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SOME MEAN-VALUE THEOREMS OF THE CAUCHY TYPE

Josip E. Pečarić [1], Mirna Rodić Lipanović [2], H. M. Srivastava [3]

Dedicated to Professor Megumi Saigo on the Occasion of his Seventieth Birthday

Abstract

Some mean-value theorems of the Cauchy type, which are connected with Jensen's inequality, are given in [2] in discrete form and in [5] in integral form. Several further generalizations and applications of these results are presented here.

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1. Introduction

Some mean-value theorems of the Cauchy type, which are connected with Jensen's inequality, are given in the recent papers by Mercer [2] and Pečarić *et al.* [5]. In [2] there is given the discrete form and in [5] the integral form, and there were also given some applications of these results on power means.

In this paper we propose to give some further generalizations of some of the aforementioned results and also discuss several interesting applications of the various general results presented here.

2. A set of main results

THEOREM 1. *Let I be a compact real interval and $\varphi, \psi \in C^2(I)$. Let Ω be a convex set equipped with a probability measure μ and let $h : \Omega \to I$ be an integrable function with respect to the probability measure μ. Then, for some $\xi \in I$,*

$$\frac{\int_\Omega \varphi(h(\mathbf{u}))d\mu(\mathbf{u}) - \varphi(\int_\Omega h(\mathbf{u})d\mu(\mathbf{u}))}{\int_\Omega \psi(h(\mathbf{u}))d\mu(\mathbf{u}) - \psi(\int_\Omega h(\mathbf{u})d\mu(\mathbf{u}))} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{1}$$

provided that the denominator on the left-hand side (1) *is non-zero.*

P r o o f. Define

$$A := \int_\Omega h(\mathbf{u})d\mu(\mathbf{u})$$

and

$$(Q\varphi)(t) := \int_\Omega \varphi(th(\mathbf{u}) + (1-t)A)d\mu(\mathbf{u}) - \varphi(A).$$

Analogously, for the function ψ, we define $(Q\psi)(t)$.

It is easily observed that

$$(Q\varphi)'(t) = \int_\Omega (h(\mathbf{u}) - A)\varphi'(th(\mathbf{u}) + (1-t)A)d\mu(\mathbf{u})$$

and

$$(Q\varphi)''(t) = \int_\Omega (h(\mathbf{u}) - A)^2\varphi''(th(\mathbf{u}) + (1-t)A)d\mu(\mathbf{u}).$$

Let us now consider the function $W(t)$ defined by

$$W(t) = (Q\psi)(1)(Q\varphi)(t) - (Q\varphi)(1)(Q\psi)(t),$$

so that, obviously,

$$W(0) = W(1) = W'(0) = 0.$$

Then, with two applications of the Lagrange mean-value theorem, we find that there is $\lambda \in (0,1)$ so that

$$W''(\lambda) = 0.$$

This implies that

$$\begin{aligned}\int_\Omega (h(\mathbf{u}) - A)^2 \big[&(Q\psi)(1) \cdot \varphi''(\lambda h(\mathbf{u}) + (1-\lambda)A) \\ &- (Q\varphi)(1) \cdot \psi''(\lambda h(\mathbf{u}) + (1-\lambda)A)\big] d\mu(\mathbf{u}) = 0.\end{aligned} \tag{2}$$

For any fixed λ, the expression in the square brackets in (2) is a continuous function of $\mathbf{u}$, so it vanishes for some value of $\mathbf{u} \in \Omega$. Corresponding to that value of $\mathbf{u} \in \Omega$, we get a number

$$\xi = \lambda h(\mathbf{u}) + (1-\lambda)A \qquad (\xi \in I)$$

so that

$$(Q\psi)(1) \cdot \varphi''(\xi) - (Q\varphi)(1) \cdot \psi''(\xi) = 0.$$

The assertion (1) of Theorem 1 now follows directly. ∎

Remark 1. By setting $\Omega = [a,b] \subset \mathbb{R}$ in Theorem 1, we get the result given earlier by Pečarić *et al.* [5].

Theorem 2. *Let I be a compact real interval and $\varphi \in C^2(I)$. Let Ω be a convex set equipped with a probability measure μ and let $h : \Omega \to I$ be an integrable function with respect to the probability measure μ. Then there is $\xi \in I$ such that the following equality holds true*:

$$\int_\Omega \varphi\left(h(\mathbf{u})\right) d\mu(\mathbf{u}) - \varphi\left(\int_\Omega h(\mathbf{u}) d\mu(\mathbf{u})\right)$$
$$= \frac{\varphi''(\xi)}{2}\left[\int_\Omega (h(\mathbf{u}))^2 d\mu(\mathbf{u}) - \left(\int_\Omega h(\mathbf{u}) d\mu(\mathbf{u})\right)^2\right]. \tag{3}$$

P r o o f. If we set $\psi(x) = x^2$ in Theorem 1, we arrive at the assertion (3) of Theorem 2. ∎

Remark 2. The assertion (3) of Theorem 2 can also be written as follows:

$$\int_\Omega \varphi\left(h(\mathbf{u})\right) d\mu(\mathbf{u}) - \varphi\left(\int_\Omega h(\mathbf{u}) d\mu(\mathbf{u})\right)$$
$$= \frac{\varphi''(\xi)}{2}\int_\Omega \left[h(\mathbf{u}) - \int_\Omega h(\mathbf{u}) d\mu(\mathbf{u})\right]^2 d\mu(\mathbf{u}). \tag{4}$$

Remark 3. If, in Theorem 2, we consider the cases when φ is a convex or concave function, we shall get Jensen's inequality.

Remark 4. For $\Omega = [a,b] \subset \mathbb{R}$, we get a known application of Theorem 1, which was considered earlier by Pečarić *et al.* [5].

Remark 5. Yet another application of these results, for power means, was given in [5]. Moreover, analogous results for quasi-arithmetic means are also seen to be valid.

Next, we let Ω be a convex set equipped with a probability measure μ. Then, for a strictly monotone continuous function g, the quasi-arithmetic mean $M_g(f;\mu)$ is defined as follows:

$$M_g(f;\mu) = g^{-1}\left(\int_\Omega (g \circ f)(\mathbf{u})d\mu(\mathbf{u})\right).$$

Throughout our present investigation, we tacitly assume, without further comment, that all the integrals involved in our results exist.

The following result is a consequence of Theorem 1.

THEOREM 3. *Let I be a compact real interval and let $f, g, k \in C^2(I)$ be strictly monotone functions. Suppose also that Ω is a convex set equipped with a probability measure μ and that $v : \Omega \to I$. Then,*

$$\frac{f(M_f(v;\mu)) - f(M_g(v;\mu))}{k(M_k(v;\mu)) - k(M_g(v;\mu))} = \frac{f''(\eta)\cdot g'(\eta) - f'(\eta)\cdot g''(\eta)}{k''(\eta)\cdot g'(\eta) - k'(\eta)\cdot g''(\eta)} \tag{5}$$

for some η in the image of $v(\mathbf{u})$, provided that the denominator on the left-hand side of (5) *is non-zero.*

P r o o f. If we choose the functions φ and ψ so that

$$\varphi = f \circ g^{-1},\ \psi = k \circ g^{-1} \quad \text{and} \quad h(\mathbf{u}) = g(v(\mathbf{u})),$$

and apply Theorem 1 to such functions, we find that there exists $\xi \in I$ such that

$$\frac{f(M_f(v;\mu)) - f(M_g(v;\mu))}{k(M_k(v;\mu)) - k(M_g(v;\mu))} = \frac{f''(g^{-1}(\xi))\cdot g'(g^{-1}(\xi)) - f'(g^{-1}(\xi))\cdot g''(g^{-1}(\xi))}{k''(g^{-1}(\xi))\cdot g'(g^{-1}(\xi)) - k'(g^{-1}(\xi))\cdot g''(g^{-1}(\xi))}.$$

Thus, by setting $g^{-1}(\xi) = \eta$, we find that there exists η in the image of $v(\mathbf{u})$ such that

$$\frac{f(M_f(v;\mu)) - f(M_g(v;\mu))}{k(M_k(v;\mu)) - k(M_g(v;\mu))} = \frac{f''(\eta)\cdot g'(\eta) - f'(\eta)\cdot g''(\eta)}{k''(\eta)\cdot g'(\eta) - k'(\eta)\cdot g''(\eta)},$$

which completes our proof of Theorem 3. ∎

REMARK 6. It follows from Theorem 3 that

$$m \leqq \left|\frac{f(M_f(v;\mu)) - f(M_g(v;\mu))}{k(M_k(v;\mu)) - k(M_g(v;\mu))}\right| \leqq M,$$

where m and M are, respectively, the minimum and the maximum values of

$$\left|\frac{f''(g^{-1}(t))\cdot g'(g^{-1}(t)) - f'(g^{-1}(t))\cdot g''(g^{-1}(t))}{k''(g^{-1}(t))\cdot g'(g^{-1}(t)) - k'(g^{-1}(t))\cdot g''(g^{-1}(t))}\right| \qquad (t \in I),$$

that is, the minimum and the maximum values of

$$\left|\frac{f''(t)\cdot g'(t)-f'(t)\cdot g''(t)}{k''(t)\cdot g'(t)-k'(t)\cdot g''(t)}\right|$$

for t in the image of $v(\mathbf{u})$.

For quasi-arithmetic means, directly from Theorem 2, we can deduce the following result.

THEOREM 4. *Let I be a compact real interval and let $f, g \in C^2(I)$ be strictly monotone functions. Suppose also that Ω be a convex set equipped with a probability measure μ and that $v : \Omega \to I$. Then,*

$$\begin{aligned} f(M_f(v;\mu)) - f(M_g(v;\mu)) &= \frac{1}{2}\cdot\frac{f''(\eta)\cdot g'(\eta)-f'(\eta)\cdot g''(\eta)}{[g'(\eta)]^3} \\ &\cdot \left[\int_\Omega ((g\circ v)(\mathbf{u}))^2\, d\mu(\mathbf{u}) - \left(\int_\Omega (g\circ v)(\mathbf{u}) d\mu(\mathbf{u})\right)^2\right] \end{aligned} \tag{6}$$

for some η in the image of $v(\mathbf{u})$.

REMARK 7. It is easily seen from Theorem 4 that

$$\begin{aligned} &\frac{1}{2}\cdot m\cdot\left[\int_\Omega ((g\circ v)(\mathbf{u}))^2\, d\mu(\mathbf{u}) - \left(\int_\Omega (g\circ v)(\mathbf{u}) d\mu(\mathbf{u})\right)^2\right] \\ &\quad \leqq |f(M_f(v;\mu)) - f(M_g(v;\mu))| \\ &\quad \leqq \frac{1}{2}\cdot M\cdot\left[\int_\Omega ((g\circ v)(\mathbf{u}))^2\, d\mu(\mathbf{u}) - \left(\int_\Omega (g\circ v)(\mathbf{u}) d\mu(\mathbf{u})\right)^2\right], \end{aligned}$$

where m and M are, respectively, the minimum and the maximum values of

$$\left|\frac{f''(g^{-1}(t))\cdot g'(g^{-1}(t))-f'(g^{-1}(t))\cdot g''(g^{-1}(t))}{[g'(g^{-1}(t))]^3}\right| \qquad (t\in I),$$

that is, the minimum and the maximum values of

$$\left|\frac{f''(t)\cdot g'(t)-f'(t)\cdot g''(t)}{[g'(t)]^3}\right|$$

for t in the image of $v(\mathbf{u})$.

Now, from the results given above, we deduce the corresponding results for integral power means. Indeed, for $r \in \mathbb{R}$, the integral power mean is defined as follows:

$$M_r(g;\mu) = \begin{cases} \left[\int_\Omega (g(\mathbf{u}))^r d\mu(\mathbf{u})\right]^{\frac{1}{r}} & (r \neq 0) \\ \\ \exp(\int_\Omega \ln(g(\mathbf{u}))d\mu(\mathbf{u})) & (r = 0). \end{cases} \tag{7}$$

Corollary 1. *Let $r, s, l \in \mathbb{R}$ and let Ω be a convex set equipped with a probability measure μ. Then*

$$\frac{M_r^r(v;\mu) - M_s^r(v;\mu)}{M_l^l(v;\mu) - M_s^l(v;\mu)} = \frac{r(r-s)}{l(l-s)} \cdot \eta^{r-l} \tag{8}$$

for some η in the image of $v(\mathbf{u})$ $(\mathbf{u} \in \Omega)$, provided that the denominator on the left-hand side of (8) *is non-zero.*

P r o o f. If we set

$$\varphi(x) = x^{\frac{r}{s}},\ \psi(x) = x^{\frac{l}{s}} \quad \text{and} \quad h(\mathbf{u}) = (v(\mathbf{u}))^s$$

in Theorem 1, that is, for

$$\varphi = f \circ g^{-1} \quad \text{and} \quad \psi = k \circ g^{-1}$$

as in Theorem 3, with

$$f(x) = x^r,\ g(x) = x^s,\ k(x) = x^l \quad \text{and} \quad h(\mathbf{u}) = g(v(\mathbf{u})), \tag{9}$$

we get the assertion (8) of Corollary 1. ∎

Remark 8. It follows from Corollary 1 that

$$\left|\frac{r(r-s)}{l(l-s)}\right| m \leqq \left|\frac{M_r^r(v;\mu) - M_s^r(v;\mu)}{M_l^l(v;\mu) - M_s^l(v;\mu)}\right| \leqq \left|\frac{r(r-s)}{l(l-s)}\right| M,$$

where m and M are, respectively, the minimum and the maximum values of the function x^{r-l} on the image of $v(\mathbf{u})$ $(\mathbf{u} \in \Omega)$. In particular, when $\Omega = [a, b] \subset \mathbb{R}$, we get the application given already in [5].

Corollary 2. *Let $r, s \in \mathbb{R}$ and let Ω be a convex set equipped with a probability measure μ. Then*

$$M_r^r(v;\mu) - M_s^r(v;\mu) = \frac{1}{2} \cdot \frac{r(r-s)}{s^2} \cdot \eta^{r-2s} \cdot \left[M_{2s}^{2s}(v;\mu) - M_s^{2s}(v;\mu)\right] \tag{10}$$

for some η in the image of $v(\mathbf{u})$ $(\mathbf{u} \in \Omega)$.

P r o o f. By the same reasoning as in the proof of Corollary 1, from Theorem 2 (or, alternatively, from Theorem 4) we can derive the assertion (10) of Corollary 2. ∎

REMARK 9. It is readily observed from Corollary 2 that

$$\frac{1}{2} \cdot \left| \frac{r(r-s)}{s^2} \right| \cdot m \cdot \left[M_{2s}^{2s}(v;\mu) - M_s^{2s}(v;\mu) \right]$$
$$\leqq |M_r^r(v;\mu) - M_s^r(v;\mu)| \leqq \frac{1}{2} \cdot \left| \frac{r(r-s)}{s^2} \right| \cdot M \cdot \left[M_{2s}^{2s}(v;\mu) - M_s^{2s}(v;\mu) \right],$$

where m and M are, respectively, the minimum and the maximum values of the function x^{r-2s} on the image of $v(\mathbf{u})$ $(\mathbf{u} \in \Omega)$.

3. Further consequences and applications

Our results in the preceding sections can be used for deriving various related results for other classes of means. Here we shall give some of such further consequences and applications of our results.

3.1. Tobey and Stolarsky-Tobey means

Let E_{n-1} represent the $(n-1)$-dimensional Euclidean simplex given by

$$E_{n-1} := \left\{ (u_1, \ldots, u_{n-1}) : \quad u_i \geqq 0 \ (1 \leqq i \leqq n-1) \text{ and } \sum_{i=1}^{n-1} u_i \leqq 1 \right\}$$

and set

$$u_n = 1 - \sum_{i=1}^{n-1} u_i.$$

Moreover, with $\mathbf{u} = (u_1, \ldots, u_n)$, let $\mu(\mathbf{u})$ be a probability measure on E_{n-1}.

The power mean of order p $(p \in \mathbb{R})$ of the positive n-tuple

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n,$$

with the weights $\mathbf{u} = (u_1, \ldots, u_n)$, is defined by

$$\bar{M}_p(\mathbf{x}, \mathbf{u}) = \begin{cases} \left(\sum_{i=1}^{n} u_i x_i^p \right)^{\frac{1}{p}} & (p \neq 0) \\ \prod_{i=1}^{n} x_i^{u_i} & (p = 0). \end{cases}$$

Then the Tobey mean $L_{p,r}(\mathbf{x};\mu)$ is defined as follows:

$$L_{p,r}(\mathbf{x};\mu) = M_r(\bar{M}_p(\mathbf{x}, \mathbf{u});\mu),$$

where $M_r(g;\mu)$ denotes the integral power mean in (7) in which Ω is now the $(n-1)$-dimensional Euclidean simplex E_{n-1}.

We note that, since $\bar{M}_p(\mathbf{x},\mathbf{u})$ is a mean, we have

$$\min\{x_i\} \leqq \bar{M}_p(\mathbf{x},\mathbf{u}) \leqq \max\{x_i\},$$

where (*and in what follows*) $\min\{x_i\}$ and $\max\{x_i\}$ are, respectively, the minimum and the maximum values of x_i $(i = 1, \ldots, n)$. Now, setting the function v as follows:

$$v(\mathbf{u}) = \bar{M}_p(\mathbf{x},\mathbf{u})$$

and using the previous results, we get the following results:

(i)
$$\frac{L^r_{p,r}(\mathbf{x};\mu) - L^r_{p,s}(\mathbf{x};\mu)}{L^l_{p,l}(\mathbf{x};\mu) - L^l_{p,s}(\mathbf{x};\mu)} = \frac{r(r-s)}{l(l-s)} \cdot \eta^{r-l}$$

for some η such that

$$\min\{x_i\} \leqq \eta \leqq \max\{x_i\};$$

(ii)
$$\left|\frac{r(r-s)}{l(l-s)}\right| m \leqq \left|\frac{L^r_{p,r}(\mathbf{x};\mu) - L^r_{p,s}(\mathbf{x};\mu)}{L^l_{p,l}(\mathbf{x};\mu) - L^l_{p,s}(\mathbf{x};\mu)}\right| \leqq \left|\frac{r(r-s)}{l(l-s)}\right| M,$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-l} $(i = 1, \ldots, n)$;

(iii)
$$L^r_{p,r}(\mathbf{x};\mu) - L^r_{p,s}(\mathbf{x};\mu) = \frac{1}{2} \cdot \frac{r(r-s)}{s^2} \cdot \eta^{r-2s} \cdot \left[L^{2s}_{p,2s}(\mathbf{x};\mu) - L^{2s}_{p,s}(\mathbf{x};\mu)\right]$$

for some η such that

$$\min\{x_i\} \leqq \eta \leqq \max\{x_i\};$$

(iv)
$$\frac{1}{2} \cdot \left|\frac{r(r-s)}{s^2}\right| \cdot m \cdot \left[L^{2s}_{p,2s}(\mathbf{x};\mu) - L^{2s}_{p,s}(\mathbf{x};\mu)\right]$$
$$\leqq \left|L^r_{p,r}(\mathbf{x};\mu) - L^r_{p,s}(\mathbf{x};\mu)\right| \leqq \frac{1}{2} \cdot \left|\frac{r(r-s)}{s^2}\right| \cdot M \cdot \left[L^{2s}_{p,2s}(\mathbf{x};\mu) - L^{2s}_{p,s}(\mathbf{x};\mu)\right],$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-2s} $(i = 1, \ldots, n)$.

Pečarić and Šimić [6] introduced the Stolarsky-Tobey mean $\mathcal{E}_{p,q}(\mathbf{x};\mu)$ defined by

$$\mathcal{E}_{p,q}(\mathbf{x};\mu)=\begin{cases}\left[\int_{E_{n-1}}\left(\sum\limits_{i=1}^{n}u_i x_i^p\right)^{\frac{q-p}{p}}d\mu(\mathbf{u})\right]^{\frac{1}{q-p}} & \big(p(q-p)\neq 0\big)\\[2ex] \exp\left(\int_{E_{n-1}}\ln\left(\sum\limits_{i=1}^{n}u_i x_i^p\right)^{\frac{1}{p}}d\mu(\mathbf{u})\right) & (p=q\neq 0)\\[2ex] \left[\int_{E_{n-1}}\left(\prod\limits_{i=1}^{n}x_i^{u_i}\right)^{q}d\mu(\mathbf{u})\right]^{\frac{1}{q}} & (p=0;\ q\neq 0)\\[2ex] \exp\left(\int_{E_{n-1}}\ln\left(\prod\limits_{i=1}^{n}x_i^{u_i}\right)d\mu(\mathbf{u})\right) & (p=q=0)\end{cases} \tag{11}$$

or, alternatively, by

$$\mathcal{E}_{p,q}(\mathbf{x};\mu)=L_{p,q-p}(\mathbf{x};\mu)=M_{q-p}(\bar{M}_p(\mathbf{x},\mathbf{u});\mu),$$

where $L_{p,r}(\mathbf{x};\mu)$ is the Tobey mean already introduced above.

For the Stolarsky-Tobey mean, we get the following results:

(i) $$\frac{\mathcal{E}^r_{p,p+r}(\mathbf{x};\mu)-\mathcal{E}^r_{p,p+s}(\mathbf{x};\mu)}{\mathcal{E}^l_{p,p+l}(\mathbf{x};\mu)-\mathcal{E}^l_{p,p+s}(\mathbf{x};\mu)}=\frac{r(r-s)}{l(l-s)}\cdot\eta^{r-l}$$

for some η such that

$$\min\{x_i\}\leqq\eta\leqq\max\{x_i\};$$

(ii) $$\left|\frac{r(r-s)}{l(l-s)}\right|m\leqq\left|\frac{\mathcal{E}^r_{p,p+r}(\mathbf{x};\mu)-\mathcal{E}^r_{p,p+s}(\mathbf{x};\mu)}{\mathcal{E}^l_{p,p+l}(\mathbf{x};\mu)-\mathcal{E}^l_{p,p+s}(\mathbf{x};\mu)}\right|\leqq\left|\frac{r(r-s)}{l(l-s)}\right|M,$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-l} $(i=1,\ldots,n)$;

(iii) $$\mathcal{E}^r_{p,p+r}(\mathbf{x};\mu)-\mathcal{E}^r_{p,p+s}(\mathbf{x};\mu)=\frac{1}{2}\cdot\frac{r(r-s)}{s^2}\cdot\eta^{r-2s}\cdot\left[\mathcal{E}^{2s}_{p,p+2s}(\mathbf{x};\mu)-\mathcal{E}^{2s}_{p,p+s}(\mathbf{x};\mu)\right]$$

for some η such that

$$\min\{x_i\}\leqq\eta\leqq\max\{x_i\};$$

(iv) $$\frac{1}{2}\cdot\left|\frac{r(r-s)}{s^2}\right|\cdot m\cdot\left[\mathcal{E}^{2s}_{p,p+2s}(\mathbf{x};\mu)-\mathcal{E}^{2s}_{p,p+s}(\mathbf{x};\mu)\right]$$

$$\leqq\left|\mathcal{E}^r_{p,p+r}(\mathbf{x};\mu)-\mathcal{E}^r_{p,p+s}(\mathbf{x};\mu)\right|\leqq\frac{1}{2}\cdot\left|\frac{r(r-s)}{s^2}\right|\cdot M\cdot\left[\mathcal{E}^{2s}_{p,p+2s}(\mathbf{x};\mu)-\mathcal{E}^{2s}_{p,p+s}(\mathbf{x};\mu)\right],$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-2s} $(i = 1, \dots, n)$.

3.2. Functional Stolarsky means

For strictly monotone continuous functions f and g, the *functional* Stolarsky means are defined by (see [3]):

$$m_{f,g}(\mathbf{x};\mu) = f^{-1}\left(\int_{E_{n-1}} (f \circ g^{-1})(\mathbf{u}\cdot\mathbf{g})d\mu(\mathbf{u})\right),$$

where

$$\mathbf{g} = \big(g(x_1), \dots, g(x_n)\big)$$

and μ is a probability measure on E_{n-1}.

In the same way as we developed results for the quasi-arithmetic means, we can get analogous results for the functional Stolarsky means by using the fact that $\Omega = E_{n-1}$.

THEOREM 5. *Let I be a compact real interval and $f, g, k \in C^2(I)$ be strictly monotone functions. Then*

$$\frac{f(m_{f,g}(\mathbf{x};\mu)) - f(m_{g,g}(\mathbf{x};\mu))}{k(m_{k,g}(\mathbf{x};\mu)) - k(m_{g,g}(\mathbf{x};\mu))} = \frac{f''(\eta)\cdot g'(\eta) - f'(\eta)\cdot g''(\eta)}{k''(\eta)\cdot g'(\eta) - k'(\eta)\cdot g''(\eta)} \tag{12}$$

for some η (in $g^{-1}(I)$), provided that the denominator on the left-hand side of (12) *is non-zero.*

P r o o f. Let the functions φ and ψ be as in the proof of Theorem 3. Also let the function h be so defined that

$$h(\mathbf{u}) = \mathbf{u}\cdot\mathbf{g} = \sum_{i=1}^{n} u_i g(x_i).$$

With the same reasoning as in the proof of Theorem 3, by applying Theorem 1, we find that there exists $\xi \in I$ such that

$$\frac{f(m_{f,g}(\mathbf{x};\mu)) - f(m_{g,g}(\mathbf{x};\mu))}{k(m_{k,g}(\mathbf{x};\mu)) - k(m_{g,g}(\mathbf{x};\mu))}$$
$$= \frac{f''(g^{-1}(\xi))\cdot g'(g^{-1}(\xi)) - f'(g^{-1}(\xi))\cdot g''(g^{-1}(\xi))}{k''(g^{-1}(\xi))\cdot g'(g^{-1}(\xi)) - k'(g^{-1}(\xi))\cdot g''(g^{-1}(\xi))}$$

or, by setting $g^{-1}(\xi) = \eta$, we find that there exists η (in $g^{-1}(I)$) such that

$$\frac{f(m_{f,g}(\mathbf{x};\mu)) - f(m_{g,g}(\mathbf{x};\mu))}{k(m_{k,g}(\mathbf{x};\mu)) - k(m_{g,g}(\mathbf{x};\mu))} = \frac{f''(\eta)\cdot g'(\eta) - f'(\eta)\cdot g''(\eta)}{k''(\eta)\cdot g'(\eta) - k'(\eta)\cdot g''(\eta)},$$

which precisely is the assertion (12) of Theorem 5. ∎

REMARK 10. It follows from Theorem 5 that

$$m \leqq \left| \frac{f(m_{f,g}(\mathbf{x};\mu)) - f(m_{g,g}(\mathbf{x};\mu))}{k(m_{k,g}(\mathbf{x};\mu)) - k(m_{g,g}(\mathbf{x};\mu))} \right| \leqq M,$$

where m and M are, respectively, the minimum and the maximum values of

$$\left| \frac{f''(g^{-1}(t)) \cdot g'(g^{-1}(t)) - f'(g^{-1}(t)) \cdot g''(g^{-1}(t))}{k''(g^{-1}(t)) \cdot g'(g^{-1}(t)) - k'(g^{-1}(t)) \cdot g''(g^{-1}(t))} \right| \qquad (t \in I),$$

that is, the minimum and the maximum values of

$$\left| \frac{f''(t) \cdot g'(t) - f'(t) \cdot g''(t)}{k''(t) \cdot g'(t) - k'(t) \cdot g''(t)} \right|$$

for t in $g^{-1}(I)$.

For the functional Stolarsky means, by similarly applying Theorem 2, we can derive the following result.

THEOREM 6. *Let I be a compact real interval and let $f, g \in C^2(I)$ be strictly monotone functions. Then,*

$$\begin{aligned} f(m_{f,g}(\mathbf{x};\mu)) - f(m_{g,g}(\mathbf{x};\mu)) = \frac{1}{2} \cdot & \frac{f''(\eta) \cdot g'(\eta) - f'(\eta) \cdot g''(\eta)}{[g'(\eta)]^3} \\ & \cdot \left[\mathcal{E}^2_{1,3}(\mathbf{g};\mu) - \mathcal{E}^2_{1,2}(\mathbf{g};\mu) \right] \end{aligned} \tag{13}$$

for some η (in $g^{-1}(I)$).

REMARK 11. It is easily seen from Theorem 6 that

$$\begin{aligned} \frac{1}{2} \cdot m \cdot \left[\mathcal{E}^2_{1,3}(\mathbf{g};\mu) - \mathcal{E}^2_{1,2}(\mathbf{g};\mu) \right] & \leqq \left| f(m_{f,g}(\mathbf{x};\mu)) - f(m_{g,g}(\mathbf{x};\mu)) \right| \\ & \leqq \frac{1}{2} \cdot M \cdot \left[\mathcal{E}^2_{1,3}(\mathbf{g};\mu) - \mathcal{E}^2_{1,2}(\mathbf{g};\mu) \right], \end{aligned}$$

where m and M are, respectively, the minimum and the maximum values of

$$\left| \frac{f''(t) \cdot g'(t) - f'(t) \cdot g''(t)}{[g'(t)]^3} \right|$$

for t in $g^{-1}(I)$.

3.3. Various symmetric means

3.3.1. Complete symmetric polynomial means

The rth complete symmetric polynomial mean (or, simply, the complete symmetric mean) of the positive real n-tuple $\mathbf{x}$ is defined by (see [1])

$$Q_n^{[r]}(\mathbf{x}) = \left(q_n^{[r]}(\mathbf{x})\right)^{\frac{1}{r}} = \left(\frac{c_n^{[r]}(\mathbf{x})}{\binom{n+r-1}{r}}\right)^{\frac{1}{r}},$$

where

$$c_n^{[0]}(\mathbf{x}) = 1 \quad \text{and} \quad c_n^{[r]}(\mathbf{x}) = \sum \left(\prod_{i=1}^{n} x_i^{i_j}\right),$$

and the sum is taken over all

$$\binom{n+r-1}{r}$$

non-negative integer n-tuples $(i_1, \ldots, i_n)$ with

$$\sum_{j=1}^{n} i_j = r \qquad (r \neq 0).$$

The complete symmetric polynomial mean can also be written in an integral form as follows:

$$Q_n^{[r]}(\mathbf{x}) = \left(\int_{E_{n-1}} \left(\sum_{i=1}^{n} x_i u_i\right)^r d\mu(\mathbf{u})\right)^{\frac{1}{r}},$$

where μ represents a probability measure such that

$$d\mu(\mathbf{u}) = (n-1)! \cdot du_1 \cdots du_{n-1}.$$

As we can see, this is a special case of the integral power mean $M_r(v;\mu)$ given by (7), where

$$v(\mathbf{u}) = \sum_{i=1}^{n} x_i u_i,$$

μ is a probability measure such that

$$d\mu(\mathbf{u}) = (n-1)! \cdot du_1 \cdots du_{n-1},$$

and Ω is the above-defined $(n-1)$-dimensional simplex E_{n-1}. Thus we have the following results:

$$\text{(i)} \qquad \frac{\left(Q_n^{[r]}(\mathbf{x})\right)^r - \left(Q_n^{[s]}(\mathbf{x})\right)^r}{\left(Q_n^{[l]}(\mathbf{x})\right)^l - \left(Q_n^{[s]}(\mathbf{x})\right)^l} = \frac{r(r-s)}{l(l-s)} \cdot \eta^{r-l}$$

for some η such that

$$\min\{x_i\} \leqq \eta \leqq \max\{x_i\};$$

$$\text{(ii)} \qquad \left|\frac{r(r-s)}{l(l-s)}\right| m \leqq \left|\frac{\left(Q_n^{[r]}(\mathbf{x})\right)^r - \left(Q_n^{[s]}(\mathbf{x})\right)^r}{\left(Q_n^{[l]}(\mathbf{x})\right)^l - \left(Q_n^{[s]}(\mathbf{x})\right)^l}\right| \leqq \left|\frac{r(r-s)}{l(l-s)}\right| M,$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-l} $(i = 1, \ldots, n)$;

$$\text{(iii)} \qquad \begin{aligned} &\left(Q_n^{[r]}(\mathbf{x})\right)^r - \left(Q_n^{[s]}(\mathbf{x})\right)^r \\ &\qquad = \frac{1}{2} \cdot \frac{r(r-s)}{s^2} \cdot \eta^{r-2s} \cdot \left[\left(Q_n^{[2s]}(\mathbf{x})\right)^{2s} - \left(Q_n^{[s]}(\mathbf{x})\right)^{2s}\right] \end{aligned}$$

for some η such that

$$\min\{x_i\} \leqq \eta \leqq \max\{x_i\};$$

$$\text{(iv)} \qquad \begin{aligned} &\frac{1}{2} \cdot \left|\frac{r(r-s)}{s^2}\right| \cdot m \cdot \left[\left(Q_n^{[2s]}(\mathbf{x})\right)^{2s} - \left(Q_n^{[s]}(\mathbf{x})\right)^{2s}\right] \\ &\qquad \leqq \left|\left(Q_n^{[r]}(\mathbf{x})\right)^r - \left(Q_n^{[s]}(\mathbf{x})\right)^r\right| \\ &\qquad \leqq \frac{1}{2} \cdot \left|\frac{r(r-s)}{s^2}\right| \cdot M \cdot \left[\left(Q_n^{[2s]}(\mathbf{x})\right)^{2s} - \left(Q_n^{[s]}(\mathbf{x})\right)^{2s}\right], \end{aligned}$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-2s} $(i = 1, \ldots, n)$.

3.3.2. Whiteley means and generalizations

Let $\mathbf{x}$ be a positive real n-tuple, $s \in \mathbb{R}$ $(s \neq 0)$ and $r \in \mathbb{N}$. Then the sth function of degree r (that is, $t_n^{[r,s]}(\mathbf{x})$) is defined by the following generating function (see [1]):

$$\sum_{r=0}^{\infty} t_n^{[r,s]}(\mathbf{x})t^r = \begin{cases} \prod_{i=1}^{n}(1+x_it)^s & (s>0) \\ \\ \prod_{i=1}^{n}(1-x_it)^s & (s<0). \end{cases}$$

The Whiteley mean is now defined by

$$\mathcal{W}_n^{[r,s]}(\mathbf{x}) = \left(w_n^{[r,s]}(\mathbf{x})\right)^{\frac{1}{r}} = \begin{cases} \left(\frac{t_n^{[r,s]}(\mathbf{x})}{\binom{ns}{r}}\right)^{\frac{1}{r}} & (s>0) \\ \left(\frac{t_n^{[r,s]}(\mathbf{x})}{(-1)^r\binom{ns}{r}}\right)^{\frac{1}{r}} & (s<0). \end{cases} \tag{14}$$

Remark 12. For $s = -1$, the Whiteley mean becomes the complete symmetric polynomial mean.

For $s < 0$, the Whiteley mean can be further generalized if we slightly change the definition of $t_n^{[r,s]}(\mathbf{x})$ and define $h_n^{[r,\sigma]}(\mathbf{x})$ as follows:

$$\sum_{r=0}^{\infty} h_n^{[r,\sigma]}(\mathbf{x})t^r = \prod_{i=1}^{n} \frac{1}{(1-x_it)^{\sigma_i}}$$

$$\big(\sigma = (\sigma_1, \ldots, \sigma_n);\ \sigma_i \in \mathbb{R}_+ \quad (i = 1, \ldots, n)\big).$$

The following generalization of the Whiteley mean for $s < 0$ is defined by (see [4])

$$\mathcal{H}_n^{[r,\sigma]}(\mathbf{x}) = \left(\frac{h_n^{[r,\sigma]}(\mathbf{x})}{\binom{\sum_{i=1}^{n}\sigma_i + r - 1}{r}} \right)^{\frac{1}{r}}. \tag{15}$$

Remark 13. If we put

$$\sigma_1 = \cdots = \sigma_n = -s \qquad (s<0)$$

in (15), we get the sth function of degree r (that is, $t_n^{[r,s]}(\mathbf{x})$) and the Whiteley mean $\mathcal{W}_n^{[r,s]}(\mathbf{x})$ given by (14).

If we denote by μ a measure on the simplex $\Delta_{n-1} = \{(u_1, \ldots, u_{n-1}) : u_i \geq 0,\ i = 1, \ldots, n-1,\ \sum_{i=1}^{n-1} u_i \leq 1\}$ such that

$$d\mu(\mathbf{u}) = \frac{\Gamma(\sum_{i=1}^{n} \sigma_i)}{\prod_{i=1}^{n} \Gamma(\sigma_i)} \prod_{i=1}^{n} u_i^{\sigma_i - 1} du_1 \ldots du_{n-1},$$

where $u_n = 1 - \sum_{i=1}^{n-1} u_i$, then we have that μ is a probability measure and we can also write the mean $\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})$ in integral form as following

$$\mathcal{H}_n^{[r,\sigma]}(\mathbf{x}) = \left(\int_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i u_i \right)^r d\mu(\mathbf{u}) \right)^{\frac{1}{r}}.$$

Finally, just as we did above in this investigation, we can develop the following analogous results:

(i)
$$\frac{\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r - \left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^r}{\left(\mathcal{H}_n^{[l,\sigma]}(\mathbf{x})\right)^l - \left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^l} = \frac{r(r-s)}{l(l-s)} \cdot \eta^{r-l}$$

for some η such that

$$\min\{x_i\} \leqq \eta \leqq \max\{x_i\};$$

(ii)
$$\left| \frac{r(r-s)}{l(l-s)} \right| m \leqq \left| \frac{\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r - \left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^r}{\left(\mathcal{H}_n^{[l,\sigma]}(\mathbf{x})\right)^l - \left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^l} \right| \leqq \left| \frac{r(r-s)}{l(l-s)} \right| M,$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-l} $(i = 1, \ldots, n)$;

(iii)
$$\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^r - \left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^r = \frac{1}{2} \cdot \frac{r(r-s)}{s^2} \cdot \eta^{r-2s} \cdot \left[\left(\mathcal{H}_n^{[2s,\sigma]}(\mathbf{x})\right)^{2s} - \left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^{2s} \right]$$

for some η such that

$$\min\{x_i\} \leqq \eta \leqq \max\{x_i\};$$

(iv) $$\frac{1}{2}\cdot\left|\frac{r(r-s)}{s^2}\right|\cdot m\cdot\left[\left(\mathcal{H}_n^{[2s,\sigma]}(\mathbf{x})\right)^{2s}-\left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^{2s}\right]$$
$$\leqq\left|\left(\mathcal{H}_n^{[r,\sigma]}(\mathbf{x})\right)^{r}-\left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^{r}\right|$$
$$\leqq\frac{1}{2}\cdot\left|\frac{r(r-s)}{s^2}\right|\cdot M\cdot\left[\left(\mathcal{H}_n^{[2s,\sigma]}(\mathbf{x})\right)^{2s}-\left(\mathcal{H}_n^{[s,\sigma]}(\mathbf{x})\right)^{2s}\right],$$

where m and M are, respectively, the minimum and the maximum values of x_i^{r-2s} $(i=1,\ldots,n)$.

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[1,2] *Faculty of Textile Technology, University of Zagreb*
Prilaz baruna Filipovića 30
HR-10000 Zagreb, CROATIA
e-mail: pecaric@hazu.hr , mrodic@ttf.hr
Received: November 10, 2006

[3] *Department of Mathematics and Statistics*
University of Victoria, Victoria
British Columbia V8W 3P4, CANADA
e-mail: harimsri@math.uvic.ca