

KRÄTZEL FUNCTION AS A FUNCTION  
OF HYPERGEOMETRIC TYPE

Anatoly A. Kilbas <sup>1</sup>, R. K. Saxena <sup>2</sup>, Juan J. Trujillo <sup>3</sup>

*Dedicated to the 70th anniversary  
of Professor Megumi Saigo*

**Abstract**

The paper is devoted to the study of the function  $Z_\rho^\nu(x)$  defined for positive  $x > 0$ , real  $\rho \in \mathbb{R}$  and complex  $\nu \in \mathbb{C}$ , being such that  $\operatorname{Re}(\nu) < 0$  for  $\rho \leq 0$ , by

$$Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} \exp\left(-t^\rho - \frac{x}{t}\right) dt.$$

Such a function was earlier investigated for  $\rho > 0$ . Using the Mellin transform of  $Z_\rho^\nu(x)$ , we establish its representations in terms of the  $H$ -function and extend this function from positive  $x > 0$  to complex  $z$ . The results obtained, being different for  $\rho > 0$  and  $\rho < 0$ , are applied to obtain the explicit forms of  $Z_\rho^\nu(z)$  in terms of the generalized Wright function. The cases, when such representations are expressed via the generalized hypergeometric functions, are given.

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### 1. Introduction

The paper deals with the special function  $Z_\rho^\nu(x)$  defined for  $x > 0$ ,  $\rho \in \mathbb{R} = (-\infty, \infty)$  and  $\nu \in \mathbb{C}$ , being such that  $\operatorname{Re}(\nu) < 0$  for  $\rho \leq 0$ , by

$$Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} \exp\left(-t^\rho - \frac{x}{t}\right) dt \quad (x > 0). \quad (1.1)$$

In particular, when  $\rho = 1$  and  $x = t^2/2$ , then in accordance with [5, 7.12(23)],

$$Z_1^\nu\left(\frac{x^2}{4}\right) = 2\left(\frac{t}{2}\right)^\nu K_\nu(t), \quad (1.2)$$

where  $K_\nu(t)$  is the modified Bessel function of the third kind, or McDonald function; see [5, Section 7.2.2]. For  $\rho \geq 1$  the function (1.1) was introduced by Krätzel [18] as a kernel of the integral transform

$$(K_\nu^\rho f)(x) = \int_0^\infty Z_\rho^\nu(xt) f(t) dt \quad (x > 0), \quad (1.3)$$

which is called by his name. Therefore we call (1.2) the Krätzel function. Krätzel in [18] established asymptotic behaviour of  $Z_\rho^\nu(x)$  for  $\rho \geq 1$  together with its composition with a special differential operator. He applied these results to prove the inversion formula for the transform (1.3), to construct its convolution and to give applications to the solution of some ordinary differential equations.

The study of the function (1.1) and the integral transform (1.3) with positive  $\rho > 0$  were continued by several authors. Rodríguez, Trujillo and Rivero [25] established compositions of the operator  $K_\nu^\rho$  with the right-hand sided Liouville fractional integrals and derivatives [26, Section 5.1], constructed the operational calculus for the operator in (1.3) and gave applications to derive the solution of two Bessel-type fractional ordinary and partial differential equations. This paper was probably the first where the explicit solution of special ordinary and partial differential equations of fractional order were obtained. Such investigations now are of a great interest in connection with applications; for example, see in this connection [13], [22], [23] and [28].

Mellin transform of  $Z_\rho^\nu(x)$  and  $(K_\nu^\rho f)(x)$  were given by Kilbas and Shlapakov [11] and [12]. They also derived the compositions of the transform (1.3) with the left- and right-hand sided Liouville fractional integrals and derivatives and applied the results obtained to the solution of the second

order ordinary differential equations. Compositions of  $(K_{\nu}^{\rho}f)(x)$  with fractional calculus operators were obtained in [11] and [12] in certain weighted spaces of locally integrable and finite differentiable functions. These results were extended by Kilbas, Bonilla, Rivero, Rodríguez and Trujillo [7] to the spaces of tested and generalized functions  $\mathcal{F}_{p,\mu}$  and  $\mathcal{F}'_{p,\mu}$  by McBride [20]. Glaeske and Kilbas in [6] and [8] proved the mapping properties such as the boundedness, the representation and the range of the transform (1.3) in weighted  $L_r$ -spaces.

All the above investigations were devoted to the function (1.1) with real positive  $x > 0$  and  $\rho > 0$ . We consider  $Z_{\rho}^{\nu}(x)$  for any real  $\rho$  and give its extension from  $x > 0$  to complex  $z \in \mathbb{C}$ . For this we establish its representations in terms of the so-called  $H$ -function. For integers  $m, n, p, q$  such that  $0 \leq m \leq q, 0 \leq n \leq p$ , for  $a_i, b_j \in \mathbb{C}$  and for  $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$  ( $i = 1, 2, \dots, p; j = 1, 2, \dots, q$ ), the  $H$ -function  $H_{p,q}^{m,n}(z)$  is defined via a Mellin-Barnes integral in the form

$$\begin{aligned}
 H_{p,q}^{m,n}(z) &\equiv H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] \\
 &= H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds, \quad (1.4)
 \end{aligned}$$

with

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)}. \quad (1.5)$$

Here  $z^{-s} = \exp[-s(\log |z| + i \arg z)], \quad z \neq 0, \quad i = \sqrt{-1}, \quad (1.6)$

$\log |z|$  represents the natural logarithm and  $\arg z$  is not necessarily the principal value. An empty product in (1.5), if it occurs, is taken to be one.  $\mathcal{L}$  in (1.4) is the infinite contour that separates all the poles of the Gamma functions  $\Gamma(b_j + \beta_j s)$  ( $j = 1, 2, \dots, m$ ) to the left and all the poles of the Gamma-functions  $\Gamma(1 - a_i - \alpha_i s)$  ( $i = 1, 2, \dots, n$ ) to the right of  $\mathcal{L}$ . Moreover, the contour  $\mathcal{L}$  has one of the following forms:

- (a)  $\mathcal{L} = \mathcal{L}_{-\infty}$  is a left loop in a horizontal strip starting at the point  $-\infty + i\varphi_1$  and terminating at the point  $-\infty + i\varphi_2$  with  $-\infty < \varphi_1 < \varphi_2 < \infty$ ;
- (b)  $\mathcal{L} = \mathcal{L}_{+\infty}$  is a right loop in a horizontal strip starting at the point  $+\infty + i\varphi_1$  and terminating at the point  $+\infty + i\varphi_2$  with  $-\infty < \varphi_1 < \varphi_2 < \infty$ ;
- (c)  $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$  is a contour starting at the point  $\gamma - i\infty$  and terminating at the point  $\gamma + i\infty + i\varphi_2$ , where  $\gamma \in \mathbb{R}$ .

See in this connection [9, 1.1.2] and [24, §8.3].

It should be noted that (1.4) generalizes most of known special functions of Bessel and hypergeometric type. The theory of the  $H$ -function is given in the books of Mathai and Saxena [19, Chapter 2], Srivastava, Gupta and Goyal [29, Chapter 1], Prudnikov, Brychkov and Marichev [24, §8.3] and Kilbas and Saigo [9, Chapters 1 and 2].

We also note that (1.4) and (1.5) mean that  $H_{p,q}^{m,n}(z)$  in (1.4) is the inverse Mellin transform of  $\mathcal{H}_{p,q}^{m,n}(s)$  in (1.5); see, for example, [3] and [27]. Therefore the Mellin transform

$$(\mathcal{M}f)(s) = \int_0^\infty f(t)t^{s-1}dt \quad (s \in \mathbb{C}, x > 0) \quad (1.7)$$

of (1.4) yields (1.5):

$$(\mathcal{M}H_{p,q}^{m,n})(s) = \mathcal{H}_{p,q}^{m,n}(s). \quad (1.8)$$

We prove the formula for the Mellin transform of  $Z_\rho^\nu(x)$ . Using such a result, we establish different representations for  $Z_\rho^\nu(z)$  as the  $H$ -function (1.4) in the cases  $\rho > 0$  and  $\rho < 0$ , and give conditions for these representations.

On the basis of the obtained formulas, we deduce the representations for  $Z_\rho^\nu(z)$  in terms of the generalized hypergeometric Wright function defined for  $z \in \mathbb{C}$ ,  $a_j, b_j \in \mathbb{C}$  and  $\alpha_j, \beta_j \in \mathbb{R}$  ( $\alpha_j, \beta_j \neq 0$ ;  $i = 1, 2, \dots, p$ ;  $j = 1, 2, \dots, q$ ) by the series:

$${}_p\Psi_q(z) \equiv_p \Psi_q \left[ \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_i, \beta_i)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}. \quad (1.9)$$

Such a function was introduced by Wright [30] who in [30], [31], [32] obtained the asymptotic behaviour of  ${}_p\Psi_q(z)$  for the values of the argument  $z$ . Some properties of  ${}_p\Psi_q(z)$  were considered in [10]. Then we present special cases when  $Z_\rho^\nu(x)$  can be expressed via the generalized hypergeometric function  ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$  defined by [4, 4.1(1)]

$${}_pF_q [a_1, \dots, a_p; b_1, \dots, b_q; z] = \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{k!} \quad (1.10)$$

$$(a_i, b_j \in \mathbb{C}; a_i \neq 0, -1, -2, \dots \quad (i = 1, \dots, p; j = 1, \dots, q)), \quad (1.11)$$

where  $(a)_k$  is the so-called Pochhammer symbol defined for  $a \in \mathbb{C}$  by [4, 4.1(2)]

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1) \quad (k = 1, 2, \dots). \quad (1.12)$$

The paper is organized as follows. Section 2 has preliminary character and contains some facts from the theory of the  $H$ -function (1.4), the generalized hypergeometric Wright function  ${}_p\Psi_p(z)$  and the Krätzel function (1.1). The representations for  $Z_\rho^\nu(z)$  in the forms of the  $H$ -function (1.4) are established in Section 3. Sections 4 and 5 contain the representation of  $Z_\rho^\nu(z)$  via the generalized Wright function (1.9) and via the generalized hypergeometric function (1.10), respectively.

### 2. Preliminaries

In this section we present conditions for the existence of the  $H$ -function (1.4), of the generalized hypergeometric Wright function (1.9) and of the Krätzel function (1.1). We shall use the notation, following [9, Section 1.1]:

$$a^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j; \tag{2.1}$$

$$\Delta = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i; \tag{2.2}$$

$$\mu = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}; \tag{2.3}$$

$$\delta = \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j}. \tag{2.4}$$

In accordance with [9, Theorem 1.1], the conditions for the existence of the  $H$ -function in (1.4) are given by the following result.

**THEOREM 2.1.** *The  $H$ -function defined by (1.4) makes sense in the following cases:*

$$\mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta > 0, \quad z \neq 0; \tag{2.5}$$

$$\mathcal{L} = \mathcal{L}_{-\infty}, \quad \Delta = 0 \text{ and either } 0 < |z| < \delta, \text{ or } |z| = \delta, \operatorname{Re}(\mu) < -1; \tag{2.6}$$

$$\mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta < 0, \quad z \neq 0; \tag{2.7}$$

$$\mathcal{L} = \mathcal{L}_{+\infty}, \quad \Delta = 0 \text{ and either } |z| > \delta, \text{ or } |z| = \delta, \operatorname{Re}(\mu) < -1; \tag{2.8}$$

$$\mathcal{L} = \mathcal{L}_{i\gamma\infty}, \quad a^* > 0, \quad |\arg z| < \frac{a^*\pi}{2}. \tag{2.9}$$

$$\mathcal{L} = \mathcal{L}_{i\gamma\infty}, \quad a^* = 0, \quad \Delta\gamma + \operatorname{Re}(\mu) < -1, \quad \arg z = 0, \quad z \neq 0. \quad (2.10)$$

Conditions for the existence of the generalized hypergeometric Wright function  ${}_p\Psi_p(z)$  in (1.9) are given by the following result [10, Theorem 1 and Corollary 1.1].

**THEOREM 2.2.** *Let  $\Delta$ ,  $\mu$  and  $\delta$  be defined by (2.2), (2.3) and (2.4).*

(a) *If  $\Delta > -1$ , then  ${}_p\Psi_p(z)$  is an entire function of  $z$ .*

(b) *If  $\Delta = -1$ , then  ${}_p\Psi_p(z)$  is defined for all values of  $|z| < \delta$  and of  $|z| = \delta$ ,  $\operatorname{Re}(\mu) > 1/2$ .*

**REMARK 2.1.** When  $\Delta > -1$ , the asymptotic behaviour of the generalized Wright function  ${}_p\Psi_p(z)$  at infinity was established in [30], [31], [32].

It is directly verified the following assertion giving conditions for the existence of the Krätzel function (1.1).

**LEMMA 2.1.** *The Krätzel function  $Z_\rho^\nu(x)$  given by (1.1), is defined for  $x > 0$  for the following parameters  $\nu$  and  $\rho$ :*

(a) *for any  $\nu \in \mathbb{C}$  and  $\rho > 0$ ;*

(b) *for  $\nu \in \mathbb{C}$ ,  $\operatorname{Re}(\nu) < 0$  and  $\rho \leq 0$ .*

*In particular,*

$$Z_0^\nu(x) = \frac{1}{e} \Gamma(-\nu) x^\nu \quad (x > 0; \nu \in \mathbb{C}, \operatorname{Re}(\nu) < 0). \quad (2.11)$$

### 3. $Z_\rho^\nu(z)$ as a $H$ -function

In this section we prove that the Krätzel function  $Z_\rho^\nu(x)$  defined by (1.1) for positive  $x > 0$ , can be extended to complex  $z \in \mathbb{C}$  ( $z \neq 0$ ) by using the  $H$ -function (1.4). Our investigation is based on the following assertion.

**LEMMA 3.1.** *Let  $\rho \in \mathbb{C}$  ( $\rho \neq 0$ ),  $\nu \in \mathbb{C}$  and  $s \in \mathbb{C}$  ( $\operatorname{Re}(s) > 0$ ) be such that  $\operatorname{Re}(s + \nu) > 0$  when  $\rho > 0$  and  $\operatorname{Re}(s + \nu) < 0$  when  $\rho < 0$ . Then the following relation holds:*

$$(\mathcal{M}Z_\rho^\nu)(s) = \frac{1}{|\rho|} \Gamma(s) \Gamma\left(\frac{s + \nu}{\rho}\right) \quad (\rho \neq 0, \nu \in \mathbb{C}). \quad (3.1)$$

**P r o o f.** Using (1.7) and (1.1), interchanging the order of integration, making the change of variable  $u = x/t$  and taking into account the relation for the Gamma-function [4, 1.1(1)], we have

$$(\mathcal{M}Z_\rho^\nu)(s) = \int_0^\infty x^{s-1} dx \int_0^\infty t^{\nu-1} \exp\left(-t^\rho - \frac{x}{t}\right) dt$$

$$= \int_0^\infty t^{\nu-1} \exp(-t^\rho) dt \int_0^\infty x^{s-1} e^{-x/t} dx = \Gamma(s) \int_0^\infty t^{s+\nu-1} \exp(-t^\rho) dt.$$

Since  $\rho \neq 0$ , by the change  $y = t^\rho$  we obtain

$$(\mathcal{M}Z_\rho^\nu)(s) = \frac{1}{\rho} \Gamma(s) \Gamma\left(\frac{s+\nu}{\rho}\right) \quad (\rho > 0) \tag{3.2}$$

for  $\rho > 0$ , while for  $\rho < 0$

$$(\mathcal{M}Z_\rho^\nu)(s) = -\frac{1}{\rho} \Gamma(s) \Gamma\left(\frac{s+\nu}{\rho}\right) \quad (\rho < 0). \tag{3.3}$$

Thus (3.1) is proved. The above operations are valid according to the conditions of the theorem. This completes the proof of lemma. ■

REMARK 3.1. The relation (3.2) was proved in [12, Lemma 2].

By (3.1) and (1.7), taking (1.4) and (1.5) into account, the function  $Z_\rho^\nu(z)$  with  $\rho \neq 0$  can be represented as the  $H$ -function in the following forms:

$$Z_\rho^\nu(z) = \frac{1}{\rho} H_{0,2}^{2,0} \left[ z \left| \begin{array}{c} \text{-----} \\ (0, 1), (\nu/\rho, 1/\rho) \end{array} \right. \right] \quad (z \in \mathbb{C}, z \neq 0; \rho > 0, \nu \in \mathbb{C}), \tag{3.4}$$

when  $\rho > 0$ , while for  $\rho < 0$ ,

$$Z_\rho^\nu(z) = -\frac{1}{\rho} H_{1,1}^{1,1} \left[ z \left| \begin{array}{c} (1 - \nu/\rho, -1/\rho) \\ (0, 1) \end{array} \right. \right] \quad (z \in \mathbb{C}, z \neq 0; \rho < 0, \text{Re}(\nu) < 0). \tag{3.5}$$

REMARK 3.2. The relation (3.4) was given in [9, (2.9.31)].

REMARK 3.3. The representation (3.4) of the Krätzel function  $Z_\rho^\nu(z)$ ,  $\rho > 0$  as a Fox's  $H_{0,2}^{2,0}$ -function reveals it as a special case ( $m = 2$ ) of the kernel-function  $H_{0,m}^{m,0}$ ,  $m > 1$ , of the so-called “multiple Borel-Dzrbashjan transform” introduced by Kiryakova [15], [16] and studied in whole details by Al-Mussalam, Kiryakova and Tuan [1]. It is a very general  $H$ -transform generalizing the Laplace and Meijer transforms and serving as integral transform basis for operational calculus of the Gelfond-Leontiev integration and differentiation operators with respect to the multi-index Mittag-Leffler functions from [15],[16]. These are generalized operators of integration and differentiation of fractional multi-order, typical cases of the

generalized fractional calculus of Kiryakova [14]. It is interesting to mention also the following. For  $\rho = 1$ , the Krätzel function (3.4), as mentioned before, is the modified Bessel function of third kind (McDonald function)(1.2) representable as a Meijer's  $G_{0,2}^{2,0}$ -function. This shows that the transformation (1.3) is a special case (with  $m = 2$  and peculiar choice of other parameters) of the Obrechhoff integral transform, studied by Dimovski and Kiryakova [2], [14, Ch. 3] for the purposes of the hyper-Bessel differential operators. This fact clarifies also the relation of Krätzel's paper [18] with a series of his earlier studies (1965-1967) on another generalization of the Laplace and Meijer transforms, see for example, Krätzel [17], and for the details - Kiryakova [14, Chapter 3, pp.197-198].

If  $\rho > 0$ , then in accordance with (3.4) and (1.4), (1.8) we have the representation of  $Z_\rho^\nu(z)$  via the Mellin-Barnes integral in the form

$$Z_\rho^\nu(z) = \frac{1}{2\rho\pi i} \int_{\mathcal{L}} \Gamma(s)\Gamma\left(\frac{\nu+s}{\rho}\right) z^{-s} ds \quad (z \in \mathbb{C}, z \neq 0; \rho > 0, \nu \in \mathbb{C}), \quad (3.6)$$

where  $\mathcal{L}$  is the infinite contour which separates the poles

$$s_k = -k \quad (k = 0, 1, 2, \dots) \quad (3.7)$$

of the Gamma functions  $\Gamma(s)$  and the poles

$$s_m = -\nu - \rho m \quad (m = 0, 1, 2, \dots) \quad (3.8)$$

of the Gamma function  $\Gamma([\nu + s]/\rho)$  to the left.

It follows from (3.4) that for such a function  $Z_\rho^\nu(z)$  with  $\rho > 0$ , the constants defined by (2.1)-(2.4) take the forms:

$$a^* = \Delta = 1 + \frac{1}{\rho} > 0, \quad \mu = \frac{\nu}{\rho} - 1, \quad \delta = \rho^{-1/\rho} \quad (\rho > 0). \quad (3.9)$$

Thus, Theorem 2.1 gives the conditions for the existence of  $Z_\rho^\nu(z)$  in (3.4).

**THEOREM 3.1.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  and  $\rho > 0$ . Then the function  $Z_\rho^\nu(z)$ , given by (3.4) and (3.6), exists in the following cases:*

$$\mathcal{L} = \mathcal{L}_{-\infty}, \quad z \neq 0; \quad (3.10)$$

$$\mathcal{L} = \mathcal{L}_{i\gamma\infty}, \quad |\arg z| < (\rho + 1)\pi/(2\rho). \quad (3.11)$$

**COROLLARY 3.1.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ) and  $\nu \in \mathbb{C}$ . Then the function  $Z_1^\nu(z)$ , given by (3.4) and (3.6) with  $\rho = 1$ , exists in the cases (3.10) and*

$$\mathcal{L} = \mathcal{L}_{i\gamma\infty}, \quad |\arg z| < \pi. \quad (3.12)$$



If  $\rho < 0$ , then by (1.4) and (3.5),

$$Z_\rho^\nu(z) = -\frac{1}{2\rho\pi i} \int_{\mathcal{L}} \Gamma(s)\Gamma\left(\frac{\nu+s}{\rho}\right) z^{-s} ds \quad (\operatorname{Re}(\nu) < 0), \quad (3.13)$$

where  $\mathcal{L}$  is the infinite contour which separates the poles (3.7) of the Gamma function  $\Gamma(s)$  to the left and the poles (3.8) of the Gamma function  $\Gamma([\nu + s]/\rho)$  to the right. It follows from (3.5) that for the function  $Z_\rho^\nu(z)$  with negative  $\rho < 0$  the constants (2.1)-(2.4) take the forms:

$$a^* = 1 - \frac{1}{\rho} > 0, \quad \Delta = 1 + \frac{1}{\rho}, \quad \mu = \frac{\nu}{\rho} - 1, \quad \delta = (-\rho)^{-1/\rho} \quad (\rho < 0). \quad (3.14)$$

Since  $\rho < 0$ , then  $\Delta > 0$  for  $\rho < -1$ ,  $\Delta < 0$  for  $-1 < \rho < 0$  and  $\Delta = 0$ ,  $a^* = 2 > 0$  for  $\rho = -1$ . Then from Theorem 2.1 we deduce the existence result.

**THEOREM 3.2.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\operatorname{Re}(\nu) < 0$ ) and  $\rho < 0$ . The function  $Z_\rho^\nu(z)$ , given by (3.5) and (3.13), exists in the following cases:*

$$\mathcal{L} = \mathcal{L}_{-\infty}, \quad \rho < -1, \quad z \neq 0; \quad (3.15)$$

$$\mathcal{L} = \mathcal{L}_{-\infty}, \quad \rho = -1, \quad 0 < |z| \leq 1; \quad (3.16)$$

$$\mathcal{L} = \mathcal{L}_{+\infty}, \quad -1 < \rho < 0, \quad z \neq 0; \quad (3.17)$$

$$\mathcal{L} = \mathcal{L}_{+\infty}, \quad \rho = -1, \quad |z| \geq 1, \quad (3.18)$$

$$\mathcal{L} = \mathcal{L}_{i\gamma\infty}, \quad \rho < 0, \quad |\arg z| < \frac{(\rho - 1)\pi}{2\rho}. \quad (3.19)$$

**COROLLARY 3.2.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ) and  $\nu \in \mathbb{C}$  ( $\operatorname{Re}(\nu) < 0$ ). The function  $Z_\rho^\nu(z)$ , given by (3.5) and (3.13) with  $\rho = -1$ , exists in cases (3.16), (3.18) and (3.12).*

#### 4. $Z_\rho^\nu(z)$ as the generalized hypergeometric Wright function

In this section we use the relations (3.6) and (3.13) in order to give representations of the function  $Z_\rho^\nu(z)$  in terms of the generalized hypergeometric Wright function (1.9). The results will be different when  $\rho > 0$ ,  $-1 \leq \rho < 0$  and  $\rho \leq -1$ . We begin from the first case.

There holds the following statement.

**THEOREM 4.1.** Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ ,  $\rho > 0$ , and let the function  $Z_\rho^\nu(z)$  is given by (3.4) and (3.6) with conditions (3.10) or (3.11).

(a) If  $k \neq \nu + \rho m$  for any  $k, m \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$ , then  $Z_\rho^\nu(z)$  is represented by

$$Z_\rho^\nu(z) = \frac{1}{\rho} {}_1\Psi_0 \left[ \begin{matrix} (\nu/\rho, -1/\rho) \\ - \end{matrix} \middle| -z \right] + z^\nu {}_1\Psi_0 \left[ \begin{matrix} (-\nu, -\rho) \\ - \end{matrix} \middle| -z^\rho \right]. \quad (4.1)$$

(b) If there exists  $k, m \in \mathbb{N}_0$  such that  $k = \nu + \rho m$ , then  $Z_\rho^\nu(z)$  is represented by

$$Z_\rho^\nu(z) = -\frac{1}{\rho} \log(z) {}_0\Psi_1 \left[ \begin{matrix} - \\ (1, 1) \end{matrix} \middle| z \right]. \quad (4.2)$$

**P r o o f.** Let  $\nu \in \mathbb{C}$  and  $\rho > 0$  be such that  $k \neq \nu + \rho m$  for any  $k, m \in \mathbb{N}_0$ . By Theorem 3.1, we use the formula (3.6) and choose the contour  $\mathcal{L} = \mathcal{L}_{-\infty}$  or  $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ . Since the simple poles  $s_k = -k$  ( $k \in \mathbb{N}_0$ ) of  $\Gamma(s)$  and  $s_m = -\nu - \rho m$  ( $m \in \mathbb{N}_0$ ) of  $\Gamma([\nu + s]/\rho)$  do not coincide and lie to the left of  $\mathcal{L}$ , we can apply the usual technique based on the residue theorem, see [4]. Using such a method, we evaluate the integral in (3.6) as the sum of residues of the integrand at the above poles:

$$Z_\rho^\nu(z) = \frac{1}{\rho} \sum_{k=0}^{\infty} \text{res}_{s=-k} \left[ \Gamma(s) \Gamma \left( \frac{\nu + s}{\rho} \right) z^{-s} \right] + \frac{1}{\rho} \sum_{m=0}^{\infty} \text{res}_{s=-\nu-\rho m} \left[ \Gamma(s) \Gamma \left( \frac{\nu + s}{\rho} \right) z^{-s} \right]. \quad (4.3)$$

By the asymptotic estimate of the Gamma function  $\Gamma(z)$  near its simple poles  $z = -k$  ( $k = 0, 1, 2, \dots$ ) [4, 1.17(11)]:

$$\Gamma(z) = \frac{(-1)^k}{k!(z+k)} \left[ 1 + O \left( \frac{1}{z} \right) \right] \quad (z \rightarrow -k; k = 0, 1, 2, \dots), \quad (4.4)$$

we have

$$\frac{1}{\rho} \sum_{k=0}^{\infty} \text{res}_{s=-k} \left[ \Gamma(s) \Gamma \left( \frac{\nu + s}{\rho} \right) z^{-s} \right] = \frac{1}{\rho} \sum_{k=0}^{\infty} \Gamma \left( \frac{\nu - k}{\rho} \right) \frac{(-z)^k}{k!} \quad (4.5)$$

and

$$\frac{1}{\rho} \sum_{m=0}^{\infty} \text{res}_{s=-\nu-\rho m} \left[ \Gamma(s) \Gamma \left( \frac{\nu + s}{\rho} \right) z^{-s} \right] = z^\nu \sum_{m=0}^{\infty} \Gamma(-\nu - \rho m) \frac{(-z^\rho)^m}{m!}. \quad (4.6)$$

Substituting (4.5) and (4.6) into (4.4) and taking (1.11) into account we obtain (4.1). By Theorem 2.2, the functions  ${}_1\Psi_0(-z)$  and  ${}_1\Psi_0(-z^\rho)$  in (4.1) are defined under conditions of the theorem.

Let now there exists  $k, m \in \mathbb{N}_0$  such that  $k = \nu + \rho m$ . Then (4.3) takes the form

$$Z_\rho^\nu(z) = \frac{1}{\rho} \sum_{k=0}^{\infty} \operatorname{res}_{s=-k} \left[ \Gamma(s) \Gamma\left(\frac{\nu+s}{\rho}\right) z^{-s} \right],$$

where the double poles  $s_k = -k$  ( $k = 0, 1, 2, \dots$ ) lie to the left of  $\mathcal{L}$ . By the residue theory,

$$\begin{aligned} Z_\rho^\nu(z) &= \frac{1}{\rho} \sum_{k=0}^{\infty} \lim_{s \rightarrow -k} \left[ (s+k)^2 \Gamma(s) \Gamma\left(\frac{\nu+s}{\rho}\right) z^{-s} \right]' \\ &= \frac{1}{\rho} \sum_{k=0}^{\infty} \lim_{s \rightarrow -k} \left\{ \left[ -(s+k)^2 \Gamma(s) \Gamma\left(\frac{\nu+s}{\rho}\right) z^{-s} \right] \log(z) \right. \\ &\quad \left. + \left[ (s+k)^2 \Gamma(s) \Gamma\left(\frac{\nu+s}{\rho}\right) \right]' z^{-s} \right\}. \end{aligned} \tag{4.7}$$

It is known [21, (3.39)], that the Gamma function  $\Gamma(s)$  has the following representation near the simple pole  $s = -k$  ( $k \in \mathbb{N}_0$ ):

$$\Gamma(s) = \frac{(-1)^k}{k!} [1 + \psi(1+k)(s+k) + O((s+k)^2)] \quad (s \rightarrow -k; k = 0, 1, 2, \dots), \tag{4.8}$$

where  $\psi(z) = \Gamma'(z)/\Gamma(z)$  is the logarithmic derivative of the Gamma function [4, 1.7(1)]. Applying (4.8), we calculate the limit in (4.7) and obtain

$$Z_\rho^\nu(z) = -\frac{1}{\rho} \log(z) \sum_{k=0}^{\infty} \frac{1}{(k!)^2} z^k = -\frac{1}{\rho} \log(z) \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \frac{z^k}{k!},$$

which yields (4.2), in accordance with (1.9). Thus theorem is proved. ■

**COROLLARY 4.1.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ , and let the function  $Z_1^\nu(z)$  is given by (3.4) and (3.6) with  $\rho = 1$  and conditions (3.10) or (3.12).*

(a) *If  $\nu \notin \mathbb{Z} = \{0, \pm 1 \pm 2, \dots\}$ , then  $Z_1^\nu(z)$  is represented by*

$$\begin{aligned} Z_1^\nu(z) &= {}_1\Psi_0 \left[ \begin{matrix} (\nu, -1) \\ - \end{matrix} \middle| -z \right] + z^\nu {}_1\Psi_0 \left[ \begin{matrix} (-\nu, -1) \\ - \end{matrix} \middle| -z \right] \\ &= \sum_{k=0}^{\infty} \Gamma(\nu - k) \frac{(-z)^k}{k!} + z^\nu \sum_{m=0}^{\infty} \Gamma(-\nu - m) \frac{(-z)^m}{m!}. \end{aligned} \tag{4.9}$$

(b) *If  $\nu \in \mathbb{Z}$ , then  $Z_1^\nu(z)$  is represented by*

$$Z_1^\nu(z) = -\log(z) {}_0\Psi_1 \left[ \begin{matrix} - \\ (1, 1) \end{matrix} \middle| z \right]. \tag{4.10}$$

COROLLARY 4.2. *Let the conditions of Theorem 4.1 be satisfied.*

(a) *If  $k \neq \nu + \rho m$  for any  $k, m \in \mathbb{N}_0$ , then the function  $Z_\rho^\nu(z)$  has the following asymptotic behaviour at zero*

$$Z_\rho^\nu(z) = \frac{1}{\rho} \Gamma\left(\frac{\nu}{\rho}\right) [1 + O(z)] + \Gamma(-\nu) z^\nu [1 + O(z^\rho)] \quad (z \rightarrow 0; \rho > 0). \quad (4.11)$$

(b) *If there exists  $k, m \in \mathbb{N}_0$  such that  $k = \nu + \rho m$ , then  $Z_\rho^\nu(z)$  has the asymptotic estimate*

$$Z_\rho^\nu(z) = -\frac{1}{\rho} \log(z) [1 + O(z)] \quad (z \rightarrow 0; \rho > 0). \quad (4.12)$$

REMARK 4.1. When  $\nu \neq 0$  and  $\rho > 0$  be such that  $k \neq \nu + \rho m$  for any  $k, m \in \mathbb{N}_0$ , then it follows from (4.11) the asymptotic estimate

$$Z_\rho^\nu(z) \sim \begin{cases} \frac{1}{\rho} \Gamma\left(\frac{\nu}{\rho}\right), & \text{if } \operatorname{Re}(\nu) > 0, \\ \frac{1}{\rho} \Gamma\left(\frac{\nu}{\rho}\right) + \Gamma(-\nu) z^\nu, & \text{if } \operatorname{Re}(\nu) = 0, \text{ but } \nu \neq 0, \\ \Gamma(-\nu) z^\nu, & \text{if } \operatorname{Re}(\nu) < 0, \end{cases} \quad (4.13)$$

as  $z \rightarrow 0$ . Such a relation for  $\rho \geq 1$  and positive  $x \rightarrow 0+$  was proved in [18].

Next we consider the case  $-1 \leq \rho < 0$ .

THEOREM 4.2. *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\operatorname{Re}(\nu) < 0$ ),  $-1 \leq \rho < 0$ , and let the function  $Z_\rho^\nu(z)$  is given by (3.5) and (3.13) with conditions (3.17), or (3.18) or (3.19). Then the function  $Z_\rho^\nu(z)$  is represented by*

$$Z_\rho^\nu(z) = z^\nu {}_1\Psi_0 \left[ \begin{matrix} (-\nu, -\rho) \\ - \end{matrix} \middle| -\frac{1}{z^{-\rho}} \right]. \quad (4.14)$$

Proof. By Theorem 3.2, we use formula (3.13) and choose the contour  $\mathcal{L} = \mathcal{L}_{+\infty}$  or  $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ . Since the simple poles  $s_k = -k$  ( $k \in \mathbb{N}_0$ ) of  $\Gamma(s)$  lie to the left of  $\mathcal{L}$ , while the simple poles  $s_m = -\nu - \rho m$  ( $m \in \mathbb{N}_0$ ) of  $\Gamma([\nu + s]/\rho)$  lie to the right of  $\mathcal{L}$ , we use the same arguments as in the proof of the Theorem 4.1 and evaluate the integral in (3.13) as the sum of residues of the integrand at  $s = s_m$ . Using (4.4), we have

$$\begin{aligned} Z_\rho^\nu(z) &= \frac{1}{\rho} \sum_{m=0}^{\infty} \operatorname{res}_{s=-\nu-\rho m} \left[ \Gamma(s) \Gamma\left(\frac{\nu+s}{\rho}\right) z^{-s} \right] \\ &= z^\nu \sum_{m=0}^{\infty} \Gamma(-\nu - \rho m) \left(-\frac{1}{z^{-\rho}}\right)^m \frac{1}{m!}, \end{aligned}$$

which proves (4.14) if we take (1.9) into account. By Theorem 2.2, the function  ${}_1\Psi_0(-z^\rho)$  in (4.14) is defined under conditions of the theorem. Thus theorem is proved.  $\blacksquare$

COROLLARY 4.3. Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\text{Re}(\nu) < 0$ ), and let the function  $Z_{-1}^\nu(z)$  be given by (3.5) and (3.13) with  $\rho = -1$  and conditions (3.18) or (3.12). Then  $Z_{-1}^\nu(z)$  is represented by

$$Z_{-1}^\nu(z) = z^\nu {}_1\Psi_0 \left[ \begin{matrix} (-\nu, 1 \\ - \end{matrix} \middle| -\frac{1}{z} \right] = z^\nu \sum_{k=0}^{\infty} \Gamma(-\nu + k) \left(-\frac{1}{z}\right)^k \frac{1}{k!}. \quad (4.15)$$

COROLLARY 4.4. Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\text{Re}(\nu) < 0$ ),  $-1 \leq \rho < 0$  be such that conditions of Theorem 4.2 hold. Then the function  $Z_\rho^\nu(z)$  has the following asymptotic behaviour at infinity

$$Z_\rho^\nu(z) = \Gamma(-\nu)z^\nu \left[ 1 + O\left(\frac{1}{z^{-\rho}}\right) \right] \quad (z \rightarrow \infty). \quad (4.16)$$

In particular,

$$Z_{-1}^\nu(z) = \Gamma(-\nu)z^\nu \left[ 1 + O\left(\frac{1}{z}\right) \right] \quad (z \rightarrow \infty). \quad (4.17)$$

Finally we consider the case  $\rho \leq -1$ .

THEOREM 4.3. Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\text{Re}(\nu) < 0$ ),  $\rho \leq -1$ , and let the function  $Z_\rho^\nu(z)$  be given by (3.5) and (3.13) with conditions (3.15), (3.16) or (3.19). Then the function  $Z_\rho^\nu(z)$  is represented by

$$Z_\rho^\nu(z) = -\frac{1}{\rho} {}_1\Psi_0 \left[ \begin{matrix} (\nu/\rho, -1/\rho \\ - \end{matrix} \middle| -z \right]. \quad (4.18)$$

P r o o f. By Theorem 3.2, we use formula (3.5) and choose the contour  $\mathcal{L} = \mathcal{L}_{-\infty}$  or  $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ . Since the simple poles  $s_k = -k$  ( $k \in \mathbb{N}_0$ ) of  $\Gamma(s)$  lie to the left of  $\mathcal{L}$ , while the simple poles  $s_m = -\nu - \rho m$  ( $m \in \mathbb{N}_0$ ) of  $\Gamma([\nu + s]/\rho)$  lie to the right of  $\mathcal{L}$ , we use the same arguments as in the proof of the Theorems 4.1 and 4.2, and evaluate the integral in (3.5) as the sum of residues of the integrand at  $s = s_k$ :

$$\begin{aligned} Z_\rho^\nu(z) &= -\frac{1}{\rho} \sum_{k=0}^{\infty} \text{res}_{s=-k} \left[ \Gamma(s) \Gamma\left(\frac{\nu + s}{\rho}\right) z^{-s} \right] \\ &= -\frac{1}{\rho} \sum_{k=0}^{\infty} \Gamma\left(\frac{\nu - k}{\rho}\right) \frac{(-z)^k}{k!}, \end{aligned} \quad (4.19)$$

which yields (4.18) in accordance with (1.9). By Theorem 2.2, the function  ${}_1\Psi_0(z)$  in (4.18) is defined under the conditions of the theorem. Thus the theorem is proved. ■

COROLLARY 4.5. Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\operatorname{Re}(\nu) < 0$ ), and let the function  $Z_{-1}^\nu(z)$  be given by (3.5) and (3.13) with conditions (3.16) or (3.12). Then the function  $Z_{-1}^\nu(z)$  is represented by

$$Z_{-1}^\nu(z) = {}_1\Psi_0 \left[ \begin{matrix} (-\nu, 1) \\ - \end{matrix} \middle| -z \right] = \sum_{k=0}^{\infty} \Gamma(-\nu + k) \frac{(-z)^k}{k!}. \quad (4.20)$$

COROLLARY 4.6. Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$  ( $\operatorname{Re}(\nu) < 0$ ),  $\rho \leq -1$  be such that conditions of Theorem 4.3 hold. Then the function  $Z_\rho^\nu(z)$  has the following asymptotic behaviour at zero

$$Z_\rho^\nu(z) = -\frac{1}{\rho} \Gamma\left(\frac{\nu}{\rho}\right) [1 + O(z)] \quad (z \rightarrow 0). \quad (4.21)$$

In particular,

$$Z_{-1}^\nu(z) = \Gamma(\nu) [1 + O(z)] \quad (z \rightarrow 0). \quad (4.22)$$

The series in (4.18) is also defined for  $z = 0$ . Using (3.13), from Theorem 2.2 we deduce the following result.

THEOREM 4.4. Let  $z \in \mathbb{C}$ ,  $\nu \in \mathbb{C}$  ( $\operatorname{Re}(\nu) < 0$ ) and  $\rho \leq -1$ .

(a) If  $\rho < -1$ , then the function  $Z_\rho^\nu(z)$ , given by (4.18), is an entire function of  $z$ .

(b) If  $\rho = -1$ , then the function  $Z_{-1}^\nu(z)$ , given by (4.20), is defined for all of  $|z| < 1$  and of  $|z| = 1$ ,  $\operatorname{Re}(\nu) < -3/2$ .

REMARK 4.2. Relations (4.15) and (4.20) yield different representations for  $Z_{-1}^\nu(z)$ . This is due (4.15) gives the representation for  $Z_{-1}^\nu(z)$  at infinity, while (4.20) at zero.

According to (4.1), (4.14) and (4.18), the function  $Z_\rho^\nu(z)$  is expressed in terms of the Wright function  ${}_1\Psi_0(z)$  for real  $\rho \neq 0$ . Taking the limit in (4.14), as  $\rho \rightarrow 0-$ , we obtain

$$\lim_{\rho \rightarrow 0-} Z_\rho^\nu(z) = z^\nu {}_1\Psi_0 \left[ \begin{matrix} (-\nu, 0) \\ - \end{matrix} \middle| -1 \right] = z^\nu \sum_{k=0}^{\infty} \Gamma(-\nu) \frac{(-1)^k}{k!} = \frac{\Gamma(-\nu)}{e} z^\nu. \quad (4.23)$$

Using this fact we can define  $Z_\rho^\nu(z)$  for  $\rho = 0$  by

$$Z_0^\nu(z) = \frac{\Gamma(-\nu)}{e} z^\nu \quad (z \in \mathbb{C}, z \neq 0; \nu \in \mathbb{C}, \operatorname{Re}(\nu) < 0). \quad (4.24)$$

In particular, when  $z = x > 0$ , we obtain (2.11).

**5.  $Z_\rho^\nu(z)$  as a generalized hypergeometric function**

In this section we apply the results in Section 4 to present the cases when  $Z_\rho^\nu(z)$  is expressed via the generalized hypergeometric function (1.10). First we consider the case  $\rho > 0$ . From Theorem 4.1(b) we deduce the first result.

**THEOREM 5.1.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ ,  $\rho > 0$  and let the function  $Z_\rho^\nu(z)$  is given by (3.4) and (3.6) with conditions (3.10) or (3.11).*

*If there exists  $k, m \in \mathbb{N}_0$  such that  $k = \nu + \rho m$ , then  $Z_\rho^\nu(z)$  is given by*

$$Z_\rho^\nu(z) = -\frac{1}{\rho} \log(z) {}_0F_1[-; 1; z]. \tag{5.1}$$

**P r o o f.** By Theorem 4.1(b), there holds the representation (4.2). Since

$$\Gamma(k + 1) = k! = (1)_k \quad (k \in \mathbb{N}_0), \tag{5.2}$$

then, in accordance with (1.9) and (1.10),

$${}_0\Psi_1 \left[ \begin{matrix} - & - & - & - & - \\ (1, 1) \end{matrix} \middle| z \right] = {}_0F_1[-; 1; z]. \tag{5.3}$$

Thus (4.2) yields (5.1), and the theorem is proved. ■

**COROLLARY 5.1.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ , and let the function  $Z_1^\nu(z)$  be given by (3.4) and (3.6) with  $\rho = 1$  and conditions (3.10) or (3.12).*

*If  $\nu \in \mathbb{Z}$ , then*

$$Z_1^\nu(z) = -\log(z) {}_0F_1[-; 1; z]. \tag{5.4}$$

Now we consider the case  $\rho = 1$  when  $\nu \notin \mathbb{Z}$ . There holds the following result.

**THEOREM 5.2.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ , and let the function  $Z_1^\nu(z)$  is given by (3.4) and (3.6) with  $\rho = 1$  and conditions (3.10) or (3.12).*

*If  $\nu \notin \mathbb{Z}$ , then the function  $Z_1^\nu(z)$  is represented by*

$$Z_1^\nu(z) = \Gamma(\nu) {}_0F_1[-; 1 - \nu; z] + \Gamma(-\nu) z^\nu {}_0F_1[-; 1 + \nu; z]. \tag{5.5}$$

**P r o o f.** By Corollary 4.1, formula (4.9) holds. Applying the functional equation for the Gamma function [4, 1.2(1)]

$$\Gamma(z + 1) = z\Gamma(z) \quad (z \in \mathbb{C}), \tag{5.6}$$

we have

$$\Gamma(\nu - k) = \frac{\Gamma(\nu - k)(\nu - k) \cdots (\nu - 1)}{(\nu - k) \cdots (\nu - 1)} = \frac{(-1)^k \Gamma(\nu)}{(1 - \nu)_k} \quad (\nu \neq 0, -1, -2, \dots),$$
(5.7)

$(1 - \nu)_k$  being the Pochhammer symbol (1.12), and similarly,

$$\Gamma(-\nu - m) = \frac{(-1)^m \Gamma(-\nu)}{(1 + \nu)_m} \quad (\nu \neq 0, 1, 2, \dots).$$
(5.8)

Substituting these relations into (4.9) and taking (1.10) into account, we obtain (5.5). ■

Next we consider the case  $\rho = 2$ .

**THEOREM 5.3.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ) and  $\nu \in \mathbb{C}$ , and let the function  $Z_2^\nu(z)$  is given by (3.4) and (3.6) with  $\rho = 2$  and conditions (3.10) or (3.11).*

*If  $k \neq \nu + 2m$  for any  $k, m \in \mathbb{N}_0$ , then the function  $Z_2^\nu(z)$  is represented by*

$$\begin{aligned} Z_2^\nu(z) = & \frac{1}{2} \Gamma\left(\frac{\nu}{2}\right) {}_0F_2\left[-; 1 - \frac{\nu}{2}, \frac{1}{2}; -\frac{z^2}{4}\right] \\ & - \frac{z}{2} \Gamma\left(\frac{\nu - 1}{2}\right) {}_0F_2\left[-; \frac{3 - \nu}{2}, \frac{3}{2}; -\frac{z^2}{4}\right] \\ & + \frac{2^{-\nu-1}}{\sqrt{\pi}} \Gamma\left(-\frac{\nu}{2}\right) \Gamma\left(\frac{1 - \nu}{2}\right) z^\nu {}_0F_2\left[-; 1 + \frac{\nu}{2}, \frac{1 + \nu}{2}; -\frac{z^2}{4}\right]. \end{aligned}$$
(5.9)

**P r o o f.** By Theorem 4.1(a) and (1.9) we have

$$Z_2^\nu(z) = \frac{1}{2} \sum_{k=0}^{\infty} \Gamma\left(\frac{\nu - k}{2}\right) \frac{(-z)^k}{k!} + z^\nu \sum_{k=0}^{\infty} \Gamma(-\nu - 2k) \frac{(-z^2)^k}{k!} = S_1 + S_2.$$
(5.10)

The series in  $S_1$  we split in two series for even  $k = 2n$  and odd  $k = 2n + 1$ :

$$S_1 = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \Gamma\left(\frac{\nu}{2} - n\right) \frac{(-z)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \Gamma\left(\frac{\nu - 1}{2} - n\right) \frac{(-z)^{2n+1}}{(2n+1)!} \right]$$

Using (5.7), the direct verified formulas

$$(2n)! = 2^{2n} (1/2)_n n!, \quad (2n+1)! = 2^{2n} (3/2)_n n!,$$
(5.11)



and taking (1.10) into account we have

$$\begin{aligned}
 S_1 &= \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{\Gamma(\nu/2)}{(1-\nu/2)_n(1/2)_n} \frac{(-z^2)^n}{2^{2n}n!} - z \sum_{n=0}^{\infty} \frac{\Gamma([\nu-1]/2)}{[3-\nu]/2)_n(3/2)_n} \frac{(-z^2)^n}{2^{2n}n!} \right] \\
 &= \frac{1}{2} \Gamma\left(\frac{\nu}{2}\right) {}_0F_2 \left[ -; 1 - \frac{\nu}{2}, \frac{1}{2}; -\frac{z^2}{4} \right] - \frac{z}{2} \Gamma\left(\frac{\nu-1}{2}\right) {}_0F_2 \left[ -; \frac{3-\nu}{2}, \frac{3}{2}; -\frac{z^2}{4} \right]. \tag{5.12}
 \end{aligned}$$

Applying the Legendre duplication formula for the Gamma function [4, 1.2(15)]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \tag{5.13}$$

and using (5.8) for the series in the second term of (5.10) we have

$$\begin{aligned}
 S_2 &= z^\nu \sum_{k=0}^{\infty} \frac{2^{-\nu-2k-1}}{\sqrt{\pi}} \Gamma\left(-\frac{\nu}{2} - k\right) \Gamma\left(\frac{1-\nu}{2} - k\right) \frac{(-z^2)^k}{k!} \\
 &= z^\nu \frac{2^{-\nu-1}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma(-\nu/2)}{(1+\nu/2)_k} \frac{\Gamma([1-\nu]/2)}{[1+\nu]/2)_k} \frac{(-z^2)^k}{2^{2k}k!} \\
 &= \frac{2^{-\nu-1}}{\sqrt{\pi}} \Gamma\left(-\frac{\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2}\right) z^\nu {}_0F_2 \left[ -; 1 + \frac{\nu}{2}, \frac{1+\nu}{2}; -\frac{z^2}{4} \right]. \tag{5.14}
 \end{aligned}$$

Substituting (5.12) and (5.14) into (5.10), we obtain (5.9). ■

The following assertion is proved similarly to Theorem 5.3 on the basis of Theorem 4.1 with  $\rho = 1/2$  with using formulas (5.8), (5.11) and (5.13).

**THEOREM 5.4.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ) and  $\nu \in \mathbb{C}$ , and let the function  $Z_{1/2}^\nu(z)$  is given by (3.4) and (3.6) with  $\rho = 1/2$  and conditions (3.10) or (3.11).*

*If  $k \neq \nu+m/2$  for any  $k, m \in \mathbb{N}_0$ , then the function  $Z_{1/2}^\nu(z)$  is represented by*

$$\begin{aligned}
 Z_{1/2}^\nu(z) &= \frac{2^{2\nu}}{\sqrt{\pi}} \Gamma(\nu) \Gamma\left(\nu + \frac{1}{2}\right) {}_0F_2 \left[ -; 1 - \nu, \frac{1}{2} - \nu; -\frac{z}{4} \right] \\
 &\quad + \Gamma(-\nu) z^\nu {}_0F_2 \left[ -; 1 + \nu, \frac{1}{2}; -\frac{z}{4} \right] \\
 &\quad - \Gamma\left(-\nu - \frac{1}{2}\right) z^{\nu+1/2} {}_0F_2 \left[ -; \frac{3}{2} + \nu, \frac{3}{2}; -\frac{z}{4} \right]. \tag{5.15}
 \end{aligned}$$

Further consider several cases of negative  $\rho < 0$ , and begin from  $\rho = -1$ .

**THEOREM 5.5.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ,  $|z| < 1$ ),  $\nu \in \mathbb{C}$ ,  $\operatorname{Re}(\nu) < 0$  and let the function  $Z_{-1}^{\nu}(z)$  is given by (3.5) and (3.13) with  $\rho = -1$  and conditions (3.16) or (3.12). Then,*

$$Z_{-1}^{\nu}(z) = \Gamma(-\nu)(1+z)^{\nu}. \quad (5.16)$$

**P r o o f.** By Corollary 4.5 there holds relation (4.20). Applying the formula for the Gamma function [4, 4.1(2)]

$$\Gamma(a+k) = (a)_k \Gamma(a) \quad (a \in \mathbb{C}, a \neq 0, -1, -2, \dots; k = 0, 1, 2, \dots), \quad (5.17)$$

we have

$$\Gamma(-\nu+k) = \Gamma(-\nu)(-\nu)_k. \quad (5.18)$$

Then the series in (4.20) takes the form of the hypergeometric function (1.10):

$$\sum_{k=0}^{\infty} \Gamma(-\nu+k) \frac{(-z)^k}{k!} = \Gamma(-\nu) {}_1F_0[-\nu; -; -z],$$

and hence, by the known formula [4, 2.8(4)]

$${}_1F_0[\mu; -; z] = {}_2F_1[\mu, b; b; z] = (1-z)^{-\mu} \quad (\mu, b \in \mathbb{C}; |z| < 1), \quad (5.19)$$

we obtain

$$\sum_{k=0}^{\infty} \Gamma(-\nu+k) \frac{z^k}{k!} = \Gamma(-\nu) \left(1 + \frac{1}{z}\right)^{\nu}.$$

Substituting this relation into (4.20), we obtain (5.16). ■

Finally we present the cases  $\rho = -1/2$  and  $\rho = -2$ .

**THEOREM 5.6.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ ,  $\operatorname{Re}(\nu) < 0$ , and let the function  $Z_{-1/2}^{\nu}(z)$  be given by (3.5) and (3.13) with  $\rho = -1/2$  and conditions (3.17) or (3.19).*

*Then  $Z_{-1/2}^{\nu}(z)$  is represented by*

$$\begin{aligned} Z_{-1/2}^{\nu}(z) = & \Gamma(-\nu) z^{\nu} {}_1F_1 \left[ -\nu; \frac{1}{2}; \frac{1}{4z} \right] \\ & - \Gamma \left( -\nu + \frac{1}{2} \right) z^{\nu-1/2} {}_1F_1 \left[ \frac{1}{2} - \nu; \frac{3}{2}; \frac{1}{4z} \right]. \end{aligned} \quad (5.20)$$

P r o o f. Applying Theorem 4.2 with  $\rho = -1/2$ , (1.9) and taking the same arguments as in the proof of Theorem 5.3, we have

$$\begin{aligned} Z_{-2}^\nu(z) &= z^\nu {}_1\Psi_0 \left[ \begin{matrix} (-\nu, 1/2) \\ - \end{matrix} \middle| -\frac{1}{\sqrt{z}} \right] \\ &= z^\nu \sum_{k=0}^{\infty} \Gamma\left(-\nu + \frac{k}{2}\right) \frac{1}{k!} \left(-\frac{1}{\sqrt{z}}\right)^k \\ &= z^\nu \left[ \sum_{n=0}^{\infty} \frac{\Gamma(-\nu + n)}{(2n)!} \left(\frac{1}{z}\right)^n - \frac{1}{\sqrt{z}} \sum_{n=0}^{\infty} \frac{\Gamma(-\nu + n + 1/2)}{(2n+1)!} \left(\frac{1}{z}\right)^n \right] \\ &= z^\nu \left[ \sum_{n=0}^{\infty} \frac{\Gamma(-\nu)(\nu)_n}{2^{2n}(1/2)_n n!} \left(\frac{1}{z}\right)^n - \frac{1}{\sqrt{z}} \sum_{n=0}^{\infty} \frac{\Gamma(-\nu + 1/2)(-\nu + 1/2)_n}{2^{2n}(3/2)_n n!} \left(\frac{1}{z}\right)^n \right] \\ &= \Gamma(-\nu) z^\nu {}_1F_1 \left[ -\nu; \frac{1}{2}; \frac{1}{4z} \right] - \Gamma\left(-\nu + \frac{1}{2}\right) z^{\nu-1/2} {}_1F_1 \left[ \frac{1}{2} - \nu; \frac{3}{2}; \frac{1}{4z} \right], \end{aligned}$$

and (5.20) is proved. ■

**THEOREM 5.7.** *Let  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\nu \in \mathbb{C}$ ,  $\text{Re}(\nu) < 0$ , and let the function  $Z_{-2}^\nu(z)$  is given by (3.5) and (3.13) with  $\rho = -2$  and conditions (3.15) or (3.19).*

*Then  $Z_{-2}^\nu(z)$  is represented by*

$$Z_{-2}^\nu(z) = \frac{1}{2} \Gamma\left(-\frac{\nu}{2}\right) {}_1F_1 \left[ -\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{4} \right] - \frac{z}{2} \Gamma\left(\frac{1-\nu}{2}\right) {}_1F_1 \left[ \frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{4} \right]. \tag{5.21}$$

P r o o f. Applying Theorem 4.2 with  $\rho = -2$ , (1.9), (5.17) and (1.10) and taking the same arguments as in the proof of Theorem 5.3, we have

$$\begin{aligned} Z_{-2}^\nu(z) &= \frac{1}{2} {}_1\Psi_0 \left[ \begin{matrix} (-\nu/2, 1/2) \\ - \end{matrix} \middle| -z \right] = \frac{1}{2} \sum_{k=0}^{\infty} \Gamma\left(\frac{-\nu + k}{2}\right) \frac{(-z)^k}{k!} \\ &= \frac{1}{2} \left[ \sum_{n=0}^{\infty} \Gamma\left(-\frac{\nu}{2} + n\right) \frac{(-z)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \Gamma\left(\frac{1-\nu}{2} + n\right) \frac{(-z)^{2n+1}}{(2n+1)!} \right] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left[ \Gamma\left(-\frac{\nu}{2}\right) \left(-\frac{\nu}{2}\right)_n \frac{z^{2n}}{2^{2n} \left(\frac{1}{2}\right)_n n!} \right. \\ &\quad \left. - z \sum_{n=0}^{\infty} \Gamma\left(\frac{1-\nu}{2}\right) \left(\frac{1-\nu}{2}\right)_n \frac{z^{2n}}{2^{2n} \left(\frac{3}{2}\right)_n n!} \right] \end{aligned}$$

$$= \frac{1}{2}\Gamma\left(-\frac{\nu}{2}\right) {}_1F_1\left[-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{4}\right] - \frac{z}{2}\Gamma\left(\frac{1-\nu}{2}\right) {}_1F_1\left[\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{4}\right],$$

and thus (5.21) is proved.  $\blacksquare$

REMARK 5.1. In Theorems 5.2-5.4 and 5.5-5.7 we have proved that if  $\rho = 1, 2, 1/2$  and  $\rho = -1, -1/2, -2$ , respectively, then  $Z_\rho^\nu(z)$  can be represented via the generalized hypergeometric series (1.10). We give a conjecture that by using the same arguments, as in the proofs of the above theorems with applying the Gauss-Legendre multiplication formula [4, 1.2(11)]

$$\Gamma(mz) = \frac{m^{mz-1}}{(2\pi)^{(m-1)/2}} \Gamma(z) \prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) \quad (m \in \mathbb{N}), \quad (5.22)$$

generalizing (5.13), it is possible to prove that  $Z_\rho^\nu(z)$  in the cases  $\rho = m, 1/m$  and  $\rho = -1/m, -m$  ( $m = 3, 4, \dots$ ) can be also represented in terms of the generalized hypergeometric functions (1.10).

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### References

- [1] F. Al-Musallam, V. Kiryakova, Vu Kim Tuan, A multi-index Borel-Dzrbashjan transform. *Rocky Mountain J. Math.* **32**, No 2 (2002), 409-428.
- [2] I. Dimovski, V. Kiryakova, The Obrechhoff integral transform: Properties and relation to a generalized fractional calculus. *Numerical Functional Analysis and Optimization* **21**, No 1-2 (2000), 121-144.
- [3] V.A. Ditkin, A.P. Prudnikov, *Integral Transforms and Operational Calculus*. Pergamon Press, Oxford (1965).
- [4] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, Vol. I. McGraw-Hill, New York - Toronto - London (1953); Reprinted: Krieger, Melbourne, Florida (1981).
- [5] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, Vol. II. McGraw-Hill, New York - Toronto -

- London (1953); Reprinted: Krieger, Melbourne, Florida (1981).
- [6] H.-J. Glaeske, A.A. Kilbas, Bessel-type integral transforms on  $\mathcal{L}_{\nu,r}$ -spaces. *Result. Math.* **34**, No. 3-4 (1998), 320-329.
  - [7] A.A. Kilbas, B. Bonilla, M. Rivero, J. Rodríguez, J.J. Trujillo, Compositions of Bessel type integral transform with fractional operators on spaces  $\mathcal{F}_{p,\mu}$  and  $\mathcal{F}'_{p,\mu}$ . *Fract. Calc. Appl. Anal.* **1**, No. 2 (1998), 135-150.
  - [8] A.A. Kilbas, H.-J. Glaeske, Bessel-type integral transforms in weighted  $\mathcal{L}_{\nu,r}$ -spaces (in Russian). *Dokl. Nats. Akad. Nauk Belarusi* **42**, No. 4 (1998), 33-39.
  - [9] A.A. Kilbas, M. Saigo, *H-Transforms. Theory and Applications*. Chapman and Hall/CRC, Boca Raton, Florida (2004).
  - [10] A.A. Kilbas, M. Saigo, J.J. Trujillo, On the generalized Wright function. Dedicated to the 60th anniversary of Prof. Francesco Mainardi. *Fract. Calc. Appl. Anal.* **5**, No. 4 (2002), 437-460.
  - [11] A.A. Kilbas, S.A. Shlapakov, On Bessel-type integral transformation and its composition with integral and differential operators (in Russian). *Dokl. Akad. Nauk Belarusi* **37**, No. 4 (1993), 10-14.
  - [12] A.A. Kilbas, S.A. Shlapakov, On the composition of a Bessel-type integral operator with operators of fractional integro-differentiation and the solution of differential equations. *Differential Equations* **30**, No. 2 (1994), 235-246; EN translation from *Differentsialnye Uravneniya* **30**, No.2 (1994), 256-268.
  - [13] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies 204, Elsevier, Amsterdam (2006).
  - [14] V. Kiryakova, *Generalized Fractional Calculus and Applications*. Longman Sci. & Techn., Harlow and J. Wiley & Sons Inc, N. York (1994).
  - [15] V. Kiryakova, Multiindex Mittag-Leffler functions, related Gelfond-Leontiev operators and Laplace type transforms. *Fract. Calc. Appl. Anal.* **2**, No 4 (1999), 445-462.
  - [16] V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *J. Comput. Appl. Mathematics* **118**(2000), 241-259.
  - [17] E. Krätzel, Eine Verallgemeinerung der Laplace- und Meijer-Transformation. *Wiss. Z. Friedrich-Shiller-Univ. Math.-naturwiss. Reihe* **14**, No 5 (1965), 369-381.
  - [18] E. Krätzel, Integral transformations of Bessel type, In: *Generalized*

- Functions and Operational Calculus (Proc. Conf. Varna 1975)*. Bulg. Acad. Sci., Sofia (1979), 148-155.
- [19] A.M. Mathai, R.K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*. Halsted Press [John Wiley and Sons], New York-London-Sydney (1978).
- [20] A.C. McBride, *Fractional Calculus and Integral Transforms of Generalized Functions*, Pitman, London (1979).
- [21] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions. Theory and Algorithmic Tables*. Ellis Horwood, Chichester; John Wiley and Sons, New York (1983).
- [22] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Reports* **339** (2000), 1-77.
- [23] I. Podlubny, *Fractional Differential Equations*, Mathematics in Sciences and Engineering, **198**. Academic Press, San Diego (1999).
- [24] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series, Vol.3, More Special Functions*. Gordon and Breach Sci. Publ., New York (1990).
- [25] J. Rodríguez, J.J. Trujillo, M. Rivero, Operational fractional calculus of Krätzel transformation. In: *Differential Equations (Xanthi, 1987)*, Lect. Notes Pure Appl. Math. **118**, Denver, New York (1989), 613-620.
- [26] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, New York *et al.* (1993).
- [27] I.N. Sneddon, *Fourier Transforms*. McGraw-Hill, New York - Toronto - London (1951); Reprinted in Dover Publication, New York (1995).
- [28] I.M. Sokolov, J. Klafter, A. Blumen, Fractional kinetics. *Physics Today* (November 2002), 48-54.
- [29] H.M. Srivastava, K.C. Gupta, S.L. Goyal, *The H-function of One and Two Variables with Applications*. South Asian Publishers, New Delhi-Madras (1982).
- [30] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function. *J. London Math. Soc.* **10** (1935), 286-293.
- [31] E.M. Wright, The asymptotic expansion of integral functions defined by Taylor series. *Philos. Trans. Roy. Soc. London, Ser. A*, **238** (1940), 423-451.
- [32] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function. *Proc. London Math. Soc. (2)* **46** (1940), 389-408.

<sup>1</sup> *Department of Mathematics and Mechanics*  
*Belarusian State University*  
*220050 Minsk, BELARUS*  
*e-mail: anatolykilbas@gmail.com*

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<sup>2</sup> *Department of Mathematics and Statistics*  
*Jai Narain Vyas University*  
*342011 Jodhpur, INDIA*  
*e-mail: ram.saxena@yahoo.com*

<sup>3</sup> *Departamento de Análisis Matemático*  
*University of La Laguna*  
*38271 La Laguna - Tenerife, SPAIN*  
*e-mail: jtrujill@ullmat.es*