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HERZ-TYPE HARDY SPACES FOR THE DUNKL OPERATOR ON THE REAL LINE *

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Dedicated to Professor Khalifa Trimèche,
for his 60th birthday

Abstract

We introduce some new weighted Herz spaces associated with the Dunkl operator on $\mathbb{R}$. Also we characterize by atomic decompositions the corresponding Herz-type Hardy spaces. As applications we investigate the Dunkl transform on these spaces and establish a version of Hardy inequality for this transform.

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1. Introduction

In the last years Herz and Herz-type Hardy spaces in the Euclidean case have been intensively considered in [11,12,13]. These spaces turn out to be very useful in the study of the sharpness of multiplier theorems on H^p spaces (see [14]).

In this work, we consider certain weighted Herz spaces, next we define the corresponding Hardy spaces in terms of the Dunkl analysis.

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The Dunkl analysis with respect to $\alpha \geq -1/2$ concerns the Dunkl operator Λ_α, the Dunkl transform $\mathcal{F}_\alpha$, the multiplication $*_\alpha$, and a certain measure μ_α on $\mathbb{R}$. In the limit case $\alpha = -1/2$, then Λ_α, $\mathcal{F}_\alpha$, $*_\alpha$ and μ_α agree with the operator d/dx, the Fourier transform, the standard convolution and the weighted Lebesque measure $\frac{1}{\sqrt{2\pi}}dx$, respectively.

The Dunkl operators on $\mathbb{R}^n$ in [7] are differential-difference operators associated with some finite reflection groups. They are important in pure mathematics and in certain parts of quantum mechanics and one expects that the results in this paper will be useful when discussing continuity properties in Dunkl analysis. Furthermore, these operators provide a useful tool in the study of special functions associated with root systems (cf. [8,9,20,24]). They are closely related to certain representations of degenerated affine Hecke algebras (see [4,16]). Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in one dimensional space (cf. [1,10]).

The paper is organized as follows. In Section 2 we recall some results about harmonic analysis associated with the Dunkl operator on $\mathbb{R}$. Then, we define the α-grand maximal function of N-order $G_{\alpha,N}$.

In Section 3, using the α-grand maximal function $G_{\alpha,N}$, we define for $0 < p \leq 1 < q \leq \infty$, $\beta \geq 1 - 1/q$ and large $N \in \mathbb{N}$:

- The homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$, by the space of functions f in $L^q_{loc}(\mu_\alpha)$ such that $\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \|f\chi_k\|^p_{q,\alpha} < \infty$, where $L^q_{loc}(\mu_\alpha)$ is the space of functions f such that $|f|^q$ is locally integrable with respect to the measure $d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}dx$ and χ_k is the characteristic function of the set $\{x \in \mathbb{R} \,/\, 2^{k-1} \leq |x| \leq 2^k\}$.

- The Herz-type Hardy spaces $H\dot{K}^{\beta,p,N}_{\alpha,q}$ are as follows:
$$H\dot{K}^{\beta,p,N}_{\alpha,q} := \Big\{ f \in \mathcal{S}'(\mathbb{R}) \;/\; G_{\alpha,N}(f) \in \dot{K}^{\beta,p}_{\alpha,q} \Big\}.$$
We study the continuity property of the operator $G_{\alpha,N}$ on these spaces. Next we establish their characterizations in terms of decompositions into central atoms.

In Section 4, the atomic decomposition allows us to study the Dunkl transform $\mathcal{F}_\alpha$ on the Herz-type Hardy spaces $H\dot{K}^{\beta,p,N}_{\alpha,q}$. In particular, we establish the following version of Hardy inequality for $\mathcal{F}_\alpha$:
$$\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(y)| \frac{dy}{|y|} \leq C \, \|f\|_{H\dot{K}^{1/2,1}_{\alpha,2}}.$$
In the classical case this property is studied in [5,6].

Throughout the paper we use the classic notation. Thus $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are the Schwartz space on $\mathbb{R}$ and the space of tempered distributions on $\mathbb{R}$ respectively. Finally, C denotes a positive constant whose value may vary from line to line.

2. Preliminaries

We recall first some basic definitions and facts. We consider the Dunkl operator Λ_α, $\alpha \geq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha f(x) := \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\Big[\frac{f(x)-f(-x)}{2}\Big].$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial problem:

$$\Lambda_\alpha f(x) = \lambda f(x), \quad f(0) = 1, \tag{1}$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel given by

$$E_\alpha(\lambda x) = \Im_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha+1)}\Im_{\alpha+1}(\lambda x), \quad x \in \mathbb{R},$$

where $\Im_\alpha$ is the modified Bessel function of order α given by

$$\Im_\alpha(\lambda x) := \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(\lambda x/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$. See [8,9,18] and [25].

Furthermore, the Dunkl kernel $E_\alpha(\lambda x)$ can be expanded in a power series in the form:

$$E_\alpha(\lambda x) = \sum_{n=0}^{\infty}\frac{(\lambda x)^n}{b_n(\alpha)}\ , \quad b_n(\alpha) = \frac{2^n(\{n/2\})!}{\Gamma(\alpha+1)}\Gamma(\{\frac{n+1}{2}\}+\alpha+1), \tag{2}$$

where $\{a\}$ is the integer part of $a \in [0,\infty[$ (see [18]).

Let

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}dx.$$

We denote by $L^p(\mu_\alpha)$, $p \in [1,\infty]$, the Lebesque space on $\mathbb{R}$ with respect to the measure μ_α. In the following we use the shorter notation $\|f\|_{p,\alpha}$ instead of $\|f\|_{L^p(\mu_\alpha)}$.

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on $\mathbb{R}$, which was introduced and studied in [9].

The Dunkl transform of a function $f \in L^1(\mu_\alpha)$, is given by

$$\mathcal{F}_\alpha(f)(y) := \int_{\mathbb{R}} E_\alpha(-ixy)f(x)d\mu_\alpha(x), \quad y \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix)| \leq 1$ for every $x \in \mathbb{R}$ ([17, p.295]). Note that $\mathcal{F}_{-1/2}$ agrees with the Fourier transform $\mathcal{F}$, given by:

$$\mathcal{F}(f)(y) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ixy} f(x)\, dx, \quad y \in \mathbb{R}.$$

Proposition 1. (See [24, p.25,26])

i) For all $f \in L^1(\mu_\alpha)$, we have $\|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \leq \|f\|_{1,\alpha}$.

ii) For all $f \in \mathcal{S}(\mathbb{R})$, we have $\mathcal{F}_\alpha(\Lambda_\alpha f)(y) = iy\, \mathcal{F}_\alpha(f)(y)$, $y \in \mathbb{R}$.

iii) $\mathcal{F}_\alpha$ is a topological isomorphism on $\mathcal{S}(\mathbb{R})$ which extends to a topological isomorphism on $\mathcal{S}'(\mathbb{R})$.

Theorem 1. (See[9,24])

i) Plancherel theorem: The Dunkl transform $\mathcal{F}_\alpha$ is an isometric isomorphism of $L^2(\mu_\alpha)$. In particular, $\|f\|_{2,\alpha} = \|\mathcal{F}_\alpha(f)\|_{2,\alpha}$.

ii) Inversion formula: Let f be a function in $L^1(\mu_\alpha)$, such that $\mathcal{F}_\alpha(f) \in L^1(\mu_\alpha)$, then

$$\mathcal{F}_\alpha^{-1}(f)(x) = \mathcal{F}_\alpha(f)(-x), \quad a.e.\ x \in \mathbb{R}.$$

Notation. For all $x, y, z \in \mathbb{R}$, we put:

$$W_\alpha(x,y,z) := \Big[1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\Big] \Delta_\alpha(|x|,|y|,|z|), \tag{3}$$

where

$$\sigma_{x,y,z} := \begin{cases} \frac{x^2+y^2-z^2}{2xy}, & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\Delta_\alpha(|x|,|y|,|z|) := \begin{cases} d_\alpha \dfrac{\Big[\Big((|x|+|y|)^2 - z^2\Big)\Big(z^2 - (|x|-|y|)^2\Big)\Big]^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y} \\ 0, & \text{otherwise }, \end{cases}$$

$$d_\alpha = 2^{1-\alpha}(\Gamma(\alpha+1))^2/\sqrt{\pi}\,\Gamma(\alpha+1/2), \quad A_{x,y} = \Big[\Big||x|-|y|\Big|, |x|+|y|\Big].$$

Remark. (See [17]). The signed kernel W_α is even and satisfies:

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z),$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\, d\mu_\alpha(z) \leq 4.$$

Theorem 2. (See [17])

i) Let $\alpha > -1/2,\ \lambda \in \mathbb{C}$. The Dunkl kernel E_α satisfies the following product formula:

$$E_\alpha(\lambda x)E_\alpha(\lambda y) = \int_{\mathbb{R}} E_\alpha(\lambda z) d\nu_{x,y}(z); \quad x, y \in \mathbb{R},$$

where $\nu_{x,y}$ is a signed measures given by

$$d\nu_{x,y}(z) = \begin{cases} W_\alpha(x,y,z)d\mu_\alpha(z), & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0. \end{cases}$$

ii) The measures $\nu_{x,y}$ have the following properties:

$$\text{supp } (\nu_{x,y}) = A_{x,y} \cup (-A_{x,y}), \quad \|\nu_{x,y}\| := \int_{\mathbb{R}} d|\nu_{x,y}|(z) \leq 4.$$

The Dunkl translation operators $\tau_x, x \in \mathbb{R}$ are defined for a continuous function f on $\mathbb{R}$, by

$$\tau_x f(y) := \int_{\mathbb{R}} f(z) d\nu_{x,y}(z), \quad y \in \mathbb{R}.$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the Dunkl multiplication $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The multiplication $*_\alpha$ is associative and commutative ([17]). Note that $*_{-1/2}$ agrees with the standard convolution $*$.

The following two propositions are shown in [19].

Proposition 2.

i) For all $x \in \mathbb{R}$ and $f \in L^q(\mu_\alpha)$, $q \in [1, \infty]$:

$$\|\tau_x f\|_{q,\alpha} \leq 4\, \|f\|_{q,\alpha}.$$

ii) For all $x \in \mathbb{R}$ and $f \in L^1(\mu_\alpha)$:

$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_\alpha(ix\lambda)\, \mathcal{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R}.$$

Proposition 3.

i) Assume that $q, q', r \in [1, \infty]$ *satisfy the Young condition* $1/q + 1/q' = 1 + 1/r$. *Then the map* $(f, g) \to f *_\alpha g$ *extends to a continuous map from* $L^q(\mu_\alpha) \times L^{q'}(\mu_\alpha)$ *to* $L^r(\mu_\alpha)$, *and we have*

$$\|f *_\alpha g\|_{r,\alpha} \leq 4 \, \|f\|_{q,\alpha} \|g\|_{q',\alpha}.$$

ii) For all $f \in L^1(\mu_\alpha)$ *and* $g \in L^2(\mu_\alpha)$, *we have*

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f) \, \mathcal{F}_\alpha(g).$$

Proposition 4.

i) The operators τ_x, $x \in \mathbb{R}$, *are continuous from* $\mathcal{S}(\mathbb{R})$ *into itself.*

ii) For all $f \in \mathcal{S}(\mathbb{R})$ *and* $x \in \mathbb{R}$, *we have* $\Lambda_\alpha(\tau_x f) = \tau_x(\Lambda_\alpha f)$.

iii) For all $f \in \mathcal{S}'(\mathbb{R})$ *and* $\phi \in \mathcal{S}(\mathbb{R})$ *such that* $\int_{\mathbb{R}} \phi(x) d\mu_\alpha(x) = 1$, *we have*

$$\lim_{t \to 0} f *_\alpha \phi_t = f, \quad \text{in } S'(\mathbb{R}),$$

where ϕ_t *is the dilation of* ϕ *given by*

$$\phi_t(x) := t^{-2(\alpha+1)} \phi\Big(\frac{x}{t}\Big), \quad x \in \mathbb{R}. \tag{4}$$

We will make use the Hardy-Littlewood maximal function. For a locally integrable function f on $\mathbb{R}$, we define its maximal function $\mathcal{M}_\alpha(f)$, by

$$\mathcal{M}_\alpha(f)(x) := \sup_{t>0} \Big\{ \frac{1}{\mu_\alpha(]-t, t[)} \int_{-t}^{t} |\tau_x(f)(y)| \, d\mu_\alpha(y) \Big\}, \quad x \in \mathbb{R}.$$

This operator satisfies the following properties.

Proposition 5. *For all* $q \in]1, \infty[$, *the operator* $\mathcal{M}_\alpha$ *is continuous from* $L^q(\mu_\alpha)$ *into itself.*

P r o o f. Since the operator $\mathcal{M}_\alpha$ is sub-linear it suffices to show the result for non-negative functions only. From (3), we have

$$|W(x, y, z)| \leq 4 \, \Delta_\alpha(|x|, |y|, |z|), \quad |z| \in A_{x,y},$$

where Δ_α is the Bessel kernel introduced in (3).

We write $f = f_e + f_o$ with f_e even and f_o odd, then

$$|\tau_x f(y)| \leq 8 \int_{\big||x|-|y|\big|}^{|x|+|y|} f_e(z) \Delta_\alpha(|x|, |y|, |z|) d\mu_\alpha(z). \tag{5}$$

Thus

$$\mathcal{M}(f)(x) \leq 4\, f_e^*(|x|),$$

where f_e^* is the maximal function of f_e on the Bessel-Kingman hypergroups [3,23]. Therefore by using [3, p.58] (see also [23]), there exists a constant $C_q > 0$, so that

$$\|\mathcal{M}_\alpha(f)\|_{q,\alpha} \leq C_q\, \|f_e\|_{q,\alpha} \leq C_q\, \|f\|_{q,\alpha},$$

which proves the result. ∎

For all $N \in \mathbb{N}$, we denote by F_N the subset of $\mathcal{S}(\mathbb{R})$ constituted by all those $\phi \in \mathcal{S}(\mathbb{R})$ such that $\mathrm{supp}(\phi) \subset [-1,1]$ and for all $m, n \in \mathbb{N}$ such that $m, n \leq N$, we have

$$\rho_{m,n}(\phi) := \sup_{x\in\mathbb{R}} (1+|x|)^m |\Lambda_\alpha^n \phi(x)| \leq 1. \tag{6}$$

Moreover the system of semi-norms $\{\rho_{m,n}\}_{m,n\in\mathbb{N}}$ generates the topology of $S(\mathbb{R})$ (see [2]).

Let $f \in \mathcal{S}'(\mathbb{R})$ and $N \in \mathbb{N}$. We define the α-grand maximal function of N-order $G_{\alpha,N}(f)$ of f, by

$$G_{\alpha,N}(f)(x) := \sup_{t>0,\, \phi\in F_N} |\phi_t *_\alpha f(x)|, \quad x \in \mathbb{R},$$

where ϕ_t is the dilation of ϕ given by (4).

According to Proposition 4 and by proceeding in a standard way as in [21,22], we obtain the following.

Corollary 1. *The α-grand maximal function $G_{\alpha,N}$ is a bounded continuous operator from $L^q(\mu_\alpha)$ into itself, for every $q \in]1,\infty]$, provided that $N > 2(\alpha+1)$.*

3. Herz-type Hardy spaces

In this section we describe certain weighted Herz-type Hardy spaces in terms of the Dunkl analysis.

Definition 1. Let $\beta \in \mathbb{R}$, $p \in]0,\infty[$ and $q \in [1,\infty]$.

i) The homogeneous weighted Herz space $\dot{K}_{\alpha,q}^{\beta,p}$ is the space constituted by all the functions $f \in L_{loc}^q(\mu_\alpha)$, such that

$$\|f\|_{\dot{K}_{\alpha,q}^{\beta,p}} := \Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp} \|f\chi_k\|_{q,\alpha}^p\Big]^{1/p} < \infty,$$

where χ_k is the characteristic function of $A_k := \{x \in \mathbb{R} \,/\, 2^{k-1} \leq |x| \leq 2^k\}$.

ii) The non-homogeneous weighted Herz space $K^{\beta,p}_{\alpha,q}$ is defined, as usual, by $K^{\beta,p}_{\alpha,q} := L^q(\mu_\alpha) \cap \dot{K}^{\beta,p}_{\alpha,q}$. Moreover, $\|f\|_{K^{\beta,p}_{q,\alpha}} := \|f\|_{q,\alpha} + \|f\|_{\dot{K}^{\beta,p}_{\alpha,q}}$.

Note that $\dot{K}^{0,q}_{\alpha,q} = K^{0,q}_{\alpha,q} = L^q(\mu_\alpha)$.

Remark. By proceeding as in [13], we can obtain blocks decompositions by the elements of the Herz spaces $\dot{K}^{\beta,p}_{\alpha,q}$ and $K^{\beta,p}_{\alpha,q}$.

We now study some properties for $\dot{K}^{\beta,p}_{\alpha,q}$. It is remarkable that we can establish similar results for $K^{\beta,p}_{\alpha,q}$. For simplicity, we prove our results in the homogeneous version.

Proposition 6. *Let $p \in]0,\infty[$, $q \in]1,\infty]$ and $-1/q < \beta < 1 - 1/q$. Then the operator $G_{\alpha,N}$, $N > 2(\alpha+1)$ is continuous from $\dot{K}^{\beta,p}_{\alpha,q}$ into itself.*

Proof. Assume that f is a compactly supported and integrable function on $\mathbb{R}$. For $x \in \mathbb{R}$ and $t > 0$, we have

$$\phi_t *_\alpha f(x) = \int_{\mathbb{R}} \tau_x \phi_t(y) f(-y) d\mu_\alpha(y),$$

where ϕ_t is the dilation of ϕ given by (4).

Using the fact $\tau_x\phi_t(y) = t^{-2(\alpha+1)}\tau_{x/t}\phi\Big(\frac{y}{t}\Big)$, we obtain

$$\phi_t *_\alpha f(x) = t^{-2(\alpha+1)} \int_{\mathbb{R}} \tau_{x/t}\phi\Big(\frac{y}{t}\Big) f(-y) d\mu_\alpha(y).$$

Let $\phi \in F_N$. Then from (5) we have

$$\Big|\tau_{x/t}\phi\Big(\frac{y}{t}\Big)\Big| \le 4 \int_{\big||x|-|y|\big|/t}^{(|x|+|y|)/t} \Delta_\alpha\Big(\frac{|x|}{t}, \frac{|y|}{t}, |z|\Big) \Big[|\phi(z)| + |\phi(-z)|\Big]\, d\mu_\alpha(z). \quad (7)$$

Thus we deduce that

$$\Big|\tau_{x/t}\phi\Big(\frac{y}{t}\Big)\Big| \le C\left(\Big||x|-|y|\Big|/t\right)^{-2(\alpha+1)}; \quad x, y \in \mathbb{R} \text{ and } t > 0.$$

Hence, we conclude for $x \notin \mathrm{supp} f$ that

$$|G_{\alpha,N}(f)(x)| \le C \int_{\mathbb{R}} \frac{|f(y)|}{\Big||x|-|y|\Big|^{2(\alpha+1)}} d\mu_\alpha(y). \quad (8)$$

To prove our result we will use a procedure similar to the one developed in the proof of [14,Theorem 1]. Assume that $p, q \in]1,\infty[$.

Let $f \in \dot{K}^{\beta,p}_{\alpha,q}$. We can write

$$\|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}} = \Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)k\beta p} \|G_{\alpha,N}(f)\chi_k\|^p_{q,\alpha}\Big]^{1/p} \le E_1 + E_2 + E_3,$$

where

$$E_1 = \Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)k\beta p} \Big(\sum_{l=-\infty}^{k-3} \|G_{\alpha,N}(f\chi_l)\chi_k\|_{q,\alpha} \Big)^p \Big]^{1/p},$$

$$E_2 = \Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)k\beta p} \Big(\sum_{l=k-2}^{k+2} \|G_{\alpha,N}(f\chi_l)\chi_k\|_{q,\alpha} \Big)^p \Big]^{1/p}$$

and

$$E_3 = \Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)k\beta p} \Big(\sum_{l=k+3}^{\infty} \|G_{\alpha,N}(f\chi_l)\chi_k\|_{q,\alpha} \Big)^p \Big]^{1/p}.$$

We now analyze E_j, $j = 1, 2, 3$.

If $k \in \mathbb{Z}$, $x \in A_k$ and $l \leq k-3$, then according to (8) and Hölder's inequality, we obtain

$$\begin{aligned} |G_{\alpha,N}(f\chi_l)(x)| &\leq C \int_{A_l} \frac{|f(y)|}{\big||x|-|y|\big|^{2(\alpha+1)}} d\mu_\alpha(y) \\ &\leq \frac{C}{(2^{\alpha+1}\Gamma(\alpha+1))^{1-1/q}} 2^{2(\alpha+1)(l-k-l/q)} \|f\chi_l\|_{q,\alpha}. \end{aligned}$$

Hence, if $0 < \gamma < 1$ and $\beta < \gamma(1-1/q)$, then

$$\begin{aligned} E_1 &\leq C \Big[\sum_{k=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{k-3} 2^{2(\alpha+1)[k\beta+(l-k)(1-1/q)]} \|f\chi_l\|_{q,\alpha} \Big)^p \Big]^{1/p} \\ &\leq C \Big[\sum_{k=-\infty}^{\infty} S_k(\gamma,p) \Big(\sum_{l=-\infty}^{k-3} 2^{2(\alpha+1)p[k\beta+\gamma(l-k)(1-1/q)]} \|f\chi_l\|_{q,\alpha}^p \Big) \Big]^{1/p}, \end{aligned}$$

where

$$S_k(\gamma,p) = \Big(\sum_{l=-\infty}^{k-3} 2^{2(\alpha+1)p'(l-k)(1-\frac{1}{q})(1-\gamma)} \Big)^{p/p'}, \quad 1/p + 1/p' = 1.$$

Since $S_k(\gamma,p) \leq C$, then

$$\begin{aligned} E_1 &\leq C \Big[\sum_{l=-\infty}^{\infty} 2^{2(\alpha+1)pl\beta} \|f\chi_l\|_{q,\alpha}^p \sum_{k=l+3}^{\infty} 2^{2(\alpha+1)p(k-l)[\beta-\gamma(1-1/q)]} \Big]^{1/p} \\ &\leq C \Big[\sum_{l=-\infty}^{\infty} 2^{2(\alpha+1)pl\beta} \|f\chi_l\|_{q,\alpha}^p \Big]^{1/p} = C\,\|f\|_{\dot{K}_{q,\alpha}^{\beta,p}}. \end{aligned}$$

To estimate E_2, from Corollary 1, we get

$$\begin{aligned} E_2 &\leq C\Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)k\beta p}\Big(\sum_{l=k-2}^{k+2}\|f\chi_l\|_{q,\alpha}\Big)^p\Big]^{1/p} \\ &\leq C\Big[\sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)k\beta p}\sum_{l=k-2}^{k+2}\|f\chi_l\|_{q,\alpha}^p\Big]^{1/p} \leq C\,\|f\|_{\dot{K}_{q,\alpha}^{\beta,p}}. \end{aligned}$$

Finally, if k, $l\in\mathbb{Z}$, $x\in A_k$ and $l\geq k+3$, from (8), we deduce that

$$|G_{\alpha,N}(f\chi_l)(x)|\leq\frac{C}{(2^{\alpha+1}\Gamma(\alpha+1))^{1-1/q}}2^{-2(\alpha+1)l/q}\|f\chi_l\|_{q,\alpha}.$$

Then, by proceeding as in the analysis of E_1, it follows that

$$E_3\leq C\,\|f\|_{\dot{K}_{\alpha,q}^{\beta,p}}.$$

Thus we conclude that $G_{\alpha,N}$ is bounded from $\dot{K}_{\alpha,q}^{\beta,p}$ into itself. ■

Definition 2. Let $N\in\mathbb{N}$, $\beta\in\mathbb{R}$, $p\in]0,\infty]$ and $q\in]1,\infty]$. The Herz-type Hardy space $H\dot{K}_{\alpha,q}^{\beta,p,N}$ is the space of distributions $f\in\mathcal{S}'(\mathbb{R})$ such that $G_{\alpha,N}(f)\in\dot{K}_{\alpha,q}^{\beta,p}$. Moreover, we define

$$\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p,N}}:=\|G_{\alpha,N}(f)\|_{\dot{K}_{\alpha,q}^{\beta,p}}.$$

Note that as in the same we define the space $HK_{\alpha,q}^{\beta,p,N}$ for the non-homogeneous case.

In particular, we have the following

Lemma 1. *Let $N>2(\alpha+1)$, $p\in]0,\infty]$, $q\in]1,\infty]$ and $-1/q<\beta<1-1/q$. Then*

$$H\dot{K}_{\alpha,q}^{\beta,p,N}=\dot{K}_{\alpha,q}^{\beta,p}.$$

P r o o f. Let $f\in\dot{K}_{\alpha,q}^{\beta,p}$, from Proposition 6 we deduce that $G_{\alpha,N}(f)\in\dot{K}_{\alpha,q}^{\beta,p}$. Hence $f\in H\dot{K}_{\alpha,q}^{\beta,p,N}$.

Conversely, let $f\in H\dot{K}_{\alpha,q}^{\beta,p,N}$ and $\phi\in F_N$ such that $\displaystyle\int_{\mathbb{R}}\phi(x)d\mu_\alpha(x)=1$. Since

$$\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p,N}}:=\Big[\sum_{k=-\infty}^{\infty}2^{2(\alpha+1)\beta kp}\|G_{\alpha,N}(f)\chi_k\|_{q,\alpha}^p\Big]^{1/p},$$

we deduce that for every $k\in\mathbb{N}$, $G_{\alpha,N}(f)\chi_k$ is bounded in $L^q(\mu_\alpha)$.

On the other hand, let $0<a<b<\infty$. Since $\mathrm{supp}(\phi)\subset[-1,1]$, we can write

$$f *_\alpha \phi_t(x) = \int_{J_{x,t}} f(-y) \int_{\left||x|-|y|\right|}^{|x|+|y|} \phi_t(z) d\nu_{x,y}(z) d\mu_\alpha(y)$$
$$+ \int_{J_{x,t}} f(-y) \int_{-|x|-|y|}^{-\left||x|-|y|\right|} \phi_t(z) d\nu_{x,y}(z) d\mu_\alpha(y),$$

where $J_{x,t} = [-|x|-t, -|x|+t] \cup [|x|-t, |x|+t]$.

Then for $|x| \in [a,b]$ and $t \in (0, a/2)$, we obtain

$$f *_\alpha \phi_t(x) = \int_{\mathcal{J}_{a,b}} f(-y) \tau_x \phi_t(y) d\mu_\alpha(y),$$

where $\mathcal{J}_{a,b} = [-a/2-b, -a/2] \cup [a/2, a/2+b]$.

Hence $f *_\alpha \phi_t(x) = g *_\alpha \phi_t(x)$, $|x| \in [a,b]$, for a certain $g \in L^q(\mu_\alpha)$, being $g(x) = f(x)$, $|x| \in [a,b]$, when t is small enough.

By a standard argument, we have $\lim_{t\to 0} g *_\alpha \phi_t = g$, a.e. $x \in \mathbb{R}$. Then,

$$\lim_{t\to 0} f *_\alpha \phi_t = f, \quad \text{a.e. } |x| \in [a,b].$$

Thus we show that

$$|f(x)| \leq G_{\alpha,N}(f)(x), \quad \text{a.e. } |x| \in [a,b].$$

From this inequality and since $G_{\alpha,N}(f)\chi_k$ is bounded in $L^q(\mu_\alpha)$, we deduce that $f \in L^q_{loc}(\mu_\alpha)$ and $\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}} \leq \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}} < \infty$. It concludes that $f \in \dot{K}^{\beta,p}_{\alpha,q}$. ■

In the sequel, we are interested in the spaces $H\dot{K}^{\beta,p,N}_{\alpha,q}$, when $\beta \geq 1-1/q$. Now, we turn to the atomic characterization of the space $H\dot{K}^{\beta,p,N}_{\alpha,q}$.

DEFINITION 3. Let $q \in]1,\infty]$ and $\beta \geq 1-1/q$. A measurable function a on $\mathbb{R}$ is called a (central) (β,q)-atom if it satisfies:

(i) $\mathrm{supp}(a) \subset [-r,r]$, for a certain $r > 0$,

(ii) $\|a\|_{q,\alpha} \leq r^{-2(\alpha+1)\beta}$,

(iii) $\int_{\mathbb{R}} a(x) x^k d\mu_\alpha(x) = 0, \ k = 0,1,...,2s+1,$

where $s = \{(\alpha+1)(\beta-1+1/q)\}$ (the integer part of $(\alpha+1)(\beta-1+1/q)$).

THEOREM 3. *Let $0 < p \leq 1 < q \leq \infty$, $\beta \geq 1-1/q$ and $N \in \mathbb{N}$, $N > 2(2s+3+\alpha)$. Then $f \in H\dot{K}^{\beta,p,N}_{\alpha,q}$ if and only if there exist, for all $j \in \mathbb{N}\backslash\{0\}$, an (β,q)-atom a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and $f = \sum_{j=1}^{\infty} \lambda_j a_j$. Moreover,*

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} \sim \inf\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p},$$

where the infimum is taken over all atomic decompositions of f.

P r o o f. We first *verify the necessity*: Suppose a is an (β, q)-atom and assume that $q < \infty$ (when $q = \infty$ we can proceed analogously). It is enough to verify that $\|G_{\alpha,N}(a)\|_{\dot{K}^{\beta,p}_{\alpha,q}} \le C$, where C is a constant independent of a. Let $\mathrm{supp}(a) \subset [-r, r]$ and $2^{k_0} < r < 2^{k_0+1}$ for some $k_0 \in \mathbb{Z}$. We write

$$\|G_{\alpha,N}(a)\|^p_{\dot{K}^{\beta,p}_{\alpha,q}} = \sum_{k=-\infty}^{\infty} 2^{2(\alpha+1)\beta kp}\|G_{\alpha,N}(a)\chi_k\|^p_{q,\alpha} := I_1(k_0) + I_2(k_0),$$

where

$$I_1(k_0) = \sum_{k=-\infty}^{k_0+3} 2^{2(\alpha+1)\beta kp}\|G_{\alpha,N}(a)\chi_k\|^p_{q,\alpha},$$

and

$$I_2(k_0) = \sum_{k=k_0+4}^{\infty} 2^{2(\alpha+1)\beta kp}\|G_{\alpha,N}(a)\chi_k\|^p_{q,\alpha}.$$

For $I_1(k_0)$ we have

$$I_1(k_0) \le \|G_{\alpha,N}(a)\|^p_{q,\alpha}\sum_{k=-\infty}^{k_0+3} 2^{2(\alpha+1)\beta kp}.$$

Applying Proposition 5 and (ii) of Definition 3, we obtain

$$I_1(k_0) \le C\,\|a\|^p_{q,\alpha}2^{2(\alpha+1)\beta k_0 p} \le C,$$

where C is a constant that does not depend on the (β, q)-atom a.

In order to estimate $I_2(k_0)$, we need a pointwise estimate of $G_{\alpha,N}(a)(x)$ on A_k for $k \ge k_0+4$. Suppose now that $\phi \in F_N$. According to [15, Theorem 2], ϕ admits a generalized Taylor formula with integral remainder

$$\phi(x) = \sum_{k=0}^{n}\frac{\Lambda^k_\alpha\phi(0)}{b_k(\alpha)}x^k + \int_{-|x|}^{|x|} w_n(x,y)\Lambda^{n+1}_\alpha\phi(y)d\mu_\alpha(y), \tag{9}$$

where $b_n(\alpha)$ given by (2) and $w_n(x, y)$ a kernel satisfying:

$$\int_{-|x|}^{|x|} w_n(x,y)d\mu_\alpha(y) \le c_n(\alpha)|x|^{n+1}, \tag{10}$$

where

$$c_n(\alpha) = \frac{1}{2^{2\alpha+1}\Gamma(\alpha+1)}\Big[\frac{1}{b_{n+1}(\alpha)} + \frac{1}{b_n(\alpha)}\Big].$$

Then, if $x \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \le 2s + 1$, (iii) of Definition 3 allows to write

$$a *_\alpha \phi_t(x) = \int_{\mathbb{R}} \int_{-|x|}^{|x|} a(-y) w_n(y,z) \tau_x(\Lambda_\alpha^{n+1}\phi_t)(z) d\mu_\alpha(z) d\mu_\alpha(y),$$

where ϕ_t is the dilation of ϕ given by (4).

Using the fact $\Lambda_\alpha^{n+1}(\phi_t)(z) = t^{-2\alpha-n-3}\phi\Big(\frac{z}{t}\Big)$, we obtain

$$a *_\alpha \phi_t(x) = t^{-2\alpha-n-3} \int_{\mathbb{R}} \int_{-|x|}^{|x|} a(-y) w_n(y,z) \tau_{x/t}(\Lambda_\alpha^{n+1}\phi)\Big(\frac{z}{t}\Big) d\mu_\alpha(z) d\mu_\alpha(y).$$

Let $(\Lambda_\alpha^{n+1}\phi)_e$ be the even part of $\Lambda_\alpha^{n+1}\phi$, then from (7), we have

$$\begin{aligned} \Big|\tau_{x/t}(\Lambda_\alpha^{n+1}\phi)\Big(\frac{z}{t}\Big)\Big| &\leq 8 \int_{\big||x|-|z|\big|/t}^{(|x|+|z|)/t} \Delta_\alpha(\frac{|x|}{t}, \frac{|z|}{t}, u) |(\Lambda_\alpha^{n+1}\phi)_e(u)| d\mu_\alpha(u) \\ &\leq 8\rho_{m,n+1}(\phi) \int_{\big||x|-|z|\big|/t}^{(|x|+|z|)/t} (1+|u|)^{-m} \Delta_\alpha(\frac{|x|}{t}, \frac{|z|}{t}, u) d\mu_\alpha(u) \\ &\leq 4\rho_{m,n+1}(\phi)\Big(1 + \Big||x|-|z|\Big|/t\Big)^{-m}. \end{aligned}$$

Here $\rho_{m,n}(\phi)$ are the semi-norms given by (6). Hence,

$$|a *_\alpha \phi_t(x)| \leq \frac{4\rho_{m,n+1}(\phi)}{t^{2\alpha+n+3}} \int_{\mathbb{R}} \int_{-|y|}^{|y|} \frac{|a(-y)||w_n(y,z)|}{\Big(1 + \Big||x|-|z|\Big|/t\Big)^m} d\mu_\alpha(z) d\mu_\alpha(y).$$

From (i) of Definition 3, there exists a constant $\theta_y \in [-|y|, |y|]$, such that $\Big||x| - |\theta_y|\Big| \leq \Big||x| - |z|\Big|$, for all $z \in [-|y|, |y|]$.

Thus,

$$|a *_\alpha \phi_t(x)| \leq \frac{4c_n(\alpha)\rho_{m,n+1}(\phi)}{t^{n-m+2\alpha+3}} \int_{\mathbb{R}} |y|^{n+1} |a(-y)| \Big(t + \Big||x| - |\theta_y|\Big|\Big)^{-m} d\mu_\alpha(y).$$

Since $c_n(\alpha) \leq \dfrac{1}{2^{2\alpha}\Gamma(\alpha+1)}$, putting $n = 2s+1$ and $m = 2(s+\alpha+2)$, then for $N \geq 2(2s+\alpha+3)$, we get

$$|a *_\alpha \phi_t(x)| \leq C\, r^{2(s+1)} \int_{\mathbb{R}} |a(-y)| \Big(t + \Big||x| - |\theta_y|\Big|\Big)^{-2(s+\alpha+2)} d\mu_\alpha(y).$$

By proceeding as in [11, p.108], we obtain

$$|a *_\alpha \phi_t(x)| \leq C \frac{r^{2(s+1)}}{|x|^{2(s+\alpha+2)}} \int_{-r}^{r} |a(y)| d\mu_\alpha(y).$$

Applying Hölder's inequality and (ii) of Definition 3, we obtain

$$
\begin{aligned}
|a *_\alpha \phi_t(x)| &\leq C\,\frac{r^{2(s+1)}}{|x|^{2(s+\alpha+2)}}\Big[\int_{-r}^{r}|a(y)|^q d\mu_\alpha(y)\Big]^{1/q}\Big[\int_{-r}^{r}d\mu_\alpha(y)\Big]^{1-1/q}\\
&\leq \frac{C\,r^u}{(2^{\alpha+1}\Gamma(\alpha+1))^{1-1/q}|x|^{2(s+\alpha+2)}},
\end{aligned}
$$

where $u = 2[s+1+(\alpha+1)(\beta+1-1/q)]$.

Using the fact that $2^{k_0} \leq r \leq 2^{k_0+1}$, we obtain

$$
|a *_\alpha \phi_t(x)| \leq \frac{C\,2^{uk_0}}{(2^{\alpha+1}\Gamma(\alpha+1))^{1-1/q}|x|^{2(s+\alpha+2)}}.
$$

Then for $x \in A_k$, $k \geq k_0+4$, we get

$$
|G_{\alpha,N}(a)(x)| \leq \frac{C\,2^{uk_0}}{(2^{\alpha+1}\Gamma(\alpha+1))^{1-1/q}|x|^{2(s+\alpha+2)}}.
$$

Hence, it follows that

$$
\begin{aligned}
I_2(k_0) &\leq C\,2^{puk_0}\sum_{k=k_0+4}^{\infty}2^{2(\alpha+1)\beta kp}\Big[2\int_{2^k}^{2^{k+1}}x^{-2(s+\alpha+2)q+2\alpha+1}dx\Big]^{p/q}\\
&\leq C\,2^{puk_0}\sum_{k=k_0+4}^{\infty}2^{2pk(\alpha+1)\beta-(\alpha+2+s)+(\alpha+1)/q}.
\end{aligned}
$$

Because $(\alpha+1)(\beta-1+1/q) < s+1$, then $I_2(k_0) \leq C$, where C a constant not depending on the (β,q)-atom a. Hence this finishes the proof of the necessity.

Now, we turn to *the proof of the sufficiency*: Suppose that $f \in H\dot{K}_{\alpha,q}^{\beta,p,N}$. To see that $f = \sum_{j=1}^{\infty}\lambda_j a_j$, where the series converges in $\mathcal{S}'(\mathbb{R})$, for certain (β,q)-atom a_j and $\lambda_j \in \mathbb{C}$, for every $j \in \mathbb{N}\backslash\{0\}$, such that $\sum_{j=1}^{\infty}|\lambda_j|^p < \infty$, we can proceed as in the proof of [11, Theorem 2.1].

We choose a positive function $\phi \in \mathcal{S}(\mathbb{R})$, such that $\mathrm{supp}(\phi) \subset [-1,1]$ and $\|\phi\|_{1,\alpha} = 1$. We define the functions:

$$
\phi_k(x) = 2^{2k(\alpha+1)}\phi(2^k x) \quad\text{and}\quad f_k(x) = f *_\alpha \phi_k, \quad k \in \mathbb{N}.
$$

It is well known that $\lim_{k\to\infty} f_k = f$, in the distribution sense. Also, we take a smooth function ψ such that $\mathrm{supp}(\psi) \subset \{x : \frac{1}{2}-\varepsilon \leq |x| \leq 1+\varepsilon\}$, for a certain $0 < \varepsilon < 1/2$ and $\psi(x) = 1$ if $1/2 \leq |x| \leq 1$. We define $\psi_k(x) = \psi(2^{-k}x)$, $k \in \mathbb{Z}$. It is easy to see that

$$\operatorname{supp}(\psi_k) \subset A_{k,\varepsilon} := \{x \,/\, 2^{k-1} - 2^k\varepsilon \leq |x| \leq 2^k + 2^k\varepsilon\}$$

and $\psi_k(x) = 1$ if $x \in A_{k,0}$. For each $k \in \mathbb{Z}$, we consider

$$\Psi_k(x) := \frac{\psi_k(x)}{\sum_{j=-\infty}^{\infty} \psi_j(x)}, \quad \text{for} \quad x \in \mathbb{R}\backslash\{0\} \quad \text{and} \quad \Psi_k(0) = 0.$$

By $\mathcal{P}_s$ we denote the space of polynomials of degree less or equal than $2s+1$. For each $k \in \mathbb{Z}$ and $l \in \mathbb{N}$, $P_{k,l}$ represents the unique polynomial in $\mathcal{P}_s$ such that

$$\int_{A_{k,\varepsilon}} x^m [f_l(x)\Psi_k(x) - P_{k,l}(x)]\chi_{\widetilde{A}_k}(x) d\mu_\alpha(x) = 0\,, \quad m = 0,\, 1,\, ...,\, 2s+1.$$

We now write $f_l = S_{1,l} + S_{2,l}$, where

$$S_{1,l}(x) = \sum_{k=-\infty}^{\infty} \{f_l(x)\Psi_k(x) - P_{k,l}(x)\} \ \text{ and } \ S_{2,l}(x) = \sum_{k=-\infty}^{\infty} P_{k,l}(x).$$

Moreover, for every $i = 1, 2$ and $j \in \mathbb{N}\backslash\{0\}$, there exist $(\beta, q)-$atom $a_{j,i}$ and $\lambda_{j,i} \in \mathbb{C}$, being $\sum_{j=1}^{\infty} |\lambda_{j,i}|^p < \infty$, such that $S_{i,l} = \sum_{j=1}^{\infty} \lambda_{j,i} a_{j,i}$. Also,

$$\sum_{j=1}^{\infty} |\lambda_{j,i}|^p \leq C \, \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}}, \quad i = 1, 2.$$

Finally, by invoking the Banach-Alaoglu theorem and (9), we can conclude that $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where the series converges in $\mathcal{S}'(\mathbb{R})$, for some (β, q)-atom a_j and $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}\backslash\{0\}$, such that

$$\sum_{j=1}^{\infty} |\lambda_j|^p \leq C \, \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}},$$

where $C > 0$ is not depending on f.

Thus the proof is finished. ■

Remark. According to Theorem 3, the space $H\dot{K}^{\beta,p,N}_{\alpha,q}$ is not depending on N provide that $N \geq 2(2s + 3 + \alpha)$. In the sequel we assume that $N \geq 2(2s + 3 + \alpha)$ and we write $H\dot{K}^{\beta,p}_{\alpha,q}$ instead of $H\dot{K}^{\beta,p,N}_{\alpha,q}$.

4. The Dunkl transform on $H\dot{K}^{\beta,p}_{\alpha,q}$

In this section we study the Dunkl transformation on the space $H\dot{K}^{\beta,p}_{\alpha,q}$. In particular, we prove a Hardy inequality for the Dunkl transform $\mathcal{F}_\alpha$. First, we establish useful estimates for the Dunkl transform of (β, q)-atoms.

LEMMA 2. *Let a be an (β, q)-atom where $\beta > 0$ and $q \in [1, \infty]$. Then for all $y \in \mathbb{R}$:*

i) $|\mathcal{F}_\alpha(a)(y)| \le C|y|^{2(s+1)}\|a\|_{q,\alpha}^A$, $A = 1 - \dfrac{1}{\beta}(1 - \dfrac{1}{q} + \dfrac{s+1}{\alpha+1})$.

ii) $|\mathcal{F}_\alpha(a)(y)| \le C\|a\|_{q,\alpha}^B$, $B = 1 - \dfrac{1}{\beta}(1 - \dfrac{1}{q})$.

P r o o f. Let a be an (β, q)-atom. Assume that $r > 0$ is such that $\mathrm{supp}(a) \subset [-r, r]$, and that $\|a\|_{q,\alpha} \le r^{-2(\alpha+1)\beta}$.

i) From (iii) of Definition 3, we have

$$\mathcal{F}_\alpha(a)(y) = \int_{-r}^{r} \Big[E_\alpha(-ixy) - \sum_{k=0}^{2s+1} \frac{(-ixy)^k}{b_k(\alpha)}\Big] a(x) d\mu_\alpha(x), \quad y \in \mathbb{R}.$$

But from (9) and (1), we have

$$E_\alpha(-ixy) = \sum_{k=0}^{2s+1} \frac{(-ixy)^k}{b_k(\alpha)} + (-1)^{s+1} \int_{-|xy|}^{|xy|} w_{2s+1}(xy, t) E_\alpha(-it) d\mu_\alpha(t).$$

Thus, by (10) we obtain

$$\Big|E_\alpha(-ixy) - \sum_{k=0}^{2s+1} \frac{(-ixy)^k}{b_k(\alpha)}\Big| \le \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} |xy|^{2s+2}.$$

Then,

$$\begin{aligned} |\mathcal{F}_\alpha(a)(y)| &\le C|y|^{2s+2} \int_{-r}^{r} |x|^{2s+2} |a(x)| d\mu_\alpha(x) \\ &\le C|y|^{2s+2} \|a\|_{q,\alpha} \Big[\int_{-r}^{r} |x|^{(2s+2)q'} d\mu_\alpha(x)\Big]^{1/q'} \\ &\le C|y|^{2s+2} \|a\|_{q,\alpha} r^{2[s+1+(\alpha+1)/q']}, \quad 1/q + 1/q' = 1. \end{aligned}$$

From (ii) of Definition 3, we obtain

$$|\mathcal{F}_\alpha(a)(y)| \le C|y|^{2(s+1)}\|a\|_{q,\alpha}^A, \quad A = 1 - \frac{1}{\beta}(1 - \frac{1}{q} + \frac{s+1}{\alpha+1}).$$

ii) We have

$$|\mathcal{F}_\alpha(a)(y)| \le \int_{-r}^{r} |a(x)| d\mu_\alpha(x) \le C\, \|a\|_{q,\alpha} r^{2(\alpha+1)(1-1/q)} \le C\, \|a\|_{q,\alpha}^B,$$

where $B = 1 - \frac{1}{\beta}(1 - \frac{1}{q})$. We complete the proof. ■

As a consequence of Lemma 2, we prove the following essential property.

PROPOSITION 7. *Let a be an (β, q)-atom, where $q \in]1, \infty]$ and $1 - \frac{1}{q} \le \beta \le 1 - \frac{1}{q} + \frac{s+1}{\alpha+1}$. Then*

$$|\mathcal{F}_\alpha(a)(y)| \le C\, |y|^{2(\alpha+1)(\beta - 1 + \frac{1}{q})}, \quad y \in \mathbb{R}.$$

P r o o f. Let a be an (β, q)-atom. Assume firstly that $|y|^{2(s+1)}\|a\|_{q,\alpha}^{A} \leq \|a\|_{q,\alpha}^{B}$, where $y \in \mathbb{R}$, A and B given in Lemma 2. Then from Lemma 2 i), it infers that

$$|\mathcal{F}_\alpha(a)(y)| \leq C\,|y|^{2(s+1)}\|a\|_{q,\alpha}^{A} \leq C\,|y|^{2(\alpha+1)(\beta-1+\frac{1}{q})}, \quad y \in \mathbb{R}.$$

On the other hand, if $|y|^{2(s+1)}\|a\|_{q,\alpha}^{A} \geq \|a\|_{q,\alpha}^{B}$, then Lemma 2 ii) leads to

$$|\mathcal{F}_\alpha(a)(y)| \leq C\,\|a\|_{q,\alpha}^{B} \leq C|y|^{2(\alpha+1)(\beta-1+\frac{1}{q})}, \quad y \in \mathbb{R}.$$

Thus, we conclude that

$$|\mathcal{F}_\alpha(a)(y)| \leq C\,|y|^{2(\alpha+1)(\beta-1+\frac{1}{q})}, \quad y \in \mathbb{R}.$$

Let $f \in \mathcal{S}'(\mathbb{R})$. The Dunkl transform $\mathcal{F}_\alpha(f)$ of f is defined by ■

$$\langle \mathcal{F}_\alpha(f), \phi\rangle = \langle f, \mathcal{F}_\alpha(\phi)\rangle, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

In the following we infer weak-type inequality for the Dunkl transform.

PROPOSITION 8. *Let* $0 < p \leq 1 < q \leq \infty$, $1 - \frac{1}{q} \leq \beta \leq 1 - \frac{1}{q} + \frac{s+1}{\alpha+1}$ *and* $f \in H\dot{K}_{\alpha,q}^{\beta,p}$. *Then,*

i) $|y|^{-2(\alpha+1)(\beta-1+\frac{1}{q})}|\mathcal{F}_\alpha(f)(y)| \leq C\|f\|_{H\dot{K}_{\alpha,q}^{\beta,p}}$, $y \in \mathbb{R}$.

ii) $\mu_\alpha\Big(\Big\{y \in \mathbb{R} \;/\; |y|^{-2(\alpha+1)(\beta-1+\frac{1}{q}+\frac{1}{p})}|\mathcal{F}_\alpha(f)(y)| > \lambda\Big\}\Big) \leq C\,\dfrac{\|f\|_{p,\alpha}^{p}}{\lambda^p}$, $\lambda > 0$.

P r o o f. i) Let $f \in H\dot{K}_{\alpha,q}^{\beta,p}$. Assume that $f = \sum_{j=1}^{\infty}\lambda_j a_j$, where the series converges in $\mathcal{S}'(\mathbb{R})$, for certain (β, q)-atom a_j and $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}\backslash\{0\}$, being $\sum_{j=1}^{\infty}|\lambda_j|^p < \infty$. Since $\mathcal{F}_\alpha$ is a continuous linear mapping from $\mathcal{S}'(\mathbb{R})$ into itself, we have

$$\mathcal{F}_\alpha(f) = \sum_{j=1}^{\infty}\lambda_j\mathcal{F}_\alpha(a_j).$$

Moreover, since $\displaystyle\sum_{j=1}^{\infty}|\lambda_j| \leq \Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p}$ from Proposition 7, we obtain

$$|\mathcal{F}_\alpha(f)(y)| \leq C\,|y|^{2(\alpha+1)(\beta-1+\frac{1}{q})}\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p}.$$

Hence we deduce i).

ii) Let $f \in H\dot{K}_{\alpha,q}^{\beta,p}$ and $\lambda > 0$. From i) it follows that

$$\mu_\alpha\Big(\Big\{y \in \mathbb{R} \;/\; |y|^{-2(\alpha+1)(\beta-1+\frac{1}{q}+\frac{1}{p})}|\mathcal{F}_\alpha(f)(y)| > \lambda\Big\}\Big) \leq 2\int_0^{C_p} d\mu_\alpha(y) \leq C\,\frac{\|f\|_{p,\alpha}^{p}}{\lambda^p},$$

where $C_p = \big(C\|f\|_{p,\alpha}/\lambda\big)^{\frac{p}{2\alpha+2}}$. We finish the proof. ■

LEMMA 3. *Let $p \in]0,1]$ and $\frac{1}{2} \le \beta \le \frac{1}{2} + \frac{s+1}{\alpha+1}$. For every $(\beta,2)$-atom a, we have*
$$\int_{\mathbb{R}} |\mathcal{F}_\alpha(a)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C.$$

P r o o f. Let a be an $(\beta,2)$-atom. Assume that $R>0$. By virtue of Lemma 2 i), we have
$$\int_{-R}^{R} |\mathcal{F}_\alpha(a)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C\, R^\sigma \|a\|_{2,\alpha}^{pA},$$
where $A = 1 - \dfrac{1}{\beta}(\dfrac{s+1}{\alpha+1} + \dfrac{1}{2})$ and $\sigma = 2p[s+1-(\alpha+1)(\beta-\frac{1}{2})]$.

Since $\sigma = \tau A$, where $\tau = -2(\alpha+1)\beta p$, we can write
$$\int_{-R}^{R} |\mathcal{F}_\alpha(a)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C \left(R\,\|a\|_{2,\alpha}^{p/\tau}\right)^\sigma. \tag{11}$$
Also according to Theorem 1, Hölder's inequality leads to
$$\int_{|y|>R} |\mathcal{F}_\alpha(a)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y)$$
$$\le \|a\|_{2,\alpha}^p \Big[\int_{|y|>R} |y|^{4\frac{(\alpha+1)p}{p-2}(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y)\Big]^{\frac{2-p}{2}} \le C\,\|a\|_{2,\alpha}^p R^\tau.$$
Thus,
$$\int_{|y|>R} |\mathcal{F}_\alpha(a)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C \left(R\,\|a\|_{2,\alpha}^{p/\tau}\right)^\tau. \tag{12}$$
By taking now $R = \|a\|_{2,\alpha}^{-p/\tau}$, from (11) and (12) we obtain the result. ∎

THEOREM 4. *Let $p \in]0,1]$ and $\frac{1}{2} \le \beta \le \frac{1}{2} + \frac{s+1}{\alpha+1}$. Then*
$$\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C\,\|f\|_{H\dot{K}_{\alpha,2}^{\beta,p}}^p,$$
for every $f \in H\dot{K}_{\alpha,2}^{\beta,p}$.

P r o o f. Assume that $f = \sum_{j=1}^\infty \lambda_j a_j$, where the series converges in $\mathcal{S}'(\mathbb{R})$, for certain $(\beta,2)$-atom a_j and $\lambda_j \in \mathbb{C},\ j \in \mathbb{N}\backslash\{0\}$, being $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Then, $\mathcal{F}_\alpha(f) = \sum_{j=1}^\infty \lambda_j \mathcal{F}_\alpha(a_j)$.

According to Lemma 3, we can write
$$\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y)$$
$$\le C \sum_{j=1}^\infty |\lambda_j|^p \int_{\mathbb{R}} |\mathcal{F}_\alpha(a_j)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C \sum_{j=1}^\infty |\lambda_j|^p.$$

Hence

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(y)|^p |y|^{-2(\alpha+1)p(\beta-\frac{1}{2}+\frac{1}{p})} d\mu_\alpha(y) \le C \|f\|^p_{H\dot{K}^{\beta,p}_{\alpha,2}}.$$

Thus the proof is completed. ■

A version of the Hardy inequality for the Dunkl transform $\mathcal{F}_\alpha$ appears when we take $\beta = 1/2$ and $p = 1$ in Theorem 4.

Corollary 2. (Hardy inequality) *Let* $f \in H\dot{K}^{1/2,1}_{\alpha,2}$, *then*

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(y)| \frac{dy}{|y|} \le C \|f\|_{H\dot{K}^{1/2,1}_{\alpha,2}}.$$

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