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CONVOLUTION PRODUCTS IN $L^1(\mathbf{R}^+)$, INTEGRAL TRANSFORMS AND FRACTIONAL CALCULUS

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Abstract

We prove equalities in the Banach algebra $L^1(\mathbf{R}^+)$. We apply them to integral transforms and fractional calculus.

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In the first section, we prove the main results: we consider two different convolution products in $L^1(\mathbf{R}^+)$ and give new equalities between them, see Theorem 1.1, Theorem 1.2 and Corollary 1.3. Moreover these convolution products are dual in some sense, see Theorem 1.4.

As applications of these formulae, we give new interesting identities involving integral transforms and fractional calculus. In the case of Laplace transform, Corollary 2.1 is a direct consequence from results of the first section, while in the case of the Stieljes transform, Theorem 2.2 is proved following the same ideas as in the first section. The Riemann-Liouville integration, Weyl fractional calculus and Doetsch derivative may be defined using convolution products. The first section is the guideline how to prove directly new results in the fractional calculus, see for example Proposition 3.1, Proposition 3.3 and Theorem 3.5.

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1. Convolution products in $L^1(\mathbf{R}^+)$

Let $(L^1(\mathbf{R}^+), +, *)$ be the Banach algebra with the usual convolution product given by

$$f * g(t) := \int_0^t f(t-s)g(s)ds, \qquad f, g \in L^1(\mathbf{R}^+), \qquad t \geq 0. \tag{1}$$

The convolution product $*$ is commutative and associative. The norm $\|.\|_1$ defined by

$$\|f\|_1 = \int_0^\infty |f(t)|dt, \qquad f \in L^1(\mathbf{R}^+),$$

holds $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$. We may consider a second convolution product in $L^1(\mathbf{R}^+)$: take $f, g \in L^1(\mathbf{R}^+)$, we defined $f \circ g \in L^1(\mathbf{R}^+)$ by

$$f \circ g(t) := \int_t^\infty f(s-t)g(s)ds, \qquad t \geq 0. \tag{2}$$

It is easy to prove that $\|f \circ g\|_1 \leq \|f\|_1 \, \|g\|_1$.

Examples. Let $\lambda \in \mathbf{C}^+$ and $e_\lambda(s) := e^{-\lambda s}$ with $s \geq 0$. Then $e_\lambda \in L^1(\mathbf{R}^+)$ and

$$e_\lambda * e_\mu = \frac{1}{\mu - \lambda}\left(e_\lambda - e_\mu\right), \qquad e_\lambda \circ e_\mu = \frac{1}{\mu + \lambda} e_\mu,$$

with $\lambda, \mu \in \mathbf{C}^+$. The convolution product $\circ$ is non commutative, $e_\lambda \circ e_\mu \neq e_\mu \circ e_\lambda$ with $\lambda, \mu \in \mathbf{C}^+$, and non associative,

$$e_\lambda \circ (e_\mu \circ e_\theta) = \frac{1}{(\lambda+\theta)(\mu+\theta)} e_\theta \neq \frac{1}{(\lambda+\mu)(\mu+\theta)} e_\theta = (e_\lambda \circ e_\mu) \circ e_\theta,$$

with $\lambda, \mu, \theta \in \mathbf{C}^+$.

We may apply convolution products $*$ and $\circ$ to some functions when the expressions (1) and (2) have sense, for example, for $\alpha > 0$, we consider the function $j_\alpha(s) := \frac{s^{\alpha-1}}{\Gamma(\alpha)}$ with $s > 0$. Then

$$j_\alpha * j_\beta = j_{\alpha+\beta}, \qquad j_\alpha \circ j_\beta = \frac{\sin(\beta\pi)}{\sin((\alpha+\beta)\pi)} j_{\alpha+\beta},$$

with $\alpha + \beta < 1$ in the second equality.

Theorem 1.1. *Take $f, g, h \in L^1(\mathbf{R}^+)$. Then*

(a) $f \circ (g \circ h) = (f * g) \circ h = (g * f) \circ h = g \circ (f \circ h)$.

(b) $(f \circ g) \circ h = g \circ (f * h) - (g \circ f) * h$.

P r o o f. *(a)* We apply the Fubini theorem to obtain

$$f \circ (g \circ h)(t) = \int_t^\infty h(u) \int_t^u f(s-t)g(u-s)dsdu = (f * g) \circ h(t),$$

for $t \geq 0$. *(b)* We use the Fubini theorem and we change variables to get that

$$(f \circ g) \circ h(u) = \int_0^\infty g(s) \int_u^{u+s} f(s-t+u)h(t)dtds$$

$$= \int_0^\infty g(s) \int_0^{u+s} f(s-t+u)h(t)dtds - \int_0^\infty g(s) \int_0^u f(s-t+u)h(t)dtds$$

$$= \int_0^\infty g(s)(f*h)(u+s)ds - \int_0^u h(t) \int_0^\infty f(s-t+u)g(s)dsdt$$

$$= \int_u^\infty g(x-u)(f*h)(x)dx - \int_0^u h(t) \int_{u-t}^\infty f(r)g(r+t-u)drdt$$

$$= g \circ (f * h)(u) - (g \circ f) * h(u),$$

for $u \geq 0$. ■

In the next result, we compare $(f \circ g) \circ h$ and $f * (g \circ h)$.

THEOREM 1.2. *Take $f, g, h \in L^1(\mathbf{R}^+)$. Then*

$$(f \circ g) \circ h(u) = f * (g \circ h)(u) + \int_u^\infty f(r) \int_u^\infty g(r+x-u)h(x)dxdr$$
$$- \int_0^u f(r) \int_{u-r}^u g(r+x-u)h(x)dxdr,$$

with $u \geq 0$.

P r o o f. We apply Fubini theorem, and change variables to obtain

$$(f \circ g) \circ h(u) = \int_u^\infty g(y-u) \int_u^y f(y-t)h(t)dtdy$$

$$= \int_u^\infty g(y-u) \int_0^{y-u} f(r)h(y-r)drdy = \int_0^\infty f(r) \int_{r+u}^\infty g(y-u)h(y-r)dydr$$

$$= \int_0^\infty f(r) \int_u^\infty g(r+x-u)h(x)dxdr$$

$$= \int_0^u f(r) \int_u^\infty g(r+x-u)h(x)dxdr + \int_u^\infty f(r) \int_u^\infty g(r+x-u)h(x)dxdr.$$

In the first summand, we have that

$$\int_0^u f(r)\int_u^\infty g(r+x-u)h(x)dxdr$$

$$= \int_0^u f(r)\int_{u-r}^\infty g(r+x-u)h(x)dxdr - \int_0^u f(r)\int_{u-r}^u g(r+x-u)h(x)dxdr$$

$$= f*(g\circ h)(u) - \int_0^u f(r)\int_{u-r}^u g(r+x-u)h(x)dxdr,$$

and we conclude the result. ∎

Corollary 1.3. *Take* $f,g,h\in L^1(\mathbf{R}^+)$*. Then*

$$g\circ(f*h)(u) = f*(g\circ h)(u) + h*(g\circ f)(u)$$

$$+\int_u^\infty f(r)\int_u^\infty g(r+x-u)h(x)dxdr - \int_0^u f(r)\int_{u-r}^u g(r+x-u)h(x)dxdr,$$

with $u\geq 0$.

P r o o f. We apply Theorem 1.1 *(b)* and Theorem 1.2. ∎

Let $L^\infty(\mathbf{R}^+)$ the Banach space with the norm $\| \quad \|_\infty$ given by

$$\|f\|_\infty = \sup \text{ess}_{t\in\mathbf{R}}|f(t)| < \infty, \qquad f\in L^\infty(\mathbf{R}^+).$$

Take $f\in L^\infty(\mathbf{R}^+)$ and $g\in L^1(\mathbf{R}^+)$ then $f*g, f\circ g, g\circ f\in L^\infty(\mathbf{R}^+)$, and

$$\|f*g\|_\infty \leq \|f\|_\infty\|g\|_1, \qquad \|f\circ g\|_\infty \leq \|f\|_\infty\|g\|_1, \qquad \|g\circ f\|_\infty \leq \|f\|_\infty\|g\|_1.$$

The next results show that the convolution products $\circ$ and $*$ are dual.

Theorem 1.4. *Take* $f\in L^\infty(\mathbf{R}^+)$ *and* $g,h\in L^1(\mathbf{R}^+)$*. Then*

(a) $\displaystyle\int_0^\infty f(t)(g*h)(t)dt = \int_0^\infty (g\circ f)(t)h(t)dt = \int_0^\infty (h\circ f)(t)g(t)dt.$

(b) $\displaystyle\int_0^\infty f(t)(g\circ h)(t)dt = \int_0^\infty (f*g)(t)h(t)dt = \int_0^\infty g(t)(f\circ h)(t)dt.$

P r o o f. *(a)* We apply the Fubini theorem to obtain

$$\int_0^\infty f(t)\int_0^t g(t-s)h(s)dsdt = \int_0^\infty h(s)\int_s^\infty g(t-s)f(t)dtds = \int_0^\infty (g\circ f)(t)h(t)dt.$$

The part *(b)* is proven in the same way. ∎

2. Integral transforms and convolution equalities

In this section we consider the Laplace transform and the Stieltjes transform. We prove new equalities for both, using results and ideas from the first section.

2.1. Laplace transform

Given $f \in L^1(\mathbf{R}^+)$, the Laplace transform of f, $\mathcal{L}(f)$, is defined by

$$\mathcal{L}(f)(z) := \int_0^\infty f(t)e^{-zt}, \qquad z \in \overline{\mathbf{C}^+},$$

see, for example [2]. The function $\mathcal{L}(f)$ is a holomorphic function on $\mathbf{C}^+$, bounded on $\overline{\mathbf{C}^+}$ such that $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$ with $f, g \in L^1(\mathbf{R}^+)$. Moreover, it is easy to check that

(i) $f \circ e_\lambda = \mathcal{L}(f)(\lambda)e_\lambda$,
(ii) $e_\lambda \circ f = \mathcal{L}(f)(\lambda)e_{-\lambda} - e_{-\lambda} * f$,

where $e_\lambda(s) = e^{-\lambda s}$ with $s \geq 0$ and $\lambda \in \mathbf{C}^+$. The incomplete Laplace transform of $f \in L^1(\mathbf{R}^+)$, $\mathcal{L}(f,t)$, is defined by

$$\mathcal{L}(f,t)(z) = \int_0^t f(s)e^{-zs}ds, \qquad t \geq 0,$$

for $z \in \mathbf{C}$. Given $f, g \in L^1(\mathbf{R}^+)$, we apply the Fubini theorem to obtain that

(i) $\mathcal{L}(f \circ g)(z) = \mathcal{L}(g\mathcal{L}(f,\,\cdot\,)(-z))(z)$, with $z \in \mathbf{C}^+$,
(ii) $(f \circ \mathcal{L}(g))(t) = \mathcal{L}(g\mathcal{L}(f))(t)$, for $t \geq 0$,
(iii) $(\mathcal{L}(f) \circ g)(t) = \int_0^\infty f(u)(e_u \circ g)(t)du$, for $t \geq 0$.

COROLLARY 2.1. *Given $f, h \in L^1(\mathbf{R}^+)$, $\lambda \in \mathbf{C}^+$ and $u \geq 0$, then*

$$\int_u^\infty f(r)e^{-\lambda r}\int_u^\infty e^{-\lambda x}h(x)dxdr - \int_0^u f(r)e^{-\lambda r}\int_{u-r}^u e^{-\lambda x}h(x)dxdr$$
$$= \mathcal{L}(f*h)(\lambda) + \mathcal{L}(f*h,u)(\lambda) - \mathcal{L}(f)(\lambda)\mathcal{L}(h,u)(\lambda) - \mathcal{L}(h)(\lambda)\mathcal{L}(f,u)(\lambda).$$

P r o o f. We apply the Theorem 1.2 with $g = e_\lambda$. ∎

2.2. Stieltjes transform

Let $\mathcal{S}_+$ be the Schwartz class on $[0,\infty)$, i.e., functions which are infinitely differentiable which verifies

$$\sup_{t\geq 0}\left|t^m \frac{d^n}{dt^n} f(t)\right| < \infty,$$

for any $m, n \in \mathbf{N} \cup \{0\}$. Given $\alpha > 0$ and $f \in \mathcal{S}_+$ the generalized Stieltjes transform of f, $\mathcal{S}_\alpha f$, is defined by

$$\mathcal{S}_\alpha f(y) = \int_0^\infty \frac{f(x)}{(x+y)^\alpha} dx, \qquad y > 0,$$

see definition and properties in [6]. If $\alpha = 1$, it is obtained the classical Stieltjes transform. Note that if $0 < \alpha < 1$, we have the equality

$$f \circ j_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{f(s)}{(s+t)^{1-\alpha}} ds = \frac{1}{\Gamma(\alpha)} \mathcal{S}_{1-\alpha}(f)(t),$$

with $f \in \mathcal{S}_+$ and $t > 0$. Following the same lines than in Theorem 1.1 *(a)* and Theorem 1.2, we prove the next result.

THEOREM 2.2. *Take $\alpha > 0$ and $f, g \in \mathcal{S}_+$, then*

(a) $f \circ \mathcal{S}_\alpha(g) = g \circ \mathcal{S}_\alpha(f) = \mathcal{S}_\alpha(f * g)$.

(b) $$\mathcal{S}_\alpha(f \circ g)(u) = f * \mathcal{S}_\alpha(g)(u) + \int_u^\infty f(r) \int_u^\infty g(r+x-u) x^{-\alpha} dx dr - \int_0^u f(r) \int_{u-r}^u g(r+x-u) x^{-\alpha} dx dr.$$

3. Fractional calculi and convolution equalities

In this section, we consider three types of fractional calculi (Riemann-Liouville, Weyl and Doetsch fractional calculus) and we obtain new equalities to them.

3.1. Riemann-Liouville fractional integration

Take $f \in L^1(\mathbf{R}^+)$, the Riemann-Liouville integral of order $\alpha > 0$ of f is defined by

$$I_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad t \geq 0.$$

Note that $I_+^{-\alpha} f = j_\alpha * f$ with $\alpha > 0$, see for example [5].

PROPOSITION 3.1. *Take $1 > \alpha > 0$, and $f, g \in \mathcal{S}_+$, then*

(a) $$\frac{1}{\Gamma(\alpha)} \mathcal{S}_{1-\alpha}(f \circ g) = g \circ (I_+^{-\alpha} f) - I_+^{-\alpha}(g \circ f).$$

(b) $\Gamma(\alpha)(g \circ I_+^{-\alpha} f - I_+^{-\alpha}(g \circ f))(u) = f * \mathcal{S}_{1-\alpha} g(u) + \int_u^\infty f(r)$
$$\times \int_u^\infty g(r+x-u)x^{\alpha-1}dxdr - \int_0^u f(r)\int_{u-r}^u g(r+x-u)x^{\alpha-1}dxdr.$$

P r o o f. The part *(a)* is proven in the same way as Theorem 1.1 *(b)* with $h = j_\alpha$. The part *(b)* is similar to Corollary 1.3 with $h = j_\alpha$. ∎

3.2. Weyl fractional calculus

Given $f \in \mathcal{S}_+$ the Weyl fractional integral of order $\alpha > 0$, $W_+^{-\alpha} f$, is defined by
$$W_+^{-\alpha} f(u) := \frac{1}{\Gamma(\alpha)}\int_u^\infty (t-u)^{\alpha-1} f(t)dt, \qquad u \geq 0,$$
see for example [5]. Note $W_+^{-\alpha} f = j_\alpha \circ f$ for $\alpha > 0$, and check
$$\int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)}(g\circ f)(t)dt = \int_0^\infty f(t)I_+^{-\alpha}g(t)dt = \int_0^\infty W_+^{-\alpha}f(t)g(t)dt,$$
with $f, g \in \mathcal{S}_+$ (see Theorem 1.4). We follow the same ideas than in the Theorem 1.1 to check the next proposition.

Proposition 3.2. *Take $\alpha > 0$, and $f, g, h \in \mathcal{S}_+$. Then:*

(a) $f \circ W_+^{-\alpha} h = I_+^{-\alpha} f \circ h = W_+^{-\alpha}(f \circ h)$.

(b) $W_+^{-\alpha} g \circ h = g \circ I_+^{-\alpha} h - \frac{1}{\Gamma(\alpha)}\mathcal{S}_{1-\alpha}(g) * h$, *with* $1 > \alpha > 0$.

(c) $\frac{1}{\Gamma(\alpha)}\mathcal{S}_{1-\alpha}(f) \circ h = W_+^{-\alpha}(f * h) - W_+^{-\alpha} f * h$, *with* $1 > \alpha > 0$.

As in the case of the Theorem 1.2 and Corollary 1.3, we obtain the following equalities.

Proposition 3.3. *Take $\alpha > 0$, and $f, g, h \in \mathcal{S}_+$. Then:*

(a) $W_+^{-\alpha} g \circ h(u) = I_+^{-\alpha}(g \circ h)(u) + \int_u^\infty \frac{r^{\alpha-1}}{\Gamma(\alpha)}\int_u^\infty g(r+x-u)h(x)dxdr$
$$- \int_0^u \frac{r^{\alpha-1}}{\Gamma(\alpha)}\int_{u-r}^u g(r+x-u)h(x)dxdr, \quad \textit{with} \quad u \geq 0.$$

(b) $W_+^{-\alpha}(f \circ h)(u) = f * W_+^{-\alpha} h(u) + h * W_+^{-\alpha} f(u)$
$$+ \int_u^\infty f(u)\int_u^\infty \frac{(r+x-u)^{\alpha-1}}{\Gamma(\alpha)}h(x)dxdr$$
$$- \int_0^u f(r)\int_{u-r}^u \frac{(r+x-u)^{\alpha-1}}{\Gamma(\alpha)}h(x)dxdr, \quad \textit{with} \quad u \geq 0.$$

The Weyl *fractional derivative* of order α is defined by

$$W_+^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s) ds, \qquad t \geq 0,$$

with $n = [\alpha] + 1$ ([5]). It is easy to check that if $\alpha \in \mathbf{N}$ then $W_+^\alpha f = (-1)^\alpha f^{(\alpha)}$, $W_+^{\alpha+\beta} f = W_+^\alpha (W_+^\beta f)$ with $\alpha, \beta \in \mathbf{R}$, $W_+^0 = Id$ and $f \in \mathcal{S}_+$. Since $W_+^{-\alpha}(W_+^\alpha g) = g$, the following "integrating by parts" formula holds:

$$\int_0^\infty f(t) g(t) dt = \int_0^\infty I_+^{-\alpha} f(t) W_+^\alpha g(t) dt. \tag{3}$$

Also, it is straightforward to prove $W_+^\alpha (f \circ g) = f \circ W_+^\alpha g$ for $\alpha \in \mathbf{R}$ and $f, g \in \mathcal{S}_+$. In [4], it is proven that

$$\int_0^\infty W_+^\alpha g(t) t^\alpha \mathcal{L}(f)(t) dt = \int_0^\infty \mathcal{L}(g)(t) t^\alpha W_+^\alpha f(t) dt, \tag{4}$$

with $f, g \in \mathcal{S}_+$, $\alpha > 0$, and

$$\begin{aligned} W_+^\alpha (f * g)(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t W_+^\alpha g(u) \int_{t-u}^t (x+u-t)^{\alpha-1} W_+^\alpha f(x) dx du \\ & - \frac{1}{\Gamma(\alpha)} \int_t^\infty W_+^\alpha g(u) \int_t^\infty (x+u-t)^{\alpha-1} W_+^\alpha f(x) dx du, \end{aligned} \tag{5}$$

with $t \geq 0$ in [3].

PROPOSITION 3.4. *Take $f, h \in \mathcal{S}_+$ and $\alpha > 0$ then*

$$(W_+^\alpha f \circ j_\alpha) \circ W_+^\alpha h = W_+^\alpha f * h - W_+^\alpha (f * h);$$

in the case $0 < \alpha < 1$, *we have that*

$$\frac{1}{\Gamma(\alpha)} S_{1-\alpha}(W_+^\alpha f) \circ W_+^\alpha h = W_+^\alpha f * h - W_+^\alpha (f * h).$$

P r o o f. It is a consequence of (5) and the Theorem 1.2. ■

3.3. Doetsch fractional derivative

G. Doetsch [2] defines the fractional derivative of f as the solution g of the integral equation

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} g(u) du = f(t), \qquad t \geq 0.$$

Let us denote it by $D^\alpha f$. This approach is closer to the Laplace transform (see [1, p.72]) and the Riemann-Liouville fractional calculus: take $f, g \in L^1(\mathbf{R}^+)$ and $\alpha > 0$, then the following are equivalent:

(i) $D^\alpha f = g$.

(ii) $\mathcal{L}(g)(s) = s^\alpha \mathcal{L}(f)(s)$ for $s > 0$.

(iii) $I_+^{-\alpha}(g) = f$.

When $f, D^\alpha f \in L^1(\mathbf{R}^+)$ with $\alpha > 0$, then

$$\int_0^\infty f(t)g(t)dt = \int_0^\infty D^\alpha f(t) W_+^{-\alpha} g(t)dt, \quad g \in \mathcal{S}_+,$$

because of $I_+^{-\alpha} D^\alpha f = f$.

THEOREM 3.5. *Take $f, g \in L^1(\mathbf{R}^+)$ and $\alpha > 0$. Then the following are equivalent:*

(i) $D^\alpha f = g$.

(ii) $\mathcal{L}(g)(s) = s^\alpha \mathcal{L}(f)(s)$ *for* $s > 0$.

(iii) $I_+^{-\alpha}(g) = f$.

(iv) $\int_0^\infty g(t)\phi(t)dt = \int_0^\infty f(t) W_+^\alpha \phi(t)dt$ *for any* $\phi \in \mathcal{S}_+$.

(v) $\int_0^\infty g(t)(\phi \circ \psi)(t)dt = \int_0^\infty f(t)(\phi \circ W_+^\alpha \psi)(t)dt$ *for any* $\phi, \psi \in \mathcal{S}_+$.

(vi) $S_1(g) = \Gamma(\alpha+1) S_{\alpha+1}(f)$.

P r o o f. *(i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* appear in [1]. *(iii)* $\Rightarrow$ *(iv)* is straighforward using the equality (3). For *(iv)* $\Rightarrow$ *(v)*, it is enough to use $W_+^\alpha(\phi \circ \psi) = \phi \circ W_+^\alpha \psi$. For *(v)* $\Rightarrow$ *(ii)*, take $\psi = e_s$, then $\phi \circ \psi(t) = e^{-st}\mathcal{L}\phi(s)$ for any $\phi \in \mathcal{S}_+$, $t, s > 0$ and $\mathcal{L}(g)(s) = s^\alpha \mathcal{L}(f)(s)$ for $s > 0$. *(vi)* $\Leftrightarrow$ *(ii)* Following equalities hold

$$\begin{aligned} S_1(g)(t) &= \int_0^\infty e^{-ts} \int_0^\infty g(x) e^{-sx} dx ds, \\ \Gamma(\alpha+1) S_{\alpha+1}(f)(t) &= \int_0^\infty e^{-ts} s^\alpha \int_0^\infty f(x) e^{-sx} dx ds. \end{aligned}$$

and the proof is finished. ■

The next result is a direct consequence of the Theorem 3.5 and (4).

PROPOSITION 3.6. *Given $f, g \in \mathcal{S}_+$ such that $D^\alpha f, D^\alpha g \in \mathcal{S}_+$, then*

$$\int_0^\infty D^\alpha f(t)\mathcal{L}(W_+^\alpha g)(t)dt = \int_0^\infty W_+^\alpha f(t)\mathcal{L}(D^\alpha g)(t)dt.$$

P r o o f. By Theorem 3.5 and (4), it is obtained that

$$\int_0^\infty W_+^\alpha g(s)s^\alpha \mathcal{L}(f)(s)ds = \int_0^\infty \mathcal{L}(W_+^\alpha g)(t)(D^\alpha f)(t)dt.$$

and

$$\int_0^\infty \mathcal{L}(g)(t)t^\alpha W_+^\alpha f(t)dt = \int_0^\infty \mathcal{L}(D^\alpha g)(t)W_+^\alpha f(t)dt,$$

and the equality is proven by (4). ∎

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