,\infty)$ that belongs to a convex cone which contains, in particular, all functions $f(x,t) = q(x)t^{-\gamma}$, $\gamma > 0$, with Borel function q in some class of functions related with the so-called Kato class $S_\infty(X^D)$. Also, with analytic method and using estimations on the Green function, we will show that solutions of (1.1) satisfy the boundary Harnack principle (BHP) without any restriction on the sign of f.

As usual, if A is a subset of $\mathbb{R}^d$, we denote by $B(A)$ the set of real Borel functions in A and $B_b(A)$ the set of bounded ones. $C(A)$ will denote the set of continuous real functions in A, $C_c(A)$ the set of ones with compact carrier and

$$C_0(A) := \{v \in C(A) : \lim_{x \to \partial A} v(x) = 0 \text{ and } \lim_{|x| \to \infty} v(x) = 0\}.$$

If $\mathcal{F}$ is a set of functions, we denote by $\mathcal{F}^+$ the set of positive elements of $\mathcal{F}$. As usual A^c is the complement of A and for any $x \in D$, let us denote by $\delta_D(x)$ the Euclidian distance between x and the boundary ∂D of D. The letter C will denote a generic positive constant which may vary from line to line. When two positive functions are defined on a set A, we write $f \simeq g$ when the two-sided inequality $\frac{1}{C} f \leq g \leq Cf$ holds on A.

2. The α-harmonic Dirichlet problem

In this section, we will recall some properties of the α-stable process in $\mathbb{R}^d$ which is associated to the infinitesimal generator $(-\Delta)^{\frac{\alpha}{2}}$.

For $\alpha \in (0,2)$, we denote by $((X_t)_{t \geq 0}, P^x)$ the standard rotation invariant (or symmetric) α-stable process in $\mathbb{R}^d$, with index of stability α, and the characteristic $E^x e^{i\xi X_t} = e^{-i|\xi|^\alpha}$, $\xi \in \mathbb{R}^d$, $t \geq 0$, (see [9] for an explicit definition). As usual E^x is the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^d$. The process $(X_t)_{t \geq 0}$ has the potential operator (see [1] or [12]), $U_\alpha f(x) = \mathcal{A}(d,\alpha) \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy$, where $\mathcal{A}(d,\gamma) = \frac{\Gamma(\frac{d-\gamma}{2})}{2^\gamma \pi^{\frac{d}{2}} \Gamma(|\frac{\gamma}{2}|)}$ and the infinitesimal generator $(-\triangle)^{\frac{\alpha}{2}}$,

$$-(-\triangle)^{\frac{\alpha}{2}} u(x) = \mathcal{A}(d,-\alpha) \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}} dy.$$

To justify the notation $(-\triangle)^{\frac{\alpha}{2}}$, we note that the Fourier transform of the generator $(-\triangle)^{\frac{\alpha}{2}}$ and the Fourier transform of the Laplacian Δ, satisfy the equation (see [12]) $\mathcal{F}((-\triangle)^{\frac{\alpha}{2}})(\xi) = |\xi|^\alpha = (\mathcal{F}(-\triangle)(\xi))^{\frac{\alpha}{2}}$.

Note that a symmetric α-stable process X on $\mathbb{R}^d$ is a Lévy process whose transition density $p(t, x-y)$ relative to the Lebesgue measure is uniquely determined by its Fourier transform $\int_{\mathbb{R}^d} e^{ix\xi} p(t,x)dx = e^{-t|\xi|^\alpha}$. When $\alpha = 2$, we get the Brownian motion.

For a Borel set $A \subset \mathbb{R}^d$, we define $T_A = inf\{t \geq 0 : X_t \in A\}$, the first entrance time of A.

Definition 2.1. Let u be a Borel function on $\mathbb{R}^d$, which is bounded from below. We say that u is α-harmonic in an open set $U \subset \mathbb{R}^d$ if $u(x) = E^x(u \circ X_{T_{B^c}}),\ x \in B$, for every bounded open set B with the closure $\overline{B}$ contained in U. We say that u is *regular* α-harmonic in U if $u(x) = E^x(u \circ X_{T_{U^c}}),\ x \in U$.

By the strong Markov property, a regular α-harmonic function u is necessarily α-harmonic. The converse is not generally true. However, by the proof of Proposition 24.10 in [13], if u is continuous on $\overline{U}$ and α-harmonic in U, then it is regular α-harmonic in U provided U is bounded.

The above definitions have their analytic counterparts (See [5] or [12]).

Let $\mathcal{U}$ be the family of all open balls $B(a,r)$. For every $U = B(a,r)$ we define a sweeping kernel H_U^α by $H_U^\alpha f(x) = \int_{U^c} p_x^U(y) f(y)dy \quad (f \in B^+(\mathbb{R}^d), x \in U)$, where the density is defined by

$$p_x^U(y) = a_\alpha \frac{(r^2 - |x-a|^2)^{\frac{\alpha}{2}}}{(|y-a|^2 - r^2)^{\frac{\alpha}{2}}} |y-x|^{-d}, \qquad |x-a| < r \leq |y-a|$$

and $a_\alpha = \pi^{-(\frac{d}{2}+1)} \Gamma(\frac{d}{2}) sin(\frac{\alpha\pi}{2})$.

For every $x \in \mathbb{R}^d$ and every open subset V of $\mathbb{R}^d$ we define

$$\mathcal{U}_x := \{U \in \mathcal{U} : x \in U\}, \quad \mathcal{U}(V) := \{U \in \mathcal{U} : \overline{U} \subset V\}.$$

In the following D denotes a domain in $\mathbb{R}^d$, $(d \geq 2)$ with compact $C^{1,1}$ boundary.

Definition 2.2. A function s is said to be α-superharmonic in D if:

(a) $s \geq 0$, $s \neq +\infty$,

(b) s is lower semicontinuous,

(c) $H_U^\alpha s \leq s$, $U \in \mathcal{U}(D)$.

It is well known that, if f is a continuous function in D^c and satisfying

$$\int_{D^c} \frac{|f(x)|}{1+|x|^{d+\alpha}} dx < \infty,$$

in the case where D^c contains the point at infinity, then there is a function $H_D^\alpha f$, defined in $\mathbb{R}^d$, α-harmonic in D and coincides with f in D^c (see [12]).

3. The 3G-theorem

In this section, we will give some estimates on the Green function of the fractional Laplacian on an unbounded domain $D \subset \mathbb{R}^d$, $(d \geq 3)$ with compact boundary such that $\overline{D}^c$ is consisting of finitely many disjoints bounded $C^{1,1}$-domains, and we will prove the Harnack principle for the exterior of the unit ball.

In [10] Chen and Song have obtained interesting estimates on the Green function G_D of the fractional Laplacian in a bounded $C^{1,1}$ domain D in $\mathbb{R}^d$ $(d \geq 3)$. In particular they showed the existence of a constant $C > 0$, such that for each $x, y, z \in D$

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq C\left((\frac{\delta_D(y)}{\delta_D(x)})^{\frac{\alpha}{2}} G_D(x,y) + (\frac{\delta_D(y)}{\delta_D(z)})^{\frac{\alpha}{2}} G_D(y,z) \right), \quad (3.1)$$

where $\delta_D(x)$ denotes the Euclidien distance between x and ∂D, and using the Kelvin transformation Bachar, Maagli and Zeddini in [3] obtained a 3G-theorem for an unbounded domain D in $\mathbb{R}^d$, $(d \geq 3)$ with compact boundary such that $\overline{D}^c$ is consisting of finitely many disjoints bounded $C^{1,1}$-domains, they prove that there exists $C > 0$ such that for each $x, y, z \in D$ we have

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq C\left((\frac{\rho_D(y)}{\rho_D(x)})^{\frac{\alpha}{2}} G_D(x,y) + (\frac{\rho_D(y)}{\rho_D(z)})^{\frac{\alpha}{2}} G_D(y,z) \right), \quad (3.2)$$

where $\rho_D(x) = \dfrac{\delta_D(x)}{\delta_D(x)+1}$ $for \ \ x \in D$. They also prove that there exists $C > 0$ such that for each $x, y, z \in D$,

$$\frac{\rho_D(y)^{\frac{\alpha}{2}}}{\rho_D(x)^{\frac{\alpha}{2}}} G_D(x,y) \leq \frac{C}{|x-y|^{d-\frac{\alpha}{2}}}. \quad (3.3)$$

Next we shall give some preliminary estimations of the Green function which will be needed later, for that we recall [3] the following lemmas:

Lemma 3.1. *There exists $C > 0$ such that for each $x, y \in D$, we have*

$$\frac{1}{C}\frac{(\delta_D(x)\delta_D(y))^{\frac{\alpha}{2}}}{|x|^{d-\frac{\alpha}{2}}|y|^{d-\frac{\alpha}{2}}} \leq G_D(x,y). \tag{3.4}$$

Moreover for $M > 1$ and $r > 0$, then there exists a constant $C > 0$ such that for each $x \in D$ and $y \in D$ satisfying $|x - y| \geq r$ and $|y| \leq M$

$$G_D(x,y) \leq C\frac{(\rho_D(x)\rho_D(y))^{\frac{\alpha}{2}}}{|x-y|^{d-\alpha}}. \tag{3.5}$$

In the sequel of this section, let $D = \overline{B}^c$ and let $x^* = \dfrac{x}{|x|^2}$ be the Kelvin transformation from D into $D^* := \{x^* : x \in D\} = B \setminus \{0_{\mathbb{R}^d}\}$.

Lemma 3.2. *There exists $C > 0$ such that for each $x \in D$, we have*

$$i)\quad (\delta_D(x) + 1) \leq |x| \leq C(\delta_D(x) + 1), \tag{3.6}$$

$$ii)\quad \frac{1}{C}\rho_D(x) \leq \delta_{D^*}(x^*) \leq C\rho_D(x). \tag{3.7}$$

Notation. Let A be a subset of $\mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}$ and let $f \in B(A^*)$. For any $x \in A$, we put $\widehat{f}(x) := |x^*|^{d-\alpha} f(x^*)$.

Theorem 3.1. *Let $\phi \in C(\overline{B})$ and let $H_D^\alpha \phi$ the α-harmonic extension of ϕ on D (as in [12] page 267) such that $\lim\limits_{|x|\to+\infty} |x|^{d-\alpha} H_D^\alpha \phi(x) = \lambda > 0$.*

Then there exists $H_B^\alpha \widehat{\phi}$ the α-harmonic extension of $\widehat{\phi}$ on B. Moreover we have $H_B^\alpha \widehat{\phi} = \widehat{H_D^\alpha \phi}$ on $B \setminus \{0_{\mathbb{R}^d}\}$.

P r o o f. First we remark that $\widehat{\phi} \in C(D)$ and $|\dfrac{\widehat{\phi}(x)}{|x|^{d+\alpha}}| \leq \dfrac{\|\phi\|_{\infty,\overline{B}}}{|x|^{2d}}$ on D, where $\|\phi\|_{\infty,\overline{B}} := \sup\limits_{x\in\overline{B}} |\phi(x)|$. So (see [12] p.267), there exists $H_B^\alpha \widehat{\phi}$ the α-harmonic extension of $\widehat{\phi}$ on B. Moreover we have $H_B^\alpha \widehat{\phi}(x) = \displaystyle\int_D \widehat{\phi}(y)\varepsilon_x^{'}(dy)$, $x \in B \setminus \{0_{\mathbb{R}^d}\}$, with Green measure of D:

$$\varepsilon_x^{'}(dy) := \chi_{(|y|>1)} P_x^B(y)dy = a_\alpha \chi_{(|y|>1)} (\frac{1-|x|^2}{|y|^2-1})^{\frac{\alpha}{2}} \frac{dy}{|x-y|^d},$$

where $a_\alpha := \Gamma(\frac{d}{2})\pi^{-\frac{d}{2}-1}\sin(\frac{\alpha\pi}{2})$. Now we fix $x \in D$, then

$$H_B^\alpha \widehat{\phi}(x^*) = a_\alpha \int_D \widehat{\phi}(y)(\frac{1-|x^*|^2}{|y|^2-1})^{\frac{\alpha}{2}} \frac{dy}{|x^*-y|^d}.$$

If we put $y = \xi^*$ in the right hand side and using the fact that (see [3]) $|\xi^* - x^*| = \dfrac{|\xi - x|}{|\xi||x|}$, we get

$$H_B^\alpha \widehat{\phi}(x^*) = a_\alpha \int_{B\setminus\{0_{\mathbb{R}^d}\}} |x|^{d-\alpha}\phi(\xi)(\frac{|x|^2-1}{1-|\xi|^2})^{\frac{\alpha}{2}} \frac{d\xi}{|x-\xi|^d} = |x|^{d-\alpha}\int_B \phi(\xi)\varepsilon_x^{''}(d\xi),$$

where $\varepsilon_x^{''}(d\xi) := a_\alpha \chi_{(|\xi|<1)}(\dfrac{|x|^2-1}{1-|\xi|^2})^{\frac{\alpha}{2}} \dfrac{d\xi}{|x-\xi|^d}$ is the Green measure of B. By ([12] page 267), we get $H_B^\alpha \widehat{\phi}(x^*) = |x|^{d-\alpha} H_D^\alpha \phi(x)$. This ends the proof.

■

Now we are ready to state the boundary Harnack inequality.

THEOREM 3.2. *Let V be an open set and let $K \subset V$ be a compact subset. Then there exists a positive constant $C = C(K, V, D)$ such that for any positive α-harmonic function u in D, vanishing on $D^c \cap V$ we have*

$$\frac{1}{C}(\frac{|y|}{|x|})^{d-\alpha}(\frac{\rho_D(x)}{\rho_D(y)})^{\frac{\alpha}{2}} \le \frac{u(x)}{u(y)} \le C(\frac{|y|}{|x|})^{d-\alpha}(\frac{\rho_D(x)}{\rho_D(y)})^{\frac{\alpha}{2}}, \quad x, y \in K \cap D.$$

P r o o f. In [4] Belhaj Rhouma and Bezzarga have proved that, if D is a bounded $C^{1,1}$ domain, V is an open set and $K \subset V$ is a compact subset, then there exists a constant $C = C(K, V, D)$ such that for any positive α-harmonic functions u in D, vanishing on $D^c \cap V$ we have

$$\frac{1}{C}(\frac{\delta_D(x)}{\delta_D(y)})^{\frac{\alpha}{2}} \le \frac{u(x)}{u(y)} \le C(\frac{\delta_D(x)}{\delta_D(y)})^{\frac{\alpha}{2}}, \quad x, y \in K \cap D.$$

By Theorem 3.1, this result is available for $D^* \cup \{0\}$, so we can write

$$\frac{1}{C}(\frac{\delta_{D^*}(x^*)}{\delta_{D^*}(y^*)})^{\frac{\alpha}{2}} \le \frac{\widehat{u}(x^*)}{\widehat{u}(y^*)} \le C(\frac{\delta_{D^*}(x^*)}{\delta_{D^*}(y^*)})^{\frac{\alpha}{2}}, \quad x^*, y^* \in K^* \cap D^*.$$

Using (3.7), we get

$$\frac{1}{C}(\frac{|y|}{|x|})^{d-\alpha}(\frac{\rho_D(x)}{\rho_D(y)})^{\frac{\alpha}{2}} \le \frac{u(x)}{u(y)} \le C(\frac{|y|}{|x|})^{d-\alpha}(\frac{\rho_D(x)}{\rho_D(y)})^{\frac{\alpha}{2}}, \quad x, y \in K \cap D.$$

■

4. The class $S_\infty(X^D)$ for $(-\triangle)^{\frac{\alpha}{2}}$

In this section we will assume that D is an unbounded domain in $\mathbb{R}^d$, $(d \geq 3)$ with compact boundary such that $\overline{D}^c$ is consisting of finitely many disjoints bounded $C^{1,1}$ domains. In [11], Chen and Song have introduced the following class of functions $S_\infty(X^D)$ as follows:

Definition 4.1. A function φ is said to be in the class $S_\infty(X^D)$ if, for every $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for any measurable set $B \subset D$ with Lebesgue measure $|B| < \delta$,

$$\sup_{(x,z)\in D\times D} \int_B \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\varphi(y)| dy \leq \varepsilon, \tag{4.1}$$

and there is a Borel subset $K = K(\varepsilon)$ of finite Lebesgue measure such that

$$\sup_{(x,z)\in D\times D} \int_{D\setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\varphi(y)| dy \leq \varepsilon. \tag{4.2}$$

Remark 4.1. From (3.2) if for every $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for all measurable sets $B \subset D$ with Lebesgue measure $|B| < \delta$ such that

$$\sup_{x\in D} \int_B \frac{(\rho_D(y))^{\frac{\alpha}{2}}}{(\rho_D(x))^{\frac{\alpha}{2}}} G_D(x,y) |\varphi(y)| dy \leq \varepsilon, \tag{4.3}$$

and there is a Borel subset $K = K(\varepsilon)$ of finite Lebesgue measure such that

$$\sup_{x\in D} \int_{D\setminus K} \frac{(\rho_D(y))^{\frac{\alpha}{2}}}{(\rho_D(x))^{\frac{\alpha}{2}}} G_D(x,y) |\varphi(y)| dy \leq \varepsilon, \tag{4.4}$$

then $\varphi \in S_\infty(X^D)$.

Remark 4.2. Note that, if φ satisfies (4.3) and (4.4), then

$$y \mapsto \delta_D(y)^\alpha \varphi(y) \in L^1_{Loc}(D) \tag{4.5}$$

Proposition 4.1. *Let $\varphi \in S_\infty(X^D)$, then*

$$\|\varphi\|_D = \sup_{(x,z)\in D\times D} \int_D \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\varphi(y)| dy < \infty.$$

P r o o f. Let $\varepsilon > 0$, then there exists a compact K such that

$$\sup_{(x,z)\in D\times D}\int_{D\setminus K}\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}|\varphi(y)|dy \le \varepsilon.$$

Also, there exists $\delta > 0$ such that for all $B \subset D$ with $|B| < \delta$, we have

$$\sup_{(x,z)\in D\times D}\int_{B}\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}|\varphi(y)|dy \le \varepsilon.$$

Let $x_1, x_2, ..., x_p$ in K such that $K \subset \bigcup_{1\le i\le p} B(x_i, r)$, where $r > 0$ is the radius of all the balls centered in x_i; $i \in \{1, 2, ..., p\}$ and satisfies $|B(x_i, r)| < \delta$ for all x_i; $i \in \{1, 2, ..., p\}$. The proof, then holds by the above two inequalities. ∎

PROPOSITION 4.2. *Let $\varphi \in S_\infty(X^D)$, $x_0 \in \overline{D}$ and h be a nonnegative α-superharmonic function in D. Then, for all $x \in D$ we have*

$$\int_D G_D(x,y)|\varphi(y)|h(y)dy \le C\|\varphi\|_D h(x). \tag{4.6}$$

Moreover, from Proposition 3.1 in [11] we have

$$\lim_{\varepsilon\to 0}(\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap B(x_0,\varepsilon)} G_D(x,y)h(y)|\varphi(y)|dy) = 0, \tag{4.7}$$

and

$$\lim_{M\to+\infty}(\sup_{x\in D}\frac{1}{h(x)}\int_{D\cap(|y|\ge M)} G_D(x,y)h(y)|\varphi(y)|dy) = 0. \tag{4.8}$$

P r o o f. Using Proposition 4.1, we get for all $x, z \in D$

$$\int_D G_D(x,y)G_D(y,z)|\varphi(y)|dy \le \|\varphi\|_D G_D(x,z).$$

On the other hand by (3.3), the kernel V^α given by $V^\alpha f = \int_D f(y)G_D(.,y)dy$, $f \in B_b(\mathbb{R}^d)$, is proper for $0 < \alpha \le 2$. Then (4.6) holds by Hunt's approximation theorem (one can see p.23 in [6]). ∎

Corollary 4.1. *Let $\varphi \in S_\infty(X^D)$. Then we have:*

$$i)\ \sup_{x\in D}\int_D G_D(x,y)|\varphi(y)|dy < \infty, \tag{4.9}$$

$$ii)\ y \mapsto \delta_D(y)^{\frac{\alpha}{2}}\varphi(y) \in L^1_{Loc}(D) \text{ and } y \mapsto \frac{\delta_D(y)^{\frac{\alpha}{2}}}{|y|^{d-\frac{\alpha}{2}}}\varphi(y) \in L^1(D)\,. \tag{4.10}$$

P r o o f. By (3.4), we have

$$\int_{D\cap(|y|\leq M)} \delta_D(y)^{\frac{\alpha}{2}}|\varphi(y)|dy \leq C\frac{|x|^{d-\frac{\alpha}{2}}}{\delta_D(x)^{\frac{\alpha}{2}}}\int_{D\cap(|y|\leq M)} G_D(x,y)|\varphi(y)|dy < \infty.$$

Using the same argument we can write

$$\int_D \frac{\delta_D(y)^{\frac{\alpha}{2}}}{|y|^{d-\frac{\alpha}{2}}}\varphi(y) \leq C\frac{|x|^{d-\frac{\alpha}{2}}}{\delta_D(x)^{\frac{\alpha}{2}}}\int_D G_D(x,y)|\varphi(y)|dy < \infty.$$

That achieves the proof of (4.10). ∎

Proposition 4.3. *Let $q(y) = \dfrac{1}{|y|^\mu(\rho_D(y))^\lambda}$, for $y \in D$, then the function q satisfies (4.3) and (4.4) if and only if $\lambda < \alpha < \mu$.*

P r o o f. Using (3.6), we can write $q(y) \sim \dfrac{1}{|y|^{\mu-\lambda}(\delta_D(y))^\lambda}$, and using [3] we end the proof. ∎

Theorem 4.1. *Let φ be a function in $S_\infty(X^D)$. Then the function $V\varphi(x) = \int_D G_D(x,y)\varphi(y)dy$ is in $C_0(D)$.*

P r o o f. Let $x_0 \in \overline{D}$ and $\varepsilon_1 > 0$, by (4.7) and (4.8), $\exists\varepsilon > 0$, $\exists M > 1$:

$$\sup_{\xi\in D}\int_{D\cap B(x_0,2\varepsilon)} G_D(\xi,y)|\varphi(y)|dy + \sup_{\xi\in D}\int_{D\cap(|y|\geq M)} G_D(\xi,y)|\varphi(y)|dy \leq \frac{\varepsilon_1}{4}.$$

Let $x, x' \in B(x_0,\varepsilon)\cap D$, then we have

$$|V\varphi(x) - V\varphi(x')| \leq \frac{\varepsilon_1}{2} + \int_{D\cap B^c(x_0,2\varepsilon)\cap B(0,M)} |G_D(x,y) - G_D(x',y)||\varphi(y)|dy.$$

On the other hand, for every $y \in B^c(x_0, 2\varepsilon) \cap B(0, M) \cap D$, $x, x' \in B(x_0, \varepsilon) \cap D$ we get using (3.5), that

$$|G_D(x,y) - G_D(x',y)| \leq C[\frac{\rho_D(x)^{\frac{\alpha}{2}}\rho_D(y)^{\frac{\alpha}{2}}}{|x-y|^{d-\alpha}} + \frac{\rho_D(x')^{\frac{\alpha}{2}}\rho_D(y)^{\frac{\alpha}{2}}}{|x'-y|^{d-\alpha}}] \leq \frac{C\rho_D(y)^{\frac{\alpha}{2}}}{\varepsilon^{d-\alpha}}.$$

Now since G_D is continuous outside the diagonal, we deduce by the dominated convergence theorem and (4.10) that

$$\int_{D\cap B^c(x_0,2\varepsilon)\cap B(0,M)} |G_D(x,y) - G_D(x',y)||\varphi(y)|dy \to 0 \quad as \quad |x-x'| \to 0.$$

Hence $V\varphi \in C(\overline{D})$. Finally, we need to prove that $V\varphi(x) \to 0$ as $|x| \to \infty$. Let $x \in D$ such that $|x| \geq M+1$. Then we have

$$|V\varphi(x)| \leq \int_{D\cap B^c(0,M)} G_D(x,y)|\varphi(y)|dy + \int_{D\cap B(0,M)} G_D(x,y)|\varphi(y)|dy.$$

For $y \in D \cap B(0,M)$, we have $|x-y| \geq 1$. Hence by (3.5) we get

$$|V\varphi(x)| \leq \frac{\varepsilon_1}{4} + \frac{C}{(|x|-M)^{d-\alpha}} \int_{D\cap(|y|\leq M)} \delta_D(y)^{\frac{\alpha}{2}}|\varphi(y)|dy.$$

Using (4.10) we obtain $V\varphi(x) \to 0$ as $|x| \to +\infty$. ■

5. Existence of solutions of (1.1)

In this section, we are concerned with the existence of solutions of (1.1). Moreover, when the function f is non increasing in u, we show the uniqueness of the solution. We also show that such solutions satisfy the Boundary Harnack Principle.

5.1. α-harmonic measure. Let ε_x, $x \in \mathbb{R}^d$, be the Dirac measure, and let V be an open set in $\mathbb{R}^d$. For each point $x \in \mathbb{R}^d$, the P^x distribution of $X_{T_{V^c}}$ is a probability measure on V^c, called *α-harmonic measure* (in x with respect to V) and denoted by ω_V^x which is usually supported on V^c and $\omega_V^x = \varepsilon_x$ for $x \in V^c$. In our case we remark that $\omega_B^x = \varepsilon_x^{'}$ and $\omega_{\overline{B}^c}^x = \varepsilon_x^{''}$. Also, we recall that for a measure μ on $\mathbb{R}^d$, we define its *Riesz potential* by

$$U_\alpha^\mu(x) = \mathcal{A}(d,\alpha) \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-\alpha}}\,.$$

We recall that the Green function satisfies

$$G_D(x,y) = U_\alpha^{\varepsilon_x}(y) - U_\alpha^{\omega_D^x}(y), \quad x,y \in \mathbb{R}^d. \tag{5.1}$$

It is well known that the first term on the right hand side of (5.1) is α-harmonic in $\mathbb{R}^d \setminus \{y\}$ (see [12]) and the second term is regular α-harmonic in $x \in D$. Moreover, we have, in the sense of distributions,

$$(-\triangle)^{\frac{\alpha}{2}}\left(\frac{\mathcal{A}(d,\alpha)}{|x-.|^{d-\alpha}}\right) = \varepsilon_x, \quad x \in \mathbb{R}^d \tag{5.2}$$

(see Lemma 1.11 in [12]). Thus, we get the following lemma:

LEMMA 5.1. *For any measurable function g such that $x \to \int_D G_D(x,y)|g(y)|dy$ in $L^1(D)$ and such $g = 0$ in D^c, we have*

$$(-\triangle)^{\frac{\alpha}{2}} \int_D G_D(x,y)g(y)dy = g(x), \quad x \in D$$

in the distributional sense.

Proof. Let $\varphi \in C_0^\infty(D) = C_0(D) \cap C^\infty(D)$. Since $\int_D G_D(x,y)g(y)dy = 0$ in D^c, we get

$$\int_{\mathbb{R}^d}\int_D G_D(x,y)g(y)dy(-\triangle)^{\frac{\alpha}{2}}\varphi(x)dx = \int_D\int_D G_D(x,y)g(y)dy(-\triangle)^{\frac{\alpha}{2}}\varphi(x)dx.$$

Using the fact that $|(-\triangle)^{\frac{\alpha}{2}}\varphi(y)| \leq C(1+|y|)^{-d-\alpha}, \quad y \in \mathbb{R}^d$, we obtain, by Fubini's theorem and (5.2) the following identity:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_D G_D(x,y)g(y)(-\triangle)^{\frac{\alpha}{2}}\varphi(x)dydx \\ &= \int_{\mathbb{R}^d}\varphi(y)g(y)dy - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(z)g(y)d\omega_D^y(z)dy.\end{aligned}$$

Since $\int_{\mathbb{R}^d}\varphi(z)d\omega_D^y(z) = 0$, it follows that

$$\int_{\mathbb{R}^d}\int_D G_D(x,y)g(y)(-\triangle)^{\frac{\alpha}{2}}\varphi(x)dydx = \int_{\mathbb{R}^d}\varphi(y)g(y)dy.$$

■

In the remaining of this paper we will assume that $D = \overline{B}^c$.

5.2. The global results. We assume that the following assumptions hold:

$\mathbf{H_1}$. $\phi \in C(D^c)$ which is zero on a neighborhood of ∂D and positive on the complement.

$\mathbf{H_2}$. f is a measurable function defined on $D \times (0, \infty)$ which is continuous with respect to the second variable.

Let h_0 be a nonnegative continuous function which is α-harmonic in D such that $Z = \{x : h_0(x) = 0\}$ is a nonempty connected subset contained in a neighborhood of ∂D and $h_0(x_0) = 1$ for some $x_0 \in D$.

In the sequel, let us consider the function h which solves the Dirichlet problem

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}} h = 0, \text{ in } D, \\ h = \phi, \text{ on } D^c, \\ \lim\limits_{|x| \to +\infty} |x|^{d-\alpha} h(x) = \lambda > 0. \end{cases} \tag{5.3}$$

For any $a > 0$, we set $F_a = \{u \in C(D) : u \geq a\}$.

Our main existence results are the following:

THEOREM 5.1. *Assume H_1 and H_2 hold. For some $a > 0$, we suppose that there exists a nonnegative function $q_a \in S_\infty(X^D)$, such that for every $u \in F_a$*

$$|f(x, u(x)h(x))| \leq q_a(x)h(x), \forall x \in D. \tag{5.4}$$

Then there exists $b_0 = b(\phi, a) > 0$ such that for any $b \in [b_0, \infty)$ there exists a solution u of

$$\begin{cases} -(-\triangle)^{\frac{\alpha}{2}} u = f(., u), \text{ in } D, \\ u = b\phi, \text{ on } D^c, \\ \lim\limits_{|x| \to \infty} |x|^{d-\alpha} u(x) = \lambda > 0. \end{cases} \tag{5.5}$$

Moreover, $u \geq ah$.

In the sequel, the following result will be used later to proof theorems. First we remark that it follows from Theorem 3.2 and the assumptions on h and D that there exists c_1 such that

$$h(x) \geq c_1 \frac{\rho_D(x)^{\frac{\alpha}{2}}}{|x|^{d-\alpha}}, \quad \text{for all } x \in D, \tag{5.6}$$

and

$$h_0(x) \geq c_1 \frac{\rho_D(x)^{\frac{\alpha}{2}}}{|x|^{d-\alpha}}, \quad \text{for all } x \in D. \tag{5.7}$$

For each $w \in F_a$, define $T_b w$, by

$$T_b w(x) = b - \frac{1}{h(x)} \int_D G_D(x,y) f(y, w(y)h(y)) dy \quad , \text{for all } x \in D.$$

Proposition 5.1. *The family of functions*

$$\mathcal{K} = \{ \frac{1}{h(x)} \int_D G_D(x,y) f(y, w(y)h(y)) dy : w \in F_a \}$$

is uniformly bounded and equicontinuous in $C(\overline{D})$, *and, consequently, it is relatively compact in* $C(\overline{D})$.

P r o o f. Set $Tw(x) = \frac{1}{h(x)} \int_D G_D(x,y) f(y, w(y)h(y)) dy$. By (5.4), we have for all $w \in F_a$, $|Tw(x)| \le \frac{1}{h(x)} \int_D G_D(x,y) q_a(y) h(y) dy$. Since $q_a \in S_\infty(X^D)$, then by proposition () we get

$$\|Tw\|_\infty \le C \|q_a\|_D. \tag{5.8}$$

Hence, the family $\mathcal{K}$ is uniformly bounded. Now, we propose to prove the equicontinuity of $\mathcal{K}$. Indeed, fix $x_0 \in \overline{D}$ and $\varepsilon > 0$.

Using (4.7) and (4.8), for all $\varepsilon_1 > 0$, there exists $\varepsilon > 0$ and $M > 1$ such that

$$\sup_{x \in D} \frac{1}{h(x)} \int_{D \cap B(x_0, 2\varepsilon)} G_D(x,y) q_a(y) h(y) dy \le \frac{\varepsilon_1}{8},$$
$$\sup_{x \in D} \frac{1}{h(x)} \int_{D \cap B^c(x_0, 2\varepsilon) \cap (|y| \ge M)} G_D(x,y) q_a(y) h(y) dy \le \frac{\varepsilon_1}{8}.$$

Then for any $x, x' \in D \cap B(x_0, \varepsilon)$ and $w \in F_a$, we have

$$|Tw(x) - Tw(x')| \le \frac{\varepsilon_1}{2}$$
$$+ \int_{D \cap B^c(x_0, 2\varepsilon) \cap (|y| \le M)} \left| \frac{G_D(x,y)}{h(x)} - \frac{G_D(x',y)}{h(x')} \right| q_a(y) h(y) dy.$$

Moreover, if $|x_0 - y| \ge 2\varepsilon$ and $|x - x_0| \le \varepsilon$, then $|x - y| \ge \varepsilon$. Using (5.6) and (3.5) for all $x, y \in D$ such that $|x - y| \ge \varepsilon$ and $|y| \le M$, it follows that

$$\frac{G_D(x,y)}{h(x)} q_a(y) h(y) \le \frac{C \rho_D(y)^{\frac{\alpha}{2}}}{\varepsilon^{d-\alpha}} |x|^{d-\alpha} \|h\|_\infty q_a(y) \le C' \delta_D(y)^{\frac{\alpha}{2}} \|h\|_\infty q_a(y).$$

Since the map $x \to \dfrac{G_D(x,y)}{h(x)}$ is continuous in $B(x_0,\varepsilon)\cap D$, whenever $y \in B^c(x_0,2\varepsilon)\cap D\cap(|y|\le M)$, then we conclude from (4.10) and the Lebesgue's dominated convergence theorem that

$$\int_{D\cap B^c(x_0,2\varepsilon)\cap(|y|\le M)} \Big|\frac{G_D(x,y)}{h(x)} - \frac{G_D(x',y)}{h(x')}\Big| q_a(y)h(y)dy \to 0, \text{ as } |x-x'|\to 0.$$

Finally, we deduce that $|Tw(x) - Tw(x')| \to 0$, as $|x-x'|\to 0$ uniformly for all $w \in F_a$. The last assertion then holds by Ascoli's theorem. ∎

Proof of Theorem 5.1. From (5.8) we have that $T_b w \ge b - C\|q_a\|_D$. Thus, for any $b \ge b_0 := a + C\|q_a\|_D$, we have $T_b w \ge a$. Hence

$$T_b(F_a) \subset F_a.$$

On the other hand, we note that if $(w_n)_n$ is a sequence in F_a such that $\|w_n - w\|_\infty \to 0$, then $f(x,h(x)w_n(x))$ converges to $f(x,h(x)w(x))$ for all $x\in D$. An application of the Lebesgue's theorem implies that $Tw_n(x) \to Tw(x)$, for all $x \in D$ and by Proposition , the convergence holds in the uniform norm. Thus we have shown that $T_b : F_a \to F_a$ is continuous.

Since $T_b(F_a)$ is relatively compact, then the Shauder fixed point theorem implies the existence of $w\in F_a$ such that

$$w(x) = b - \frac{1}{h(x)}\int_D G_D(x,y)f(y,w(y)h(y))dy. \tag{5.9}$$

For any $x\in D$, put $u(x) = w(x)h(x)$. Thus, u is a solution of

$$u(x) = bh(x) - \int_D G_D(x,y)f(y,u(y))dy, \tag{5.10}$$

i.e. u is a solution of (5.5). Since $u = wh$ where w is the function given in (5.9) and $w\ge a$, then $u \ge ah$. ∎

Theorem 5.2. *Assume that the conditions of Theorem 5.1 hold and that the mapping $u \to f(.,u)$ is nondecreasing. Moreover, we assume that for any $c>0$, there exists a nonnegative measurable function q_c such that:*
i) $\displaystyle\int_D G_D(x,y)q_c(y)dy < \infty$,
ii) $|f(x,y) - f(x,y')| \le q_c(x)|y-y'|, \quad y,y' \in [0,c]$.
Then there exists an unique solution of (5.5).

P r o o f. Let u_1 and u_2 be two solutions of (5.5) and let $c = max(\|u_1\|_\infty, \|u_2\|_\infty)$. Set

$$\psi(x) = \begin{cases} \dfrac{f(x,u_1(x)) - f(x,u_2(x))}{u_1(x) - u_2(x)}; \text{ if } u_1(x) \neq u_2(x), \\ 0; \text{ if } u_1(x) = u_2(x). \end{cases}$$

Then $0 \leq \psi \leq q_c$ and by (1.2) we get $u_1(x) - u_2(x) + V_\psi^\alpha(u_1 - u_2) = 0$, where for any Borel function g, $V_\psi^\alpha g(x) = \int_D G_D(x,y)g(y)\psi(y)dy$.
Since $u_1 - u_2 + V_\psi^\alpha(u_1 - u_2)^+ = V_\psi^\alpha(u_1 - u_2)^-$, we obtain $V_\psi^\alpha(u_1 - u_2)^- \geq V_\psi^\alpha(u_1 - u_2)^+$ on the set $[(u_1 - u_2)^+ > 0]$. We get from the complete maximum principle that $V_\psi^\alpha(u_1 - u_2)^- \geq V_\psi^\alpha(u_1 - u_2)^+$ on D and therefore $u_1 \geq u_2$ on D. Similarly, by interchanging u_1 by u_2 we get $u_1 = u_2$ on D. Since $u_1 = u_2$ on D^c, we obtain $u_1 = u_2$ on $\mathbb{R}^d$. ■

We follow the proof of the boundary Harnack principle.

Theorem 5.3. *Suppose that the assumptions of Theorem 5.1 hold and let V be an open set. Then, for every compact $K \subset V$, there exist constants c_1, $c_2 > 0$ depending only on K, V and D such that for any solution u of (1.1) given in Theorem 5.1 such that $u(x_0) = 1$ we have*

$$c_1 \frac{\rho_D(x)^{\frac{\alpha}{2}}}{|x|^{d-\alpha}} \leq u(x) \leq c_2 \frac{\rho_D(x)^{\frac{\alpha}{2}}}{|x|^{d-\alpha}}, \quad x \in K \cap D.$$

P r o o f. Let u and w as defined above. Then, from (5.4) and (4.6) we get

$$\int_D G_D(x,y)|f(y, w(y)h(y))|dy \leq \int_D G_D(x,y)q_a(y)h(y)dy \leq C\|q_a\|_D h(x).$$

Finally, from (5.10) we get $u(x) \leq (b + 2\|q_a\|_D)h(x)$. Since

$$ah(x) \leq u(x) \leq (C\|q_a\|_D + b)h(x), \quad x \in D$$

and h vanishes continuously on $V \cap D^c$, then Theorem 3.2 ends the proof. ■

Corollary 5.1. *Assume H_1 and H_2 hold. Moreover we suppose that there exist $\beta > 0$, $\gamma > 0$ and two nonnegative functions q and q_1 satisfying:*

a: $|f(x,t)| \leq q(x)t^{-\gamma}$, *for* $0 < t \leq \beta$,

b: $|f(x,t)| \leq q_1(x)$, *for* $t \geq \beta$,

c: *The maps* $x \to q(x)\rho_D(x)^{\frac{-\alpha}{2}(1+\gamma)}|x|^{(d-\alpha)(1+\gamma)}$
and $x \to q_1(x)|x|^{d-\alpha}\rho_D(x)^{\frac{-\alpha}{2}}$ *are in* $S_\infty(X^D)$.

Then, there exists $b_\phi > 0$ *such that for every* $b \in [b_\phi, \infty)$ *there exists a solution of* (5.5) *satisfying* $u \geq ah$.

P r o o f. From (5.6), we have

$$q(x)h(x)^{-1-\gamma} \leq c_1 q(x)\rho_D(x)^{\frac{-\alpha}{2}(1+\gamma)}|x|^{(d-\alpha)(1+\gamma)}$$

and $|q_1(x)h(x)^{-1}| \leq c_1 q_1(x)|x|^{d-\alpha}\rho_D(x)^{\frac{-\alpha}{2}}$, which yields that $qh^{-1-\gamma}$ and $q_1 h^{-1}$ are in $S_\infty(X^D)$. Set $A_h = C\|qh^{-1-\gamma}\|_D$ and $B_h = C\|q_1 h^{-1}\|_D$. Then, the mapping $a \to a + A_h a^{-\gamma} + B_h$ attains its minimal value b_0 for a positive number a_0. Setting $q_{a_0} = sup(a_0^{-\gamma} qh^{-1-\gamma}, q_1 h^{-1})$, we get that for every $w \in F_{a_0}$, $|f(x, w(y)h(y))| \leq q_{a_0}(x)h(y)$. The conclusion follows from the previous theorem. ∎

Example 5.1. Under the conditions of Corollary 5.1, we suppose that there exists $C > 0$ and γ quite small such that $q(x) \leq \dfrac{C}{|x|^\mu(\rho_D(x))^\lambda}$ and $q_1(x) \leq \dfrac{C}{|x|^\mu(\rho_D(x))^\lambda}$ for $\lambda < \frac{\alpha}{2}$ and $d < \mu$, then using Proposition 4.3, the result of Theorem 5.1 holds.

Theorem 5.4. *Assume* H_2 *is true. Suppose that there exist* $\beta > 0$, $\gamma > 0$ *and two nonnegative functions* q *and* q_1 *satisfying the same conditions of Corollary 5.1, then there exists* $b_0 > 0, a_0 > 0$ *such that for any* $\phi \in C_c(D^c)$ *with* $\phi \geq b_0 h_0$*, there exists a solution* u *of (1.1) such that* $u \geq a_0 h_0$.

P r o o f. By (5.7), we get that $qh_0^{-1-\gamma}$ and $q_1 h_0^{-1}$ are in $S_\infty(X^D)$. So let $A = C\|qh_0^{-1-\gamma}\|_D$ and $B = C\|q_1 h_0^{-1}\|_D$. Then, the map $a \to a + Aa^{-\gamma} + B$ has its minimal value b_0 for a positive number a_0. Set $K(x) = sup(a_0^{-\gamma} q(x)h_0^{-1-\gamma}, q_1 h_0^{-1})$. Let $\phi \in C_c(D^c)$ be such that $\phi \geq b_0 h_0$. Set $\widetilde{\phi} = \frac{1}{b_0}\phi$ and h the solution of

$$\begin{cases} (-\triangle)^{\frac{\alpha}{2}} h = 0, \text{ in } D, \\ h = \frac{1}{b_0}\phi, \text{ on } D^c. \end{cases} \tag{5.11}$$

Then, by the maximum principle (see Theorem 1.28 in [12]), we get $h \geq h_0$. Using the fact that $\gamma > 0$ and the assumptions on q and q_1 we get that for every $w \in F_{a_0}$, we have

$$\begin{aligned} |f(x, w(y)h(y))| \leq & \quad (a_0^{-\gamma} q(x)h^{-\gamma}(x)) \vee q_1(x) \\ \leq & \quad [(a_0^{-\gamma} q(x)h_0^{-1-\gamma}(x)) \vee (q_1(x)h_0^{-1}(x))]h(x) = K(x)h(x). \end{aligned}$$

Hence

$$\frac{1}{h(x)}\int_D G_D(x,y)|f(y,w(y)h(y))|dy \leq \frac{1}{h(x)}\int_D G_D(x,y)K(y)h(y)dy$$
$$\leq C\|K\|_D \leq Aa_0^{-\gamma}+B.$$

Hence for $b \geq b_0 = Aa_0^{-\gamma}+B+a_0$, we get $T_b u \geq b - Aa_0^{-\gamma} - B \geq a_0$. As in the proof of Theorem 5.1, $T_b(F_{a_0}) \subset F_{a_0}$. Hence, we conclude that there exists a function $w \in F_{a_0}$ such that $T_b w = w$, i.e. w is a solution of

$$T_b(w) = b - \frac{1}{h(x)}\int_D G_D(x,y)f(y,w(y)h(y))dy. \tag{5.12}$$

It follows that if we take $b = b_0$ in (5.12), the function $u = wh$ is a solution of (1.1) such that $w \geq a_0 h_0$. ■

In the sequel, we shall give the general Boundary Harnack Principle (BHP) for the case $f \geq 0$.

THEOREM 5.5. *We assume H_1, H_2 and the function f is nonnegative. Let u be a solution of (1.1) which is minorized by h_0. Moreover, we suppose that there exists an open set V such that u vanishes continuously on $V \cap D^c$. Then, for every compact $K \subset V$, there exist constants c_1, $c_2 > 0$ depending only on u, h_0, K, V and D such that*

$$c_1\frac{\rho_D(x)^{\frac{\alpha}{2}}}{|x|^{d-\alpha}} \leq u(x) \leq c_2\frac{\rho_D(x)^{\frac{\alpha}{2}}}{|x|^{d-\alpha}}, \quad x \in K \cap D.$$

P r o o f. Using the assumption on u, we get $h_0 \leq u \leq h$ in D. The conclusion then follows from Theorem 3.2. ■

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