Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 9, NUMBER 1 (2006) ISSN 1311-0454

WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR INHOMOGENEOUS TIME-FRACTIONAL PSEUDO-DIFFERENTIAL EQUATIONS

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Abstract

In the present paper the Cauchy problem for partial inhomogeneous pseudo-differential equations of fractional order is analyzed. The solvability theorem for the Cauchy problem in the space $\Psi_{G,2}(R^n)$ of functions in $L_2(R^n)$ whose Fourier transforms are compactly supported in a domain $G \subseteq R^n$ is proved. The representation of the solution in terms of pseudo-differential operators is given. The solvability theorem in the Sobolev spaces $H_2^s(R^n),\ s \in R^1$ is also established.

Mathematics Subject Classification: 26A33, 45K05, 35A05, 35S10, 35S15, 33E12

Key Words and Phrases: pseudo-differential equations, Cauchy problem, Caputo fractional derivative, Mittag-Leffler function, inhomogeneous equation, time-fractional equation

1. Introduction

This paper is devoted to the Cauchy problem for time-fractional inhomogeneous pseudo-differential equations of fractional order

$$D_*^\alpha u(t,x) = A(D_x)u(t,x) + f(t,x), \quad t > 0, \quad x \in R^n, \tag{1}$$

$$u(0, x) = \varphi(x), \qquad (2)$$

where $f(t, x)$ and $\varphi(x)$ are given functions in certain spaces defined later; $D_x = (D_1, ..., D_n)$, $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, ..., n$; $A(D_x)$ is a pseudo-differential operator with a symbol $A(\xi)$, which is a real-analytic function defined in an open domain $G \subseteq R^n$, and D_*^α, $0 < \alpha < 1$ is the operator of fractional differentiation of order α in the Caputo sense (see, for example, [1, 2, 3])

$$D_*^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1, \quad t > 0.$$

In our analysis we essentially rely on the results of the paper [4], where the Cauchy problem for fractional order homogeneous pseudo-differential equations of arbitrary order $\alpha > 0$ is studied. Note that inhomogeneous case can not be directly reduced to the homogeneous case. The Duhamel principle is not applicable as well. Moreover, in that case the solution of the problem is connected with the construction and estimation of the corresponding Green's function.

As it is well-known, initial and boundary value problems for partial differential equations of fractional order in bounded and unbounded domains are important for the study of non-brownian diffusion and transport of premices in fractal media. The sphere of applications of fractional order differential equations has been expanded during the last two decades and includes such areas as computer tomography [6], finance and macroeconomics [5], biology [7, 8], hydrodynamics [9], etc.

We note that the Cauchy problem for integer order pseudo-differential equations with analytic symbols or with symbols having singularities was studied, for example, by Dubinskij [10], Umarov [11] and Tran Duc Van [12]. For studying of the Cauchy and more general multi-point value problems they used some properties of pseudo-differential operators with singular symbols. These operators were introduced by Dubinskij [10, 13] in the case of analytic symbols defined in some open domain G. Later on, Umarov [14] suggested the method of localization of singularities. This method allows to consider non-analytic symbols, which may have non-integrable or other type of singularities on the boundary of G or in its exterior. Umarov has found the solvability conditions of the multi-point problem with general boundary operators in the case of homogeneous equation. The inhomogeneous case requires construction and additional estimation of the Green function. This idea was realized by Saydamatov [15] and Saydamatov, Umarov [16] in the case of integer order pseudo-differential equations. In the present paper we will extend this idea to the fractional differential equations.

Finally, we note that Gorenflo, Luchko and Umarov in their recent work [4] have studied the Cauchy problem (1), (2) and more general multi-point boundary value problems in the case of homogeneous partial pseudo-differential equations of fractional order (i.e. $f(t,x) = 0$ in (1)). They essentially used the language of symbols and have applied the method of localization of singularities of the solution operators, which are pseudo-differential operators with singular symbols. As the space of initial data the class $\Psi_{G,p}(R^n)$ of entire functions of finite exponential type was taken. Under certain condition on G the space $\Psi_{G,p}(R^n)$ is densely embedded into the Sobolev spaces $H_p^s(R^n)$, $s \in R^1$, $1 < p < \infty$. This fact allows to transfer the obtained results to the Sobolev spaces by means of the closure of solution operators.

This paper is organized as follows. In Section 2 the pseudo-differential operators with singular symbols and the space $\Psi_{G,p}(R^n)$ introduced in [17], are defined. In Section 3 solvability theorem for the Cauchy problem (1), (2) in the space $\Psi_{G,p}(R^n)$, $p = 2$ is proved. In Section 4 the well-posedness of the Cauchy problem (1), (2) in the Sobolev spaces $H_2^s(R^n)$ is studied. In Section 5 the Cauchy problem (1), (2) in the case $A(D_x) = \Delta$, where Δ is the Laplace operator with the symbol $A(\xi) = -|\xi|^2$, as a fractional model of sub-diffusion process is considered.

2. Basic spaces of functions and pseudo-differential operators

In this section we formulate some necessary notions and results, obtained by Gorenflo, Luchko and Umarov [4].

Let G be an open domain in R^n and the system of open sets $\{g_k\}_{k=0}^{\infty}$ be a locally finite covering of G, i.e.

$$G = \bigcup_{k=0}^{\infty} g_k, \quad g_k \subset\subset G.$$

Let any compact set $K \subset G$ have a nonempty intersection with finitely many sets g_k. Denote by $\{\theta_k\}_{k=0}^{\infty}$ a smooth partition of unity of G.

Further, let $1 < p < \infty$ and a function $f(x)$ be in $L_p(R^n)$ whose Fourier transform has a compact support in G. For example, $f(x) = x^{-1}\sin x \in L_p(R^1)$ for all $p \in (1; +\infty)$ and its Fourier transform

$$F[x^{-1}\sin x](y) = \begin{cases} \frac{1}{2}, & y \in (-1;1), \\ 0, & y \in R^1 \backslash (-1;1) \end{cases}$$

has a compact support in G, where G is an arbitrary interval containing the segment $[-1;1]$. The set of all such functions endowed with the convergence defined in Definition 2.1 is denoted by $\Psi_{G,p}(R^n)$.

Definition 2.1. A sequence of functions $f_m \in \Psi_{G,p}(R^n)$, $m = 1, 2, 3, ...$ is said to converge to an element $f_0 \in \Psi_{G,p}(R^n)$ iff:

1) there exists a compact set $K \subset G$ such that the support $supp\, F f_m \subset K$ for all $m \in N$;

2) the norm $\|f_m - f_0\|_{L_p} = (\int_{R^n} |f_m(x) - f_0(x)|^p dx)^{\frac{1}{p}} \to 0$ for $m \to \infty$.

According to the Paley-Wiener-Schwartz theorem, the elements of $\Psi_{G,p}(R^n)$ are entire functions of exponential type which, restricted to R^n, are in the space $L_p(R^n)$.

The space $\Psi_{G,p}(R^n)$ can be represented as an inductive limit of some spaces. Namely, let

$$G_N = \bigcup_{k=1}^{N} g_k, \quad \chi_N(\xi) = \sum_{k=1}^{N} \theta_k(\xi).$$

Denote by Ψ_N the set of functions $f \in L_p(R^n)$ satisfying the following conditions:

a) $supp\, F f \subset G_N$;

b) $supp\, F f \cap supp\ \theta_j = \emptyset$ for $j > N$;

c) $p_N(f) = \|F^{-1}\chi_N F f\|_{L_p} < \infty$.

Here by F^{-1} we denote the operator inverse to the Fourier transform F. It is not hard to verify that (see [4])

$$\Psi_{G,p} = ind \lim_{N\to\infty} \Psi_N.$$

Let $A(D)$ be a pseudo-differential operator with a symbol $A(\xi)$, which is a real-analytic function in G. Outside of G or on its boundary $A(\xi)$ may have singularities of arbitrary type. For a function $\varphi(x) \in \Psi_{G,p}(R^n)$ the operator $A(D)$ is defined by the formula

$$A(D)\,\varphi(x) = \frac{1}{(2\pi)^n}\int_{R^n} A(\xi)F\varphi(\xi)e^{ix\xi}d\xi = \frac{1}{(2\pi)^n}\int_{G} A(\xi)F\varphi(\xi)e^{ix\xi}d\xi.$$

As shown in [17], [4], the space $\Psi_{G,p}(R^n)$ is an invariant with respect to the action of such pseudo-differential operators and these operators act continuously. The embedding

$$\Psi_{G,p}(R^n) \subset H_p^s(R^n), \quad s \in R^1, \quad 1 < p < \infty$$

is dense if the complement of G in R^n has zero measure. Moreover, if in addition the symbol $A(\xi)$ for some $\ell \in R^1$ satisfies the estimate

$$|A(\xi)| \leq C(1+|\xi|^2)^{\frac{\ell}{2}}, \quad C>0,\ \xi \in R^n,$$

then for the operator $A(D) : \Psi_{G,2}(R^n) \to \Psi_{G,2}(R^n)$, corresponding to $A(\xi)$, there exists a unique continuous closure

$$\overline{A}(D) : H_2^s(R^n) \to H_2^{s-\ell}(R^n), \quad s \in R^1.$$

3. Well-posedness of the Cauchy problem in $\Psi_{G,2}(R^n)$

In this section we prove the well-posedness of the Cauchy problem (1), (2) in the space $\Psi_{G,2}(R^n)$, although the result is true for arbitrary $p \in (1,\infty)$. We take the particular case $p=2$ only for the convenience of narrative noting the differences in the general case.

First we recall the well-known Duhamel principle for the Cauchy problem in the case $\alpha = 1,\ D_*^\alpha \equiv \frac{\partial}{\partial t}$, i.e.

$$\frac{\partial u}{\partial t}(t,x) = A(D_x)u(t,x) + f(t,x), \quad t>0,\ x \in R^n, \tag{3}$$

$$u(0,x) = \varphi(x), \quad x \in R^n. \tag{4}$$

Namely, the Duhamel principle states that to solve the problem (3), (4), it is sufficient to consider the homogeneous case. In fact, if $U(t,\tau,x)$ is a solution of the problem

$$\frac{\partial U}{\partial t} = A(D_x)U,$$

$$U(0,\tau,x) = f(\tau,x), \quad 0<\tau<t,$$

then

$$u(t,x) = \int_0^t U(t-\tau,\tau,x)d\tau$$

is a solution of the problem

$$\frac{\partial u}{\partial t} = A(D_x)u + f(t,x),$$

$$u(0,x) = 0.$$

Now we consider the Cauchy problem (1), (2). Note that in this case the Duhamel principle can not be applied directly. The Cauchy problem (1),

(2) in the homogeneous case ($f(t,x) \equiv 0$) was studied by Gorenflo, Luchko and Umarov [4]. They obtained the following representation for the solution

$$u(t,x) = E_\alpha(t^\alpha A(D_x))\varphi(x),$$

where $E_\alpha(t^\alpha A(D_x))$ is a pseudo-differential operator with the symbol $E_\alpha(t^\alpha A(\xi))$ and $E_\alpha(z)$ is the Mittag-Leffler function (see [18])

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}.$$

Denote by $C^{(m)}[t>0;\, \Psi_{G,2}(R^n)]$ and by $AC[t>0;\, \Psi_{G,2}(R^n)]$ the space of m times continuously differentiable functions and the space of absolute continuous functions on $(0;+\infty)$ accordingly ranging in the space $\Psi_{G,2}(R^n)$, respectively. It is well-known that there exists $D_*^{1-\alpha} f(t,x)$, $0<\alpha<1$, if $f(t,x) \in AC[t \geq 0;\, \Psi_{G,2}(R^n)]$ (see [19]).

Definition 3.1. A function $u(t,x) \in C^{(1)}[t>0; \Psi_{G,2}(R^n)] \cap C[t \geq 0; \Psi_{G,2}(R^n)]$ is called a solution of the problem (1), (2) if it satisfies the equation (1) and the initial condition (2) pointwise.

Theorem 3.2. *Let* $\varphi(x) \in \Psi_{G,2}(R^n)$, $f(t,x) \in AC[t \geq 0;\, \Psi_{G,2}(R^n)]$, $D_*^{1-\alpha} f(t,x) \in C[t \geq 0;\, \Psi_{G,2}(R^n)]$ *and* $f(0,x)=0$. *Then the Cauchy problem (1), (2) has a unique solution. This solution is given by the representation*

$$u(t,x) = E_\alpha(t^\alpha A(D_x))\varphi(x) + \int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f(\tau,x)\,d\tau. \quad (5)$$

P r o o f. It is sufficient to consider the case $\varphi(x)=0$ in (2)and to prove that the function

$$v(t,x) = \int_0^\infty U(t,\tau,x)\,d\tau = \int_0^t U(t,\tau,x)\,d\tau, \quad (6)$$

where

$$U(t,\tau,x) = \begin{cases} E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f(\tau,x), & t \geq \tau, \\ 0, \quad t<\tau, \end{cases}$$

is a solution of the problem

$$D_*^\alpha v(t,x) - A(D_x) v(t,x) = f(t,x), \quad (7)$$

$$v(0,x) = 0. \quad (8)$$

First we show that the equation

$$D_*^\alpha\Big(\int_0^t g(\tau,x)d\tau\Big) = f(t,x) \tag{9}$$

has a unique solution $g(t,x) = D_*^{1-\alpha} f(t,x)$, if $f(t,x) \in AC[t \geq 0;\ \Psi_{G,2}(R^n)]$ and $f(0,x) = 0$. In fact, we have

$$D_*^\alpha\Big(\int_0^t g(\tau,x)d\tau\Big) \equiv J^{1-\alpha}\frac{\partial}{\partial t}\int_0^t g(\tau,x)d\tau.$$

In this formula, J^α is the fractional integration operator

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\rho)^{\alpha-1} y(\rho)d\rho, \quad 0<\alpha<1.$$

Therefore,

$$D_*^\alpha\Big(\int_0^t g(\tau,x)d\tau\Big) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}\Big(\int_0^\rho g(\tau,x)d\tau\Big)d\rho$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(\rho,x)}{(t-\rho)^\alpha}d\rho$$

and we can rewrite the equation (9) in the form

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(\rho,x)}{(t-\rho)^\alpha}d\rho = f(t,x).$$

The last equality is an Abel type equation (see [19], [20]) and therefore, it has a unique solution $g(t,x)$, if $f(t,x) \in AC[t \geq 0;\ \Psi_{G,2}(R^n)]$. Under the condition $f(0,x) = 0$ this solution is represented in the form

$$g(t,x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\rho)^{\alpha-1}\frac{\partial}{\partial\rho} f(\rho,x)d\rho = J^\alpha\Big(\frac{\partial}{\partial t} f(t,x)\Big) = D_*^{1-\alpha} f(t,x).$$

Further, verify that $v(t,x)$ in (6) satisfies to the equation (7). Indeed, the function $U(t,\tau,x)$ is a solution of the homogeneous problem

$$D_*^\alpha U(t,\tau,x) - A(D_x)U(t,\tau,x) = 0,$$

$$U(\tau,\tau,x) = g(\tau,x).$$

Here $g(t,x)$ is the solution of the equation (9), i.e. $g(t,x) = D_*^{1-\alpha} f(t,x)$. We have

$$D_*^\alpha v(t,x) - A(D_x)v(t,x) = J^{1-\alpha}\frac{\partial}{\partial t}v(t,x) - A(D_x)v(t,x)$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}v(\rho,x)d\rho - A(D_x)\int_0^t U(t,\tau,x)d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}\int_0^\rho U(\rho,\tau,x)d\tau d\rho - \int_0^t A(D_x)U(t,\tau,x)d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}[U(\rho,\rho,x) + \int_0^\rho \frac{\partial}{\partial\rho}U(\rho,\tau,x)d\tau]d\rho - \int_0^t A(D_x)U(t,\tau,x)d\tau.$$

Taking into account that $U(\rho,\rho,x) = g(\rho,x)$ and

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha} g(\rho,x)d\rho = D_*^\alpha(\int_0^t g(\tau,x)d\tau) = f(t,x),$$

we obtain

$$D_*^\alpha v(t,x) - A(D_x)v(t,x) = f(t,x)$$

$$+ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}\int_0^\rho \frac{\partial}{\partial\rho}U(\rho,\tau,x)d\tau d\rho - \int_0^t A(D_x)U(t,\tau,x)d\tau.$$

Further, changing the order of integration we have

$$D_*^\alpha v(t,x) - A(D_x)v(t,x) = f(t,x)$$

$$+ \int_0^t \int_\tau^t \frac{1}{\Gamma(1-\alpha)}(t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}U(\rho,\tau,x)d\rho d\tau - \int_0^t A(D_x)U(t,\tau,x)d\tau$$

$$= f(t,x) + \int_0^t [\int_\tau^t \frac{1}{\Gamma(1-\alpha)}(t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}U(\rho,\tau,x)d\rho - A(D_x)U(t,\tau,x)]d\tau.$$

Since $U(\rho,\tau,x) = 0$ for $\rho < \tau$, then

$$\int_\tau^t \frac{1}{\Gamma(1-\alpha)}(t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}U(\rho,\tau,x)d\rho = \int_0^t \frac{1}{\Gamma(1-\alpha)}(t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}U(\rho,\tau,x)d\rho,$$

and finally we get that

$$D_*^\alpha v(t,x) - A(D_x)v(t,x) = f(t,x)$$

$$+\int_0^t [\int_0^t \frac{1}{\Gamma(1-\alpha)}(t-\rho)^{-\alpha}\frac{\partial}{\partial\rho}U(\rho,\tau,x)d\rho - A(D_x)U(t,\tau,x)]d\tau$$

$$= f(t,x) + \int_0^t [D_*^\alpha U(t,\tau,x) - A(D_x)U(t,\tau,x)]d\tau = f(t,x).$$

Further, it is obvious, that the function $v(t,x)$ in (6) satisfies to the initial condition (8). Consequently, $v(t,x)$ satisfies to the equation (7) and the condition (8).

Moreover, for a fixed $t > 0$ making use of the semi-norm of Ψ_N we have

$$p_{_N}^2(v(t,x)) = \|F^{-1}\chi_{_N} Fv\|_{L_2}^2 = \|\chi_{_N} Fv\|_{L_2}^2$$

$$= \int_{R^n} |\chi_{_N}(\xi)|^2 \cdot |\int_0^t E_\alpha((t-\tau)^\alpha A(\xi)) F D_*^{1-\alpha} f(\tau,\xi)d\tau|^2 d\xi. \qquad (10)$$

For $\chi_{_N}(\xi)$ there exists a compact set $K_{_N} \subset G$ such that $supp\, \chi_{_N}(\xi) \subset K_{_N}$. By using Cauchy-Bunjakowski's inequality we get the estimate

$$p_{_N}^2(v(t,x)) \le \int_{K_N} |\chi_{_N}(\xi)|^2 \cdot \int_0^t |E_\alpha((t-\tau)^\alpha A(\xi))|^2 d\tau \cdot \int_0^t |FD_*^{1-\alpha} f(\tau,\xi)|^2 d\tau d\xi.$$

The function $\int_0^t |E_\alpha((t-\tau)^\alpha A(\xi))|^2 d\tau$ is bounded on $K_{_N}$. Consequently, there exists a constant $C_{_N} > 0$ such that

$$p_{_N}^2(v(t,x)) \le C_{_N} \int_{K_N} |\chi_{_N}(\xi)|^2 \cdot \int_0^t |FD_*^{1-\alpha} f(\tau,\xi)|^2 d\tau d\xi$$

$$\le C_{_N} \int_0^t \int_{R^n} |\chi_{_N}(\xi)|^2 \cdot |FD_*^{1-\alpha} f(\tau,\xi)|^2 d\xi d\tau$$

$$= C_{_N} \int_0^t \|\chi_{_N}(\xi) F D_*^{1-\alpha} f(\tau,\xi)\|_{L_2}^2 d\tau = C_{_N} \int_0^t p_{_N}^2(D_*^{1-\alpha} f(\tau,x))d\tau.$$

It follows from the condition $D_*^{1-\alpha} f(t,x) \in C[t \ge 0; \Psi_{G,2}(R^n)]$ that the function $p_{_N}(D_*^{1-\alpha} f(\tau,x))$ is continuous with respect to $\tau \in (0;t)$ and for a fixed $t > 0$ the estimate

$$p_{_N}^2(v(t,x)) \le C_{_N} \cdot t \cdot \sup_{0<\tau<t} p_{_N}^2(D_*^{1-\alpha} f(\tau,x)) < +\infty$$

holds. Consequently, for every fixed $t \in (0;+\infty)$ the function $v(t,x)$ in (6) belongs to the space $\Psi_{G,2}(R^n)$. The analogous estimate is valid for $\frac{\partial}{\partial t}v(t,x)$.

Thus $v(t,x) \in C^{(1)}[t>0;\, \Psi_{G,2}(R^n)] \cap C[t \geq 0;\, \Psi_{G,2}(R^n)]$. Therefore this function is a solution of the problem (7), (8). The uniqueness of a solution follows from the uniqueness of a solution of the homogeneous Cauchy problem (1), (2) (see [4]). Finally, we deduce that the function $u(t,x)$ in (5) is a unique solution of the Cauchy problem (1), (2). ■

REMARK 3.3. An analogous theorem is valid not only for $p=2$, but for all $1<p<\infty$ as well. In the general case, in the estimation (10) it should be used the theorem on multipliers in $L_p(R^n)$.

Further, consider the equation

$$A(D_x)u_0(x) = -f(0,x), \tag{11}$$

with a pseudo-differential operator $A(D_x)$ whose symbol $A(\xi)$ is a real-analytic function in G and has no zeros in G. Formally applying the Fourier transform, we get the algebraic equation

$$A(\xi)Fu_0(\xi) = -Ff(0,\xi).$$

Obviously, if $f(0,x) \in \Psi_{G,2}(R^n)$, then there exists a unique solution of the equation (11) in the form

$$u_0(x) = -\frac{I}{A(D_x)} f(0,x),$$

where $\frac{I}{A(D_x)}$ is the pseudo-differential operator with the symbol $\frac{1}{A(\xi)}$. This solution belongs to the space $\Psi_{G,2}(R^n)$, too.

THEOREM 3.4. *Let $A(D_x)$ be a pseudo-differential operator with a symbol $A(\xi) \neq 0,\ \xi \in G$ and $\varphi(x) \in \Psi_{G,2}(R^n)$, $f(t,x) \in AC[t \geq 0;\, \Psi_{G,2}(R^n)]$, $D_*^{1-\alpha} f(t,x) \in C[t \geq 0;\, \Psi_{G,2}(R^n)]$. Then there exists a unique solution of the Cauchy problem (1), (2). This solution is represented in the form*

$$u(t,x) = E_\alpha(t^\alpha A(D_x))[\varphi(x) + \frac{I}{A(D_x)} f(0,x)]$$
$$+ \int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f(\tau,x)\,d\tau \;-\; \frac{I}{A(D_x)} f(0,x). \tag{12}$$

P r o o f. First we consider the Cauchy problem

$$D_*^\alpha v(t,x) = A(D_x)v(t,x) + f(t,x) - f(0,x), \tag{13}$$

$$v(0,x) = \varphi(x) + \frac{I}{A(D_x)} f(0,x). \tag{14}$$

It follows from Theorem 3.2 that the problem (13), (14) has a unique solution. This solution is given by the formula

$$v(t,x) = E_\alpha(t^\alpha A(D_x))[\varphi(x) + \frac{I}{A(D_x)} f(0,x)]$$

$$+ \int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha}[f(\tau,x) - f(0,x)]d\tau$$

$$= E_\alpha(t^\alpha A(D_x))[\varphi(x)+\frac{I}{A(D_x)} f(0,x)]+\int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f(\tau,x)d\tau.$$

Then it can be easily verified that the function

$$u(t,x) = v(t,x) - \frac{I}{A(D_x)} f(0,x)$$

is a unique solution of the Cauchy problem (1), (2) and for this solution the representation (12) is valid. ■

4. Well-posedness of the Cauchy problem (1), (2) in the Sobolev space $H_2^s(R^n)$

Let G be a domain whose complement $R^n \backslash G$ has n-dimensional zero measure. Then the embedding

$$\Psi_{G,2}(R^n) \subset H_2^s(R^n), \quad s \in R^1$$

is continuous and dense (see [14], [4]).

Theorem 4.1. *Suppose that the following conditions are fulfilled:*

1) $\varphi(x) \in H_2^s(R^n),\ s \in R^1$;

2) for a fixed $\ell \in R^1$ *and* $T > 0$ *the estimate*

$$|E_\alpha(t^\alpha A(\xi))| \le C(1 + |\xi|^2)^{\frac{\ell}{2}},$$

where C *is a positive constant, is valid for all* $\xi \in R^n$ *and* $t \in [0; T]$;

3) $f(t,x) \in AC[0 \le t \le T; H_2^s(R^n)], D_*^{1-\alpha} f(t,x) \in C[0 \le t \le T; H_2^s(R^n)]$;

and $f(0,x) = 0$. *Then there exists a unique solution of the Cauchy problem (1), (2) in the space*

$$C^{(1)}[0 < t < T;\ H_2^{s-\ell}(R^n)] \cap C[0 \le t \le T;\ H_2^{s-\ell}(R^n)].$$

Remark 4.2. The conditions 1) and 2) in Theorem 4.1 provide the solvability of the Cauchy problem (1), (2) in the homogeneous case.

P r o o f. For $\varphi(x) \in H_2^s(R^n)$ there exists a sequence $\varphi_N(x) \in \Psi_{G,2}(R^n)$, $N = 1, 2, ...$ approximating $\varphi(x)$ in the norm of the Sobolev space $H_2^s(R^n)$. Analogously for $f(t,x)$, such that $f(0,x) = 0$, $f(t,x) \in AC[0 \le t \le T; H_2^s(R^n)]$ and $D_*^{1-\alpha} f(t,x) \in C[0 \le t \le T; H_2^s(R^n)]$, we can choose an approximating sequence of functions $f_k(t,x)$, $k = 1, 2, 3, ...$ satisfying the conditions

$$f_k(0,x) = 0, \quad f_k(t,x) \in AC[0 \le t \le T;\, \Psi_{G,2}(R^n)],$$
$$D_*^{1-\alpha} f_k(t,x) \in C[0 \le t \le T;\, \Psi_{G,2}(R^n)].$$

According to Theorem 3.2, for a fixed N and k there exists a unique solution of the Cauchy problem (1), (2) (in which $\varphi(x)$ and $f(t,x)$ are replaced by φ_N and $f_k(t,x)$, respectively). This solution is represented in the form

$$u_{N,k}(t,x) = E_\alpha(t^\alpha A(D_x))\varphi_N(x) + \int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f_k(\tau,x)\, d\tau. \tag{15}$$

We have

$$\|u_{N,k}(t,x)\|_{H_2^{s-\ell}} \le \|E_\alpha(t^\alpha A(D_x))\varphi_N(x)\|_{H_2^{s-\ell}} + \|\int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f_k(\tau,x)\, d\tau\|_{H_2^{s-\ell}}$$
$$= \Big(\int_{R^n} |E_\alpha(t^\alpha A(\xi)) F\varphi_N(\xi)|^2 \cdot (1+|\xi|^2)^{s-\ell} d\xi\Big)^{\frac{1}{2}}$$
$$+ \Big(\int_{R^n} |\int_0^t E_\alpha((t-\tau)^\alpha A(\xi)) F D_*^{1-\alpha} f_k(\tau,\xi)\, d\tau|^2 \cdot (1+|\xi|^2)^{s-\ell} d\xi\Big)^{\frac{1}{2}}.$$

By using the condition 2) and the Cauchy-Bunjakowski inequality, we get

$$\|u_{N,k}(t,x)\|_{H_2^{s-\ell}} \le \Big(C^2 \int_{R^n} |F\varphi_N(\xi)|^2 \cdot (1+|\xi|^2)^s d\xi\Big)^{\frac{1}{2}}$$
$$+ \Big(\int_{R^n} \int_0^t |E_\alpha((t-\tau)^\alpha A(\xi))|^2 d\tau \cdot \int_0^t |F D_*^{1-\alpha} f_k(\tau,\xi)|^2 d\tau \cdot (1+|\xi|^2)^{s-\ell} d\xi\Big)^{\frac{1}{2}}$$
$$\le C\|\varphi_N(x)\|_{H_2^s} + \Big(C^2 T \int_{R^n} \int_0^t |F D_*^{1-\alpha} f_k(\tau,\xi)|^2 d\tau \cdot (1+|\xi|^2)^s d\xi\Big)^{\frac{1}{2}}$$
$$= C\|\varphi_N(x)\|_{H_2^s} + C\Big(T \int_0^t \|D_*^{1-\alpha} f_k(\tau,x)\|_{H_2^s}^2 d\tau\Big)^{\frac{1}{2}}$$

$$\leq C(\|\varphi_N(x)\|_{H_2^s} + T \cdot \sup_{0<t<T} \|D_*^{1-\alpha} f_k(t,x)\|_{H_2^s}). \tag{16}$$

This estimate yields the existence of a unique closure of the solution operator in the representation formula (15). Consequently, the Cauchy problem (1), (2) has a unique solution in the form

$$u(t,x) = E_\alpha(t^\alpha A(D_x))\varphi(x) + \int_0^t E_\alpha((t-\tau)^\alpha A(D_x)) D_*^{1-\alpha} f(\tau,x) d\tau.$$

Moreover, it follows from the estimate (16) that this solution belongs to the space

$$C^{(1)}[0<t<T;\, H_2^{s-\ell}(R^n)] \cap C[0 \leq t \leq T;\, H_2^{s-\ell}(R^n)].$$

■

Applying Theorem 3.4 we get the following result.

Theorem 4.3. *Suppose that the following conditions are fulfilled:*

1) the conditions 1) and 2) of Theorem 4.1;

2) $A(D_x)$ is a pseudo-differential operator with a symbol $A(\xi) \neq 0$, $\xi \in G$;

3) $f(t,x) \in AC[0 \leq t \leq T; H_2^s(R^n)]$, $D_*^{1-\alpha} f(t,x) \in C[0 \leq t \leq T; H_2^s(R^n)]$. Then there exists a unique solution of the Cauchy problem (1), (2) in the space

$$C^{(1)}[0<t<T;\, H_2^{s-\ell}(R^n)] \cap C[0 \leq t \leq T;\, H_2^{s-\ell}(R^n)].$$

This solution is given by the formula (12).

5. An example: fractional sub-diffusion

Let us consider the Cauchy problem for the time-fractional equation

$$D_*^\alpha u(t,x) = \Delta u(t,x) + f(t,x),\ t>0,\ x \in R^n, \tag{17}$$

$$u(0,x) = \varphi(x), \tag{18}$$

where $0 < \alpha < 1$ and Δ is the Laplace operator. The Cauchy problem (17), (18) is important in the theory of fractional brownian motion and anomalous transport of premices. In this case the solution operator $E_\alpha(t^\alpha A(D_x))$ has the symbol $E_\alpha(-|\xi|^2 t^\alpha)$. From the well-known asymptotics at infinity of the Mittag-Leffler function with a negative argument (see [19, 21]), we have

$$E_\alpha(-|\xi|^2 t^\alpha) = \underset{\sim}{O}(1+|\xi|)^{-2}, \quad |\xi| \to \infty.$$

It follows from this asymptotic formula and Theorem 4.1 that the Cauchy problem (17), (18) has a unique solution in the space

$$C^{(1)}[0 < t < T;\ H_2^{s+2}(R^n)] \cap C[0 \le t \le T;\ H_2^s(R^n)],$$

provided the conditions 1) and 3) of Theorem 4.1 are valid. This solution is given by the formula

$$u(t,x) \ = \ E_\alpha(t^\alpha \Delta)\varphi(x) \ + \int_0^t E_\alpha((t-\tau)^\alpha \Delta) D_*^{1-\alpha} f(\tau, x) d\tau.$$

Moreover, if $G = R^n \backslash \{0\}$, then the space $\Psi_{G,2}(R^n)$ is dense in , $H_2^s(R^n)$ and the symbol $A(\xi) \ = \ -|\xi|^2$ has no zeros in G. Therefore, we get the following result.

THEOREM 5.1. *Let* $\varphi(x) \in H_2^s(R^n)$ *and* $f(t,x) \in AC[0 \le t \le T; H_2^s(R^n)]$, $D_*^{1-\alpha} f(t,x) \in C[0 \le t \le T; H_2^s(R^n)]$. *Then the Cauchy problem (1), (2) (in which* $A(D_x)$ *is replaced by* $\triangle$*) has a unique solution in the space* $C^{(1)}[0 < t < T; H_2^{s+2}(R^n)] \cap C[0 \le t \le T;\ H_2^s(R^n)]$. *This solution is given by the formula*

$$u(t,x) = E_\alpha(t^\alpha \Delta)[\varphi(x) \ + \ \frac{I}{\Delta} f(0,x)]$$

$$+ \int_0^t E_\alpha((t-\tau)^\alpha \Delta) D_*^{1-\alpha} f(\tau,x) d\tau \ - \ \frac{I}{\Delta} f(0,x).$$

Here $\frac{I}{\Delta}$ *is the pseudo-differential operator with the symbol* $-\frac{1}{|\xi|^2}$, $\xi \in G$.

The author is grateful to Professor Sabir Umarov for his valuable advises and attention to the present work.

References

[1] M. Caputo, Linear models of dissipation whose Q is almost frequency independent. *Part II, Geophys. J. R. Astr. Soc.* **13** (1967), 529-539.

[2] R. Gorenflo, Yu. Luchko, F. Mainardi, Analytical properties and applications of the Wright function. *Fractional Calculus and Applied Analysis* **2** (1999), 383-415.

[3] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order. In: *Fractals and Fractional Calculus in Continuum Mechanics (Eds. A. Carpinteri and F. Mainardi).* Springer Verlag, Wien - N. York (1997), 223-276.

[4] R. Gorenflo, Yu. F. Luchko, S.R. Umarov, On the Cauchy and multi-point problems for partial pseudo-differential equations of fractional order. *Fractional Calculus and Applied Analysis* **3**, No 3 (2000), 249-277.

[5] E. Scalas, et. al., Fractional calculus and continuous-time finance. *Economics Working Paper Archive at WUSTL* (2004), 0411007 available at http://ideas.repec.org/p/wpa/wuwpfi/0411007.html

[6] F. Natterer, *The Mathematics of Computerized Tomography.* Wiley, Chichester, UK (1986).

[7] M. J. Saxton, Anomalous diffusion due to obstacles: a Monte Carlo study. *Biophys. J.*, **66**, No 2 (1994), 394-401.

[8] K. Ritchie, et al. , Detection of non-Brownian diffusion in the cell membrane in single molecule tracking. *Biophys. J.*, **88**, No 3 (2005), 2266-2277.

[9] M. Meerschaert and D. Benson, Governing equations and solutions of anomalous random walk limits. *Physical Review E*, **66** (2002), 1-4.

[10] Yu.A. Dubinskij, On a method of solving partial differential equations. *Sov. Math. Dokl.* **23** (1981), 583-587.

[11] S.R. Umarov, Boundary value problems for differential operator and pseudo-differential equations. *Izv. Acad. Sci. RU*, No 4 (1986), 38-42 (in Russian).

[12] Tran Duc Van, On the pseudo-differential operators with real analytic symbol and their applications. *J. Fac. Sci. Univ. Tokyo, IA, Math.* **36** (1989), 803-825.

[13] Yu.A. Dubinskij, The algebra of pseudo-differential operators with analytic symbols and its applications to mathematical physics. *Soviet Math. Survey* **37** (1982), 107-153.

[14] S.R. Umarov, On well-posedness of boundary value problems for pseudo-differential equations with analytic symbols. *Russ. Acad. Sci. Dokl., Math.* **45** (1992), 229-233.

[15] E.M. Saydamatov, On the well-posedness of nonhomogeneous boundary value problems for pseudo-differential equations. *Uzb. Mat. Zh.*, No 2 (1995), 78-89 (in Russian).

[16] E.M. Saydamatov, S.R. Umarov, On the well-posedness of general nonhomogeneous boundary value problems for pseudo-differential equations. *Uzb. Mat. Zh.*, No 5 (1999), 53-60 (in Russian).

[17] S.R. Umarov, Nonlocal boundary value problems for pseudo-differential and differential operator equations, I. *Differential Equations* **33**(1997), 831-840.

[18] M.M. Djrbashian, *Integral Transforms and Representations of Functions in the Complex Plane*. Nauka, Moscow (1966) (in Russian).

[19] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers, N. York - London (1993).

[20] R. Gorenflo, S. Vessella, *Abel Integral Equations. Analysis and Applications*. In: Lecture Notes in Mathematics, Vol. **1461**, Springer, New York (1991).

[21] M.M. Djrbashian, *Harmonic Analysis and Boundary Value Problems in the Complex Domain*. Basel, Birkhauser Verlag (1993).

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Received: June 28, 2005