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HANKEL TRANSFORM IN QUANTUM CALCULUS AND APPLICATIONS

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Abstract

This paper is devoted to study the q-Hankel transform associated with the third q-Bessel function called also Hahn-Exton function. We use the q-approximation of unit for establishing a q-inverse formula of this transform. Moreover, we establish the related q-Parseval theorem.

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1. Introduction

The j_α-Bessel function is defined by:

$$\begin{aligned} j_\alpha(x) &= 2^\alpha \Gamma(\alpha+1) x^{-\alpha} J_\alpha(x), \quad x \neq 0;\ \alpha > -\tfrac{1}{2}, \\ j_\alpha(0) &= 1, \end{aligned} \tag{1.1}$$

where $J_\alpha(.)$ is the Bessel function of first kind and order α (see [10]):

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\alpha+k+1)} \left(\frac{x}{2}\right)^{\alpha+2k}. \tag{1.2}$$

For λ complex, the function $x \longmapsto j_\alpha(\lambda x)$ is the unique solution of the following second order singular differential equation:

$$\begin{gathered} u'' + \frac{2\alpha+1}{x} u' = -\lambda^2 u, \\ u(0) = 1\,,\ u'(0) = 0, \end{gathered} \tag{1.3}$$

and satisfies the following property for λ real:

$$|j_\alpha(\lambda x)| \leq 1. \tag{1.4}$$

The function $\lambda \longmapsto j_\alpha(\lambda x)$ is even and analytic over R.

We recall that, for $f \in L^1_\alpha([0,+\infty[, x^{2\alpha+1}dx)$, i.e. $\int_0^{+\infty} |f(x)|x^{2\alpha+1}dx < \infty$, the Hankel transform (see [9]) is defined by

$$H_\alpha(f)(\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)} \int_0^\infty f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx. \tag{1.5}$$

In this paper we attempt to study the analogue of the Hankel transform (1.5) in quantum theory. It is well known that in the literature there are many q-extensions of the Bessel function rearranged by Ismail [5]. Here we are concerned with the third q-Bessel function called also Hahn-Exton function, and studied in details by many authors, in particular by Ismail [5], Swarttouw [8], Fitouhi [3]. To make this work easily to read, we need some notations and preliminaries about the quantum theory.

2. Notations and preliminaries

We use the notions and notations used in the q-theory given as in [4]. Let a and q be real numbers, and $0 < q < 1$.

The q-shifted factorial is defined by

$$(a;q)_0 = 1; (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)\ ; \quad n = 1,2,... \tag{2.1}$$

and

$$(a_1,\cdots,a_r;q)_n = \prod_{k=1}^{r}(a_k;q)_n. \tag{2.2}$$

We recall the q-binomial theorem:

$$_1\phi_0(a;-;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{2.3}$$

where $_1\phi_0$ is the q-hypergeometric function in [4].

The q-derivative D_qf of a function f on an open interval is given by:

$$D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0, \tag{2.4}$$

and $(D_qf)(0) = f^{'}(0)$ provided $f^{'}(0)$ exists.

The q-Jackson integrals from 0 to a and to ∞ are respectively defined by:

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{2.5}$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{-\infty}^{+\infty} f(q^n)q^n. \tag{2.6}$$

The q-integration by parts is given for suitable functions f and g by:

$$\int_a^b f(x)D_qg(x)d_qx = [g(b)f(q^{-1}b) - g(a)f(a)] - \int_a^b g(x)D_qf(q^{-1}x)d_qx. \tag{2.7}$$

Some q-functional spaces will be used in this work. We begin by putting

$$\mathbf{R}_q = \{\pm q^k, k \in \mathbf{Z}\} \cup \{0\}, \tag{2.8}$$

$$\mathbf{R}_{q,+} = \{+q^k, k \in \mathbf{Z}\}, \tag{2.9}$$

and $D_{*,q}$ the space of even functions defined on $\mathbf{R}_q$ with compact support $\in \mathbf{R}_q$. This space is equipped with the topology of uniform convergence.

Jackson [6] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1\,;\ x \neq 0, -1, -2, ... \tag{2.10}$$

The q-Beta function is defined by:

$$\beta_q(x;y) = \int_0^1 t^{y-1}\frac{(tq;q)_\infty}{(tq^x;q)_\infty}d_qt\ , \quad x > 0\,;\ y > 0, \tag{2.11}$$

and we have

$$\beta_q(x;y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{2.12}$$

We recall also the q-analogue of the exponential function, studied in details in [7]:

$$E(x;q^2) = (-(1-q^2)x;q^2)_\infty = \sum_{n=0}^{\infty}\frac{(1-q^2)^n}{(q^2;q^2)_n}q^{n(n-1)}x^n\,, \quad x \in \mathbf{R}. \tag{2.13}$$

Note that when $\frac{\log(1-q)}{\log q} \in \mathbf{Z}$, the function Γ_q has the following q-integral representation (see [2]):

$$\Gamma_q(x) = \int_0^{\infty} t^{x-1}E(-qt;q)d_qt. \tag{2.14}$$

Definition 2.1. In [3] the authors introduce the q-j_α Bessel function:

$$j_\alpha(x;q^2)=\Gamma_{q^2}(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}(\frac{x}{1+q})^{2k}. \tag{2.15}$$

Proposition 2.2. (see [3]) *For λ complex, the function $j_\alpha(\lambda x;q^2)$ is the solution of the q-problem*

$$\begin{aligned}\Delta_{q,\alpha}y(x)+\lambda^2y(x)=0,\\ y(0)=1\,,\; y\prime(0)=0.\end{aligned} \tag{2.16}$$

Here, $\Delta_{q,\alpha}$ is the q-Bessel operator, defined by

$$\Delta_{q,\alpha}f(x)=q^{2\alpha+1}\Delta_q f(x)+\frac{1-q^{2\alpha+1}}{(1-q)q^{-1}x}D_qf(q^{-1}x), \tag{2.17}$$

where $\Delta_q f(x)=(D_q^2f)(q^{-1}x)$.

In ([3]), the authors give the q-integral representation of the q-j_α Bessel function of Mehler type as

$$j_\alpha(x;q^2)=(1+q)k(\alpha;q^2)\int_0^1 W_\alpha(t;q^2)\ \cos(xt;q^2)dt\,,\ \alpha\neq-\frac{1}{2},-1,-\frac{3}{2},... \tag{2.18}$$

with $k(\alpha;q^2)=\frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\alpha+\frac{1}{2})\Gamma_{q^2}(\frac{1}{2})}$ and W_α being the q-binomial function

$$W_\alpha(x;q^2)={}_1\varphi_1(q^{1-2\alpha},-;q^2;x^2q^{2\alpha+1})\quad;\quad |x|<1,\ \alpha>-\frac{1}{2}. \tag{2.19}$$

Note that the latter q-function tends to $(1-x^2)^{\alpha-\frac{1}{2}}$ when q$\to 1^-$.

For f in $D_{*,q}$, the q-generalized Bessel translation is defined by (see [3]):

$$T^\alpha_{q,x}(f)(y)=\sum_{n=0}^{+\infty}\frac{q^{n^2}}{(q^2;q^{2\alpha+2};q^2)_n}\left(\frac{x}{y}\right)^{2n}\sum_{k=-n}^{n}(-1)^{n-k}U_k(n)f(q^ky), \tag{2.20}$$

where the sequence $U_k(n)$ satisfies for all $n\in\mathbf{N}$:

$$U_k(n{+}1)=q^{2n+1}U_{k+1}(n){+}(q{+}q^{2\alpha+1})U_k(n){+}q^{-2n+2\alpha+1}U_{k-1}(n)\text{ if }|k|\leq n, \tag{2.21}$$

$U_k(n)=0$ if $|k|>n$.

For all f,g in $D_{*,q}$ the q-product formula is given by:

$$T^\alpha_{q,x}j_\alpha(y,q^2)=j_\alpha(x,q^2)\,j_\alpha(y,q^2). \tag{2.22}$$

Recall also the definition of the q-Bessel convolution, defined for f,g in $D_{*,q}$ by

$$(f\star_\alpha g)\,(x)=\frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty}T^\alpha_{q,x}f(y)g(y)y^{2\alpha+1}d_qy. \tag{2.23}$$

3. q-Hankel transform

In the following we suppose that

$$\frac{\log(1-q)}{\log q} \in \mathbf{Z} \text{ and denote by } L^1_\alpha(\mathbf{R}_{q,+}, x^{2\alpha+1}d_qx)$$

the space of functions f such that $\|f\|_{L^1_\alpha} = \int_0^\infty |f(x)|x^{2\alpha+1}d_qx < +\infty$.

Definition 3.1. Let f be in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$, the q-Hankel transform is defined as:

$$H_{\alpha,q}(f)(\lambda) = c(\alpha;q)\int_0^\infty f(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx, \qquad \lambda \in \mathbf{R}_q, \alpha > -\frac{1}{2}, \tag{3.1}$$

where, $c(\alpha;q) = \dfrac{1}{(1+q)^\alpha \Gamma_{q^2}(\alpha+1)}$ and $j_\alpha(\lambda x;q^2)$ is given by (2.15).

In the following, we give some interesting properties of the q-Hankel transform (see [3] and [8]) which tends to the classical case when q tends to 1^-.

Proposition 3.3.
1- Let f and g be two functions in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$. For all complex λ and μ in $\mathbf{R}_q$ we have:

$$H_{\alpha,q}(f+\mu g)(\lambda) = H_{\alpha,q}(f)(\lambda) + \mu H_{\alpha,q}(g)(\lambda). \tag{3.2}$$

2- For f in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$ and $\lambda, a \in \mathbf{R}_q$, we have:

$$H_{\alpha,q}(f(ax))(\lambda) = a^{-(\alpha+2)}H_{\alpha,q}(f)(\frac{\lambda}{a}). \tag{3.3}$$

3- Let f be in $D_{*,q}$

$$H_{\alpha,q}(\frac{1}{x}D_qf)(\lambda) = -q^{-2\alpha+1}H_{(\alpha-1),q}(f)(\lambda q^{-1}). \tag{3.4}$$

4- For f and g in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$,we have:

$$\int_0^\infty H_{\alpha,q}(f)(y)g(y)y^{2\alpha+1}d_qy = \int_0^\infty H_{\alpha,q}(g)(y)f(y)y^{2\alpha+1}d_qy. \tag{3.5}$$

5- For f in $L^1_\alpha\left(\mathbf{R}_q, x^{2\alpha+1}d_qx\right)$ and $\lambda \in \mathbf{R}_q$, we have:

$$|H_{\alpha,q}(f)(\lambda)| \le \frac{1}{(1-q)^{\frac{1}{2}}(q;q)_\infty}\|f\|_{\alpha,q}. \tag{3.6}$$

6- For f and g in $D_{*,q}$ we have:

$$H_{\alpha,q}\left(f\star_{\alpha} g\right)=H_{\alpha,q}(f).H_{\alpha,q}(g), \tag{3.7}$$

$$H_{\alpha,q}\left(T^{\alpha}_{q,x}f\right)(\lambda)=j_{\alpha}(\lambda x;q^2).H_{\alpha,q}(f)(\lambda) \quad ,\lambda\in\mathbf{R}_q. \tag{3.8}$$

7- For f in $D_{*,q}$, we have:

$$H_{\alpha,q}\left(\Delta_{q,\alpha}f\right)(\lambda)=-\lambda^2 H_{\alpha,q}\left(f\right)(\lambda). \tag{3.9}$$

P r o o f.
1- The property (3.2) is a direct consequence of the linearity of the q-Jackson integrals.
2- Let $a=q^k$. The definition of q-Jackson integral (2.5) and sample computation give the result (3.3).
3- For $f\in D_{*,q}$ we have:

$$H_{\alpha,q}(\frac{1}{x}D_q f)(\lambda)=c(\alpha;q)\int_0^{\infty}D_q f(x)j_{\alpha}(\lambda x;q^2)x^{2\alpha}d_q x.$$

The q-integration by parts leads to the result. ■

Example 3.4. In this example, we shall compute the q-Hankel transform of the following function (3.10):

$$f(x)=w_{\alpha,u}(x;q^2)1_{[0,1]}(x) \tag{3.10}$$

where $w_{\alpha,u}$ is the q-binomial function given by

$$w_{\alpha,u}(x;q^2)=\frac{(x^2q^2;q^2)_{\infty}}{(x^2q^{2(u-\alpha)};q^2)_{\infty}}={}_1\phi_0(q^{2-2(u-\alpha)},-,q^2,x^2q^{2(u-\alpha)})$$

which tends to $(1-x^2)^{u-\alpha-1}$ when $q\to 1^-$ and

$$1_{[0,1]}(q^n)=\begin{cases}1, & \text{if} \quad n\geq 0,\\ 0, & \text{if} \quad n<0.\end{cases}$$

So we have:

$$H_{\alpha,q}(f)(\lambda)=\frac{c(\alpha;q)\beta_{q^2}(u-\alpha,\alpha+1)}{(1+q)}j_u(\lambda;q^2). \tag{3.11}$$

In fact using the definitions (,,) and (,,) we obtain
$H_{\alpha,q}(f)(\lambda)=c(\alpha;q)\Gamma_{q^2}(\alpha+1)$

$$\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}(\frac{\lambda}{1+q})^{2k}\sum_{n=0}^{+\infty}\frac{(q^{2n}q^2;q^2)_{\infty}}{(q^{2n}q^{2(u-\alpha)};q^2)_{\infty}}q^{2n(\alpha+k+1)}.$$

The computation is legitimated by the fact that the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}(\frac{\lambda}{1+q})^{2k}\sum_{n=0}^{+\infty}\frac{(q^{2n}q^2;q^2)_\infty}{(q^{2n}q^{2(u-\alpha)};q^2)_\infty}q^{2n(\alpha+k+1)}$$

converges uniformly on every compact.

The q-integral (2.5) and the q-Beta formula (2.11), (2.12) give:

$$H_{\alpha,q}(f)(\lambda)=\frac{c(\alpha;q)\Gamma_{q^2}(\alpha+1)}{(1+q)}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}$$
$$\times\,\beta_{q^2}(\alpha+k+1,u-\alpha)(\frac{\lambda}{1+q})^{2k}=\frac{c(\alpha;q)\beta_{q^2}(u-\alpha,\alpha+1)}{(1+q)}j_u(\lambda;q^2).$$

When $u=\alpha+1$, we obtain

$$H_{\alpha,q}\left[1_{[0,1]}(x)\right](\lambda)=c(\alpha+1;q)j_{\alpha+1}(\lambda;q^2). \tag{3.12}$$

and $u=\alpha+\frac{1}{2}$, we obtain

$$H_{\alpha,q}\left[\frac{(x^2q^2;q^2)_\infty}{(x^2q;q^2)_\infty}1_{[0,1]}(x)\right](\lambda)=\frac{\Gamma_{q^2}(\frac{1}{2})}{(1+q)^{\alpha+1}\Gamma_{q^2}(\alpha+\frac{3}{2})}j_{\alpha+\frac{1}{2}}(\lambda;q^2). \tag{3.13}$$

Indeed: $H_{\alpha,q}\left[1_{[0,1]}(x)\right](\lambda)=c(\alpha;q)\int_0^{\infty}1_{[0,1]}(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx$

$$=\frac{c(\alpha;q)\Gamma_{q^2}(\alpha+1)}{(1+q)}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}$$
$$\times(\frac{\lambda}{1+q})^{2k}(1-q^2)\sum_{n=0}^{+\infty}q^{2n(\alpha+k+1)}$$
$$=\frac{c(\alpha;q)\Gamma_{q^2}(\alpha+1)}{(1+q)}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+2)}(\frac{\lambda}{1+q})^{2k}$$
$$=c(\alpha+1;q)j_{\alpha+1}(\lambda;q^2),$$

and, $H_{\alpha,q}\left[\frac{(x^2q^2;q^2)_\infty}{(x^2q;q^2)_\infty}1_{[0,1]}(x)\right](\lambda)$

$$=c(\alpha;q)\int_0^{\infty}\frac{(x^2q^2;q^2)_\infty}{(x^2q;q^2)_\infty}1_{[0,1]}(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx$$
$$=\frac{1}{(1+q)^{\alpha+1}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}$$
$$\times(\frac{\lambda}{1+q})^{2k}(1-q^2)\sum_{n=0}^{+\infty}q^{2n(\alpha+k+1)}\frac{(q^{2(n+1)};q^2)_\infty}{(q^{2n+1}q;q^2)_\infty},$$

and with definition of q-Jackson integral (2.5) and definition (2.12), we obtain the result. ∎

4. Relations between q-Hankel and q-Laplace transforms

The q-Laplace transform is defined (see [1]) for f on $\mathbf{R}_q$ and $\Re p > a > 0$ as:

$$\pounds_q(f(x))(p) = \int_0^{+\infty} E(-pqx;q) f(x) d_q x \tag{4.1}$$

which tends to the classical Laplace transform $\pounds(f)(p) = \int_0^{+\infty} e^{-px} f(x)dx$ when $q \to 1^-$.

Proposition 4.1. *The q-Hankel and q-Laplace transforms are linked by the following relation:*

$$H_{\alpha,q}\left[E(-qpx;q)f(x)\right](\lambda) = c(\alpha;q)\pounds_q\left[x^{2\alpha+1} f(x) j_\alpha(\lambda x;q^2)\right](p). \tag{4.2}$$

Example 4.2. For every $p, a \in \mathbf{C}$ such that $\Re p > a > 0$ and $\alpha > -\frac{1}{2}$, we have:

$$\pounds_q\left[x^{2\alpha} j_\alpha(ax;q^2)\right](p) = \frac{(1+q)^\alpha \Gamma_{q^2}(\alpha+\frac{1}{2})}{c(\alpha;q)p^{2\alpha+1}\Gamma_{q^2}(\frac{1}{2})} {}_1\phi_1\left(q^{2\alpha+1};0;q^2;\frac{a^2}{p^2}\right), \tag{4.3}$$

and

$$\pounds_q\left[x^{2\alpha+1} j_\alpha(ax;q^2)\right](p) = \frac{(1+q)^{\alpha+1} \Gamma_{q^2}(\alpha+\frac{3}{2})}{c(\alpha;q)p^{2\alpha+2}\Gamma_{q^2}(\frac{1}{2})} {}_1\phi_1\left(q^{2\alpha+3};0;q^2;\frac{a^2}{p^2}\right). \tag{4.4}$$

P r o o f. To prove (4.3), we have for $\Re p > a > 0$ and $\alpha > -\frac{1}{2}$:

$$\pounds_q\left[x^{2\alpha} j_\alpha(ax;q^2)\right](p) = \int_0^{+\infty} E(-pqx;q) x^{2\alpha} j_\alpha(ax;q^2) d_q x = \Gamma_{q^2}(\alpha+1)$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)} (\frac{a}{1+q})^{2k} \int_0^\infty E(-pqx;q) x^{2\alpha+2k} d_q x.$$

To this end we use the following result:

$$\int_0^\infty E(-pqx;q) x^{2\alpha+2k} d_q x = \frac{1}{p^{2\alpha+2k+1}} \Gamma_q(2\alpha+2k+1)$$

and the q-duplication formula. Hence,

$$\pounds_q\left[x^{2\alpha} j_\alpha(ax;q^2)\right](p)$$

$$= \frac{(1+q)^\alpha \Gamma_{q^2}(\alpha+\frac{1}{2})}{c(\alpha;q)p^{2\alpha+1}\Gamma_{q^2}(\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)} \Gamma_{q^2}(\alpha+k+\frac{1}{2})}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+\frac{1}{2})} \left(\frac{a^2}{p^2}\right)^k.$$

Finally, the use of (2.5), (2.15) and the definition of the q-hypergeometric series (2.6) to obtain the result.

Similarly, we can prove the example (4.4). So, we deduce the following results:

$$H_{\alpha,q}\left[x^{-1}E(-qpx;q)\right](\lambda)=\frac{(1+q)^{\alpha}\Gamma_{q^2}(\alpha+\frac{1}{2})}{p^{2\alpha+1}\Gamma_{q^2}(\frac{1}{2})}\,{}_1\phi_1\left(q^{2\alpha+1};0;q^2;\frac{a^2}{p^2}\right) \quad (4.5)$$

$$H_{\alpha,q}\left[E(-qpx;q)\right](\lambda)=\frac{(1+q)^{\alpha+1}\Gamma_{q^2}(\alpha+\frac{3}{2})}{p^{2\alpha+2}\Gamma_{q^2}(\frac{1}{2})}\,{}_1\phi_1\left(q^{2\alpha+3};0;q^2;\frac{a^2}{p^2}\right). \quad (4.6)$$

Example 4.3. Suppose that $\frac{\ln(1+q)}{\ln q}\in\mathbf{Z}$, then we have

$$\begin{aligned}&H_{\alpha,q}\left[E(-pq^2\frac{x^2}{(1+q)^2};q^2)f(x)\right](\lambda)\\&=\frac{(1+q)^{\alpha+1}}{\Gamma_{q^2}(\alpha+1)}\pounds_{q^2}\left[x^{\alpha}f((1+q)\sqrt{x})j_{\alpha}(\lambda(1+q)\sqrt{x};q^2)\right](p).\end{aligned} \quad (4.7)$$

It is easy to prove the last relation (4.7) since the hypothesis gives $\frac{q^k}{1+q}=q^n\in\mathbf{R}_{q,+}$ where n and k are integers numbers.

As consequence of (4.7), we have the following result:

$$H_{\alpha,q}\left[E(-pq^2\frac{x^2}{(1+q)^2};q^2)\right](\lambda)=\frac{(1+q)^{\alpha+1}}{p^{\alpha+1}}E(-\frac{\lambda^2}{p};q^2), \quad (4.8)$$

which can seen as follows:

$$H_{\alpha,q}\left[E(-pq^2\frac{x^2}{(1+q)^2};q^2)\right](\lambda)=\frac{(1+q)^{\alpha+1}}{\Gamma_{q^2}(\alpha+1)}\pounds_{q^2}\left[x^{\alpha}j_{\alpha}(\lambda(1+q)\sqrt{x};q^2)\right](p)$$

and

$$\pounds_{q^2}\left[x^{\alpha}j_{\alpha}(\lambda(1+q)\sqrt{x};q^2)\right](p)=\int_0^{\infty}E(-pq^2x;q^2)x^{\alpha}j_{\alpha}(\lambda(1+q)\sqrt{x};q^2)d_{q^2}x.$$

The exchange of the signs sum and q-integral hold via the relation (2.14) and the definition (2.13) as follows:

$$\begin{aligned}&\pounds_{q^2}\left[x^{\alpha}j_{\alpha}(\lambda(1+q)\sqrt{x};q^2)\right](p)\\&=\Gamma_{q^2}(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^kq^{k(k-1)}}{\Gamma_{q^2}(k+1)\Gamma_{q^2}(\alpha+k+1)}\lambda^{2k}\int_0^{\infty}E(-pq^2x;q^2)x^{\alpha+k}d_{q^2}x\\&=\frac{\Gamma_{q^2}(\alpha+1)}{p^{\alpha+1}}\sum_{k=0}^{\infty}\frac{(-1)^kq^{k(k-1)}}{\Gamma_{q^2}(k+1)}\left(\frac{\lambda^2}{p}\right)^k=\frac{\Gamma_{q^2}(\alpha+1)}{p^{\alpha+1}}E(-\frac{\lambda^2}{p};q^2),\end{aligned}$$

and finally, we have the result (4.6)

Remark 4.4. For $a>0$ and $\lambda \in \mathbf{R}_q$, we have:

$$H_{\alpha,q}\left[E(-a^2(qx)^2;q^2)\right](\lambda)=\frac{1}{a^{2\alpha+2}(1+q)^{\alpha+1}}E(-\frac{\lambda^2}{(a(1+q))^2};q^2). \quad (4.9)$$

The previous relation (4.9) can be written as:

$$\int_0^\infty E(-a^2(qx)^2;q^2)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx=\frac{(1+q)^\alpha\Gamma_{q^2}(\alpha+1)}{a^{2\alpha+2}(1+q)^{\alpha+1}}E(-\frac{\lambda^2}{(a(1+q))^2};q^2). \quad (4.10)$$

The last equality is the q-analogue of the Weber formula [3], we have when q tends to 1^-,

$$\int_0^\infty e^{-a^2x^2}j_\alpha(\lambda x)x^{2\alpha+1}dx=\frac{2^\alpha\Gamma(\alpha+1)}{(2a^2)^{\alpha+1}}e^{-\frac{\lambda^2}{4a^2}}. \quad (4.11)$$

5. Relations between q-Hankel and q-Mellin transforms

Definition 5.1. (see [2]) Let f be a function on $\mathbf{R}_{q,+}$, we define the q-Mellin transform of f as:

$$M_q(f)(s)=M_q\left[f(t)\right](s)=\int_0^\infty t^{s-1}f(t)d_qt \quad (5.1)$$

which tends to the classical Mellin transform $M(f)(s)=\int_0^\infty t^{s-1}f(t)dt$ when q tends to 1^-.

Proposition 5.2. *The q-Hankel and q-Mellin transforms are related by:*

$$H_{\alpha,q}\left[x^{s-2}f(x)\right](\lambda)=M_q\left[x^{2\alpha}f(x)j_\alpha(\lambda x;q^2)\right](s). \quad (5.2)$$

As a special case of the relation (5.2) we have

$$H_{\alpha,q}\left[x^{s-2-2\alpha}\right](\lambda)=M_q\left[j_\alpha(\lambda x;q^2)\right](s) \quad (5.3)$$

and

$$M_q\left[j_\alpha(\lambda x;q^2)\right](s)=\frac{(1+q)^{s-1}\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\frac{s}{2})}{\Gamma_{q^2}(\frac{3\alpha}{2}-\frac{s}{2}+1)}, \quad (5.4)$$

so,

$$H_{\alpha,q}\left[x^{s-2-2\alpha}\right](\lambda)=\frac{(1+q)^{s-1}\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\frac{s}{2})}{\Gamma_{q^2}(\frac{3\alpha}{2}-\frac{s}{2}+1)}. \quad (5.5)$$

6. The q-Hankel inversion theorem

In this section we try to give a proof of the q-Hankel inversion theorem, by the use of the q-analogue of the unit approximation.

To this end, we begin by establishing the following result.

Proposition 6.1. *Let* $(\varphi_p)_{p\in\mathbf{N}}$ *be a sequence of elements in* $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$ *satisfying the following conditions when* $d_q\mu(x) = \dfrac{x^{2\alpha+1}}{(1+q)^\alpha\Gamma_{q^2}(\alpha+1)}d_qx$*:*

1- For $p \in \mathbf{N}$*:*

$$\int_0^{+\infty} \varphi_p(x)d_q\mu(x) = 1; \tag{6.1}$$

2- There exists a constant $M > 0$ *such that for all* $p \in \mathbf{N}$*:*

$$\int_0^{+\infty} |\varphi_p(x)|d_q\mu(x) \leq M; \tag{6.2}$$

3- For $\eta > 0$*:*

$$\lim_{p\longrightarrow+\infty}\int_\eta^{+\infty} |\varphi_p(x)|d_q\mu(x) = 0. \tag{6.3}$$

Then, the sequence $(\varphi_p)_{p\in\mathbf{N}}$ *is an unity of approximation.*

Moreover, for f *in* $L^1_\alpha(\mathbf{R}_{q,+}, x^{2\alpha+1}d_qx)$*, we have*

$$\lim_{p\longrightarrow+\infty}\|f *_\alpha \varphi_p - f\|_{L^1_\alpha} = 0. \tag{6.4}$$

P r o o f. Let $f \in L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$. For all $x \in \mathbf{R}_{q,+}$ we have

$$f(x) = \int_0^{+\infty} \varphi_p(y)f(x)d_q\mu(y).$$

Then by using the definition (2.23) we have

$$(f *_\alpha \varphi_p)(x) - f(x) = \int_0^{+\infty} \left[T^\alpha_{q,x}(f)(y) - f(x)\right]\varphi_p(y)d_q\mu(y).$$

Then

$$\|f *_\alpha \varphi_p - f\|_{L^1_\alpha} \leq \int_0^{+\infty}\int_0^{+\infty} \left|T^\alpha_{q,y}(f)(x) - f(x)\right| |\varphi_p(y)|d_q\mu(y)d_q\mu(x)$$

and using the Fubini-Tonnelli theorem, we deduce that

$$\|f *_\alpha \varphi_p - f\|_{L^1_\alpha} \leq \int_0^{+\infty} \left\|T^\alpha_{q,y}f - f\right\|_{L^1_\alpha} |\varphi_p(y)|d_q\mu(y).$$

Since the map $y \longmapsto T^\alpha_{q,y}f$ on $\mathbf{R}_{q,+}$ is continuous, in particular at 0, we have

$$\forall\varepsilon > 0, \exists\eta > 0; |y| < \eta \Rightarrow \left\|T^\alpha_{q,y}f - f\right\|_{L^1_\alpha} < \frac{\varepsilon}{2M}.$$

Then,

$$\|f *_\alpha \varphi_p - f\|_{L^1_\alpha} \leq \frac{\varepsilon}{2M}\int_0^\eta |\varphi_p(y)|d_q\mu(y) + \int_\eta^{+\infty} \left\|T^\alpha_{q,y}f - f\right\|_{L^1_\alpha} |\varphi_p(y)|d_q\mu(y).$$

Therefore by the property (6.2) of the last proposition, we can write

$$\|f *_\alpha \varphi_p - f\|_{L^1_\alpha} \leq \frac{\varepsilon}{2} + \int_\eta^{+\infty} \left\|T^\alpha_{q,y}f - f\right\|_{L^1_\alpha} |\varphi_p(y)|d_q\mu(y)$$
$$\|f *_\alpha \varphi_p - f\|_{L^1_\alpha} \leq \frac{\varepsilon}{2} + c\,\|f\|_{L^1_\alpha}\int_\eta^{+\infty} |\varphi_p(y)|d_q\mu(y),$$

finally by the property (6.3) we deduce

$$\forall\varepsilon > 0, \exists p_0 \in \mathbf{N}; \forall p \geq p_0 \Rightarrow c\,\|f\|_{L^1_\alpha}\int_\eta^{+\infty} |\varphi_p(y)|d_q\mu(y) < \frac{\varepsilon}{2}$$
$$\forall p \geq p_0; \|f *_\alpha \varphi_p - f\|_{L^1_\alpha} \leq \varepsilon.$$

Theorem 6.2. *Let f be in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$ such that $H_{\alpha,q}(f)$belong in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$, then we have for $\alpha > \frac{-1}{2}$:*

$$f(x) = \frac{1}{(1+q)^\alpha\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty} H_{\alpha,q}(f)(y)j_\alpha(yx;q^2)y^{2\alpha+1}d_qy. \tag{6.5}$$

P r o o f. For the relation (4.8) we can deduce the following result

$$H_{\alpha,q}\left[\frac{1}{q^{2\alpha+2}}E(-\frac{x^2}{(1+q)^2k^2};q^2)\right](\lambda) = (1+q)^{\alpha+1}k^{2\alpha+2}E(-\lambda^2q^2k^2;q^2),\ k \in \mathbf{N}.$$

We consider the following functions

$$\varphi_k(\lambda) = (1+q)^{\alpha+1}k^{2\alpha+2}E(-\lambda^2q^2k^2;q^2)$$

and

$$\psi_k(x) = \frac{1}{q^{2\alpha+2}}E(-\frac{x^2}{(1+q)^2k^2};q^2)$$

such that

$$H_{\alpha,q}\left[\psi_k\right](\lambda) = \varphi_k(\lambda).$$

The sequence $(\varphi_k)_{k\in\mathbf{N}}$ is an unit of approximation. In fact,

$$\int_0^{+\infty} \varphi_k(x)d_q\mu(x) = \frac{(1+q)}{\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty} k^{2\alpha+2}E(-x^2q^2k^2;q^2)x^{2\alpha+1}d_qx$$
$$= \frac{1}{\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty} E(-xq^2;q^2)x^\alpha d_qx = 1,$$

then by Proposition 6.1 we show that $f *_\alpha \varphi_k \underset{k\to\infty}{\longrightarrow} f$ in $L^1_\alpha(\mathbf{R}_q, x^{2\alpha+1}d_qx)$. On the other hand by the definition of q-convolution (2.23) we have

$$\begin{aligned} f *_\alpha \varphi_k(x) &= c(\alpha, q)\int_0^{+\infty} T^\alpha_{q,x}(f)(y)\varphi_k(y)y^{2\alpha+1}d_qy \\ &= c(\alpha, q)\int_0^{+\infty} j_\alpha(yx; q^2)H_{\alpha,q}(f)(y)\psi_k(y)y^{2\alpha+1}d_qy. \end{aligned}$$

Finally by using the dominate convergence theorem we have

$$\lim_{k\longrightarrow\infty} f *_\alpha \varphi_k(x) = c(\alpha, q)\int_0^{+\infty} j_\alpha(yx; q^2)H_{\alpha,q}(f)(y)y^{2\alpha+1}d_qy.$$

7. The Parseval theorem of the q-Hankel transform

THEOREM 7.1. *Let f and g be two functions satisfying the conditions of Proposition 6.1 and denote by $H_{\alpha,q}(f)$ and $H_{\alpha,q}(g)$ their q-Hankel transforms. Then,*

$$\int_0^{+\infty} f(x)g(x)x^{2\alpha+1}d_qx = \int_0^{+\infty} H_{\alpha,q}(f)(x)H_{\alpha,q}(g)(x)x^{2\alpha+1}d_qx. \tag{7.1}$$

P r o o f. Using the definition of $H_{\alpha,q}(g)(x)$ we have

$$\begin{aligned} &\int_0^{+\infty} H_{\alpha,q}(f)(x)H_{\alpha,q}(g)(x)x^{2\alpha+1}d_qx \\ &= c(\alpha, q)\int_0^{+\infty} H_{\alpha,q}(f)(x)x^{2\alpha+1}d_qx \int_0^{+\infty} g(y)j_\alpha(yx; q^2)y^{2\alpha+1}d_qy \\ &= \int_0^{+\infty} g(y)y^{2\alpha+1}d_qy \int_0^{+\infty} H_{\alpha,q}(f)(x)j_\alpha(yx; q^2)x^{2\alpha+1}d_qx, \end{aligned}$$

then using the q-inversion theorem (6.5) the result follows immediately. ■

EXAMPLE 7.2. Let $f(x) = 1_{[0,a]}(x)$, $a \in \mathbf{R}_{q,+}$. We have for $\alpha > -\frac{1}{2}$: $H_{\alpha,q}(f)(\lambda) = a^{2\alpha+2}c(\alpha + 1; q)j_{\alpha+1}(\lambda a; q^2)$

Now, by the use of the Parseval theorem (7.1) we deduce for $a, b \in \mathbf{R}_{q,+}$ and $\alpha > -\frac{1}{2}$,

$$(a.b)^{2\alpha+2}c(\alpha+1; q)^2\int_0^{\infty} j_{\alpha+1}(bx; q^2)j_{\alpha+1}(ax; q^2)x^{2\alpha+1}d_qx = \int_0^{\min(a,b)} x^{2\alpha+1}d_qx.$$

Suppose that $0 < a < b$, we can write

$$\int_0^{\infty} j_{\alpha+1}(bx; q^2)j_{\alpha+1}(ax; q^2)x^{2\alpha+1}d_qx = \frac{(1 - q)}{b^{2\alpha+2}c(\alpha + 1; q)^2(1 - q^{2\alpha+2})}.$$

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