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HARDY INEQUALITY IN VARIABLE EXPONENT LEBESGUE SPACES

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Abstract

We prove the Hardy inequality

$$\left\| x^{\alpha(x)+\mu(x)-1} \int\limits_0^x \frac{f(y)\,dy}{y^{\alpha(y)}} \right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \leq C\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}$$

and a similar inequality for the dual Hardy operator for variable exponent Lebesgue spaces, where $0 \leq \mu(0) < \frac{1}{p(0)}, 0 \leq \mu(\infty) < \frac{1}{p(\infty)}, \frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0), \ \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty)$, and $\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)}, \quad \beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}$, not requiring local log-condition on $\mathbb{R}^1_+$, but supposing that this condition holds for $\alpha(x), \mu(x)$ and $p(x)$ only at the points $x = 0$ and $x = \infty$.

These Hardy inequalities are proved by means of the general result of independent interest stating that any convolution operator on $\mathbb{R}^n$ with the kernel $k(x-y)$ admitting the estimate $|k(x)| \leq c(1+|x|)^{-\nu}$ with $\nu > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$, is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ without local log-condition on $p(\cdot)$, only under the decay log-condition at infinity.

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1. Introduction

The classical Hardy inequalities have the form

$$\left\| x^{\alpha+\mu-1} \int_0^x \frac{\varphi(y)\,dy}{y^\alpha} \right\|_{L^q(\mathbb{R}^1_+)} \leq C\,\|f\|_{L^p(\mathbb{R}^1_+)} \tag{1.1}$$

and

$$\left\| x^{\beta+\mu} \int_x^\infty \frac{\varphi(y)\,dy}{y^{\beta+1}} \right\|_{L^q(\mathbb{R}^1_+)} \leq C\,\|f\|_{L^p(\mathbb{R}^1_+)}\,, \tag{1.2}$$

where $1 \leq p \leq q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. They hold if and only if

$$0 \leq \mu < \frac{1}{p} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \mu$$

and

$$\alpha < \frac{1}{p'} \quad \text{and} \quad \beta > -\frac{1}{p},$$

respectively, see for instance, [16], p. 6, [20], Ch.5, Lemma 3.14.

In the rapidly developing field of ”variable exponent spaces” there have already been made an essential progress in studying classical integral operators, such as maximal and singular operators, Riesz potentials and Hardy operators, we refer for example to the papers [3], [4], [5], [6], [7], [13], [14], [17] and surveys [9], [12], [18], the specifics of these spaces being in the fact that they are not invariant neither with respect to translations, nor with respect to dilations.

In particular, a Hardy inequality of type (1.1) over a finite interval $[0, \ell]$ with $\mu = 0$ was proved in [14] for variable $p(x)$ under the following assumptions on $p(x)$ and the exponent α:

$i)$ $1 \leq p(x) \leq p_+ < \infty$ on $[0, \ell]$,
$ii)$ $p(x)$ is log-continuous on $[0, \delta]$ for some small $\delta > 0$ and $p(0) > 1$,
$iii)$ $-\frac{1}{p(0)} < \alpha < \frac{1}{p'(0)}$.

It was also shown in [14] that in the case where $p(0) \leq p(x), x \in [0, \delta]$, the log-condition on the whole neighborhood $[0, \delta]$ may be replaced by a weaker log-condition holding only at the point $x = 0$. In the case $0 \leq \alpha < \frac{1}{p'(0)}$, in [10] it was proved that Hardy inequality of type (1.1) on a finite interval holds without the assumption $p(0) \leq p(x), x \in [0, \delta]$ and with the log-condition for $p(x)$ at the point $x = 0$ in the *limsup* form.

Meanwhile a natural hypothesis was that Hardy inequality (1.1) should be valid under assumptions on $p(x)$ and the exponents α much weaker than stated in *i)-iii)*.

HYPOTHESIS. *Let*

$$1 \leq p(x) \leq p_+ < \infty, \quad x \in \mathbb{R}^1_+, \tag{1.3}$$

$$|p(x) - p(0)| \leq \frac{A}{|\ln x|}, \quad 0 < x \leq \frac{1}{2}, \tag{1.4}$$

$$|p(x) - p(\infty)| \leq \frac{A}{\ln x}. \quad x > 2, \tag{1.5}$$

Then (1.1) holds, if and only if

$$\alpha < \min\left\{\frac{1}{p'(0)}, \frac{1}{p'(\infty)}\right\}, \tag{1.6}$$

while (1.2) holds, if and only if

$$\beta > \max\left\{-\frac{1}{p(0)}, -\frac{1}{p(\infty)}\right\}. \tag{1.7}$$

We prove in this paper that this hypothesis is true. The proof will be based on a certain result on boundedness of convolution operators in the spaces $L^{p(\cdot)}$. This result, given in Theorem 4.6 and its corollary for convolutions whose kernels are better than just integrable, is of interest by itself, because it does not assume that $p(\cdot)$ satisfies the log-condition. Theorem 4.6 paves the way to various applications, because many concrete convolution operators in analysis have rather "nice" kernels, see for instance example (4.12).

2.1. Variable exponent Lebesgue spaces

We refer to [19] and [15] for basic properties of the variable exponent $L^{p(\cdot)}$-spaces. We only remind that $p(x)$ is assumed to be a measurable bounded function with values in $[1, \infty)$ and the space $L^{p(\cdot)}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$, is introduced as the space of all measurable functions $f(x)$ on Ω which have finite modular

$$I_p(f) := \int\limits_{\Omega} |f(x)|^{p(x)} dx < \infty$$

and the norm in $L^{p(\cdot)}(\Omega)$ is introduced as

$$\|f\|_\Omega = \inf\left\{\lambda > 0 :\ I_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

DEFINITION 2.1.

I. In the case $\Omega = \mathbb{R}^n$, by $\mathcal{M}_\infty(\mathbb{R}^n)$ we denote the set of all measurable bounded functions $p : \mathbb{R}^n \to \mathbb{R}^1_+$ which satisfy the following conditions:

$i)$ $0 \le p_- \le p(x) \le p_+ < \infty, \quad x \in \mathbb{R}^n$,

$ii)$ there exists $p(\infty) = \lim\limits_{x\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \le \frac{A}{\ln(2 + |x|)},\ \ x \in \mathbb{R}^n. \tag{2.1}$$

By $\mathcal{P}_\infty(\mathbb{R}^n)$ we denote the subset of $\mathcal{M}_\infty(\mathbb{R}^n)$ of measurable bounded functions $p : \mathbb{R}^n \to [1, \infty)$.

II. In the case $\Omega = \mathbb{R}^1_+$, by $\mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ we denote the set of all measurable bounded functions $p(x) : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ which satisfy the following conditions:

$i)$ $0 \le p_- \le p(x) \le p_+ < \infty, \quad x \in \mathbb{R}^1_+$,

$ii_0)$ there exists $p(0) = \lim\limits_{x\to 0} p(x)$ and $|p(x) - p(0)| \le \frac{A}{\ln \frac{1}{x}},\ \ 0 < x \le \frac{1}{2}$,

$ii_\infty)$ there exists $\mu(\infty) = \lim\limits_{x\to\infty} p(x)$ and $|p(x) - p(\infty)| \le \frac{A}{\ln x},\ \ x \ge 2$.

By $\mathcal{P}_{0,\infty} = \mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ we denote the subset of functions $p(x) \in \mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ with $\inf\limits_{x\in\mathbb{R}^1_+} p(x) \ge 1$.

Observe that

$$c_1 x^{\mu(0)} \le x^{\mu(x)} \le c_2 x^{\mu(0)},\ 0 < x \le 1, \text{ and } c_1 x^{\mu(\infty)} \le x^{\mu(x)} \le c_2 x^{\mu(\infty)},\ x \ge 1$$

for every $\mu \in \mathcal{M}_{0,\infty}$ and consequently,

$$c_1 x^{\mu(0)} \le x^{\mu(x)} \le c_2 x^{\mu(0)} \tag{2.2}$$

for $\mu \in \mathcal{M}_{0,\infty}$ with $\mu(0) = \mu(\infty)$.

2.2. On kernels homogeneous of degree -1

The operators

$$H^\alpha f(x) = x^{\alpha-1}\int\limits_0^x \frac{f(y)}{y^\alpha}dy \quad \text{and} \quad \mathcal{H}_\beta f(x) = x^\beta \int\limits_x^\infty \frac{\varphi(y)\,dy}{y^{\beta+1}} \tag{2.3}$$

have the kernels

$$k^{\alpha}(x,y)=\frac{1}{x}\left(\frac{x}{y}\right)^{\alpha}\theta_{+}(x-y) \quad \text{and} \quad k_{\beta}(x,y)=\frac{1}{y}\left(\frac{x}{y}\right)^{\beta}\theta_{+}(y-x), \tag{2.4}$$

respectively, where $\theta_+(x)=\frac{1}{2}(1+\operatorname{sign}x)$. They are examples of integral operators with kernels homogeneous of degree -1:

$$k(\lambda x,\lambda y)=\lambda^{-1}k(x,y), \quad x,y\in\mathbb{R}^1_+, \;\; \lambda>0;$$

we refer to [8] for such operators, see also [11], p. 51S. As is known, any operator

$$K\varphi(x)=\int\limits_0^{\infty}k(x,y)\varphi(y)dy$$

on the half-line $\mathbb{R}^1_+$ with a homogeneous kernel $k(x,y)$ may be reduced to convolution operator on the whole line $\mathbb{R}^1$ by means of the exponential change of variables. In the case of constant p and order of homogeneity -1, the mapping

$$(W_pf)(t)=e^{-\frac{t}{p}}f(e^{-t})\;, \;\; -\infty<t<\infty \tag{2.5}$$

realizes an isometry of $L_p(\mathbb{R}^1_+)$ onto $L_p(\mathbb{R}^1):\;\; \|W_pf\|_{L_p(\mathbb{R}^1)}=\|f\|_{L_p(\mathbb{R}^1_+)}$, and

$$W_pKW_p^{-1}=H \tag{2.6}$$

where

$$H\varphi=\int\limits_{\mathbb{R}^1}h(t-\tau)\varphi(\tau)d\tau\;, \quad h(t)=e^{\frac{t}{p'}}k(1,e^t), \quad t\in\mathbb{R}^1$$

and $\|h\|_{L^1(\mathbb{R}^n)}=\int\limits_0^{\infty}y^{-\frac{1}{p}}|k(1,y)|dy$.

3. Main statements

The following statements are valid, the proofs of which are given in Sections 5 and 6.

THEOREM 3.1. *Let $p\in\mathcal{P}_{0,\infty}$. Then the operators H^{α} and $\mathcal{H}_{\beta}$ are bounded in the space $L^{p(\cdot)}(\mathbb{R}^1_+)$ if and only if conditions (1.6) and (1.7) are fulfilled, respectively.*

REMARK 3.2. One may take α and β variable, and deal with the operators

$$H^{\alpha(\cdot)}f(x) = x^{\alpha(x)-1}\int\limits_0^x \frac{f(y)}{y^{\alpha(y)}}dy \quad \text{and} \quad \mathcal{H}_{\beta(\cdot)}f(x) = x^{\beta(x)}\int\limits_x^\infty \frac{\varphi(y)\,dy}{y^{\beta(y)+1}}\,, \tag{3.1}$$

where $\alpha(x)$ is an arbitrary bounded function log-continuous at the origin and infinity: $|\alpha(x)-\alpha(0)| \le \frac{A}{|\ln x|},\ 0 < x \le \frac{1}{2},\ |\alpha(x)-\alpha(\infty)| \le \frac{A}{\ln x},\ x \ge 2$, and similarly for $\beta(x)$. Then Theorem 3.1 remains valid with conditions (1.6) and (1.7) replaced by

$$\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)} \tag{3.2}$$

and

$$\beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}\,. \tag{3.3}$$

The following theorem is a generalization of Theorem 3.1 covering the Hardy inequalities (1.1)-(1.2).

THEOREM 3.3. *Let $p \in \mathcal{P}_{0,\infty}$ and $\mu \in \mathcal{M}_{0,\infty}$ and*

$$0 \le \mu(0) < \frac{1}{p(0)} \quad \textit{and} \quad 0 \le \mu(\infty) < \frac{1}{p(\infty)}.$$

Let also $q(x)$ be any function in $\mathcal{P}_{0,\infty}$ such that

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0) \quad \textit{and} \quad \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty). \tag{3.4}$$

Then the Hardy-type inequalities

$$\left\| x^{\alpha+\mu(x)-1}\int\limits_0^x \frac{f(y)\,dy}{y^\alpha}\right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \tag{3.5}$$

and

$$\left\| x^{\beta+\mu(x)}\int\limits_x^\infty \frac{f(y)\,dy}{y^{\beta+1}}\right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}\,, \tag{3.6}$$

are valid, if and only if α and β satisfy conditions (1.6) and (1.7).

REMARK 3.4. Theorem 3.1 is a particular case of Theorem 3.3, but we prefer to give separately their formulations and proofs.

4. On convolutions in $L^{p(\cdot)}(\mathbb{R}^n)$ with kernels in $L^1 \bigcap L^r$

As is well known, see [20], if a kernel $k(x)$ has a non-increasing radial dominant in $L^1(\mathbb{R}^n)$, the convolution $k * f(x)$ is pointwise estimated via the maximal function $Mf(x)$. Therefore, for such kernels the boundedness of the convolution operator in the space $L^{p(\cdot)}(\mathbb{R}^n)$ follows immediately from that of the maximal operator. However, in terms of sufficient conditions this in fact requires the log-condition to be satisfied on the whole Euclidean space $\mathbb{R}^n$. Meanwhile, for rather "nice" kernels, this everywhere log-continuity seems to be an extra requirement and it is natural to suppose that for such kernels the log-condition is needed only at the infinite point, that is, the decay condition. Theorem 4.6 below gives a sufficient condition of such a kind for convolution operators. To prove Theorem 4.6, we need the following embedding theorem from [5]. In Theorem 4.1 and in the sequel we use the convention $\exp(-K/0) := 0$.

THEOREM 4.1. *Let p, q be bounded exponents on $\mathbb{R}^n$. Then*

$$L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^n)$$

if and only if $p(\cdot) \geq q(\cdot)$ almost everywhere and there exists $K > 0$ such that

$$\int_{\mathbb{R}^n} \exp\left(\frac{-K}{\left|\frac{1}{q(x)} - \frac{1}{p(x)}\right|}\right) dx < \infty.$$

P r o o f. The theorem is just a reformulation of Lemma 2.2 of [5] with $\lambda = \exp(-K)$. ■

REMARK 4.2. For every function $p \in \mathcal{P}_\infty(\mathbb{R}^n)$ there exists $K > 0$ such that

$$\int_{\mathbb{R}^n} \exp\left(\frac{-K}{\left|\frac{1}{p(x)} - \frac{1}{p(\infty)}\right|}\right) dx < \infty. \tag{4.1}$$

Indeed, for $K := 2\,n\,A$, where $A = \sup\limits_{x\in\mathbb{R}^n} \left|\frac{1}{p(x)} - \frac{1}{p(\infty)}\right| \ln(2 + |x|)$, we have

$$\int_{\mathbb{R}^n} \exp\left(\frac{-K}{\left|\frac{1}{p(x)} - \frac{1}{p(\infty)}\right|}\right) dx \leq \int_{\mathbb{R}^n} \exp\big(-2\,n\,\ln(2 + |x|)\big)\, dx < \infty.$$

COROLLARY 4.3. *Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$, $\lambda = \mathrm{const}$ and $\lambda \geq p(\infty)$. Then*

$$L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\min\{p(\cdot),\lambda\}}(\mathbb{R}^n) \tag{4.2}$$

unconditionally when $\lambda > p(\infty)$ *and under the decay condition* (2.1) *or its weaker form* (4.1) *when* $\lambda = p(\infty)$.

REMARK 4.4. Let the exponents p_1, p_2, p_3 satisfy (1.3) and $p_1(x) \le p_2(x) \le p_3(x)$ almost everywhere on $\mathbb{R}^n$. Then

$$L^{p_1(\cdot)}(\mathbb{R}^n) \cap L^{p_3(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_2(\cdot)}(\mathbb{R}^n). \tag{4.3}$$

This follows directly from $t^{p_2} \le t^{p_1} + t^{p_3}$.

LEMMA 4.5. *Let a bounded exponent* p *on* $\mathbb{R}^n$ *with* $1 \le p_- \le p_+ < \infty$ *either satisfy* (2.1) *or its weaker form* (4.1). *Then*

$$L^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \cong L^{p(\infty)}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \tag{4.4}$$

and

$$L^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_-}(\mathbb{R}^n) \hookrightarrow L^{p(\infty)}(\mathbb{R}^n) + L^{p_-}(\mathbb{R}^n). \tag{4.5}$$

Moreover,

$$L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_-}(\mathbb{R}^n) + L^{p(\infty)}(\mathbb{R}^n). \tag{4.6}$$

P r o o f. For simplicity we will drop the $\mathbb{R}^n$ in the calculations. Due to (2.1), Theorem 4.1, and (4.3) it follows that

$$\begin{aligned} L^{p(\cdot)} \cap L^{p_+} &\hookrightarrow L^{\min\{p(\cdot),p(\infty)\}} \cap L^{p_+} && \text{by Theorem 4.1} \\ &\hookrightarrow L^{p(\infty)} \cap L^{p_+} && \text{by (4.3)} \\ &\hookrightarrow L^{\max\{p(\cdot),p(\infty)\}} \cap L^{p_+} && \text{by (4.3)} \\ &\hookrightarrow L^{p(\cdot)} \cap L^{p_+} && \text{by Theorem 4.1.} \end{aligned}$$

This proves (4.4). Analogously, by Theorem 4.1,

$$\begin{aligned} L^{p(\cdot)} \cap L^{p_-} &\hookrightarrow L^{\min\{p(\cdot),p(\infty)\}} \cap L^{p_-} \\ &\hookrightarrow L^{p(\infty)} + L^{p_-}, \end{aligned}$$

where we have used in the last step that $t^{\min\{p(\cdot),p(\infty)\}} \le t^{p(\infty)} + t^{p_-}$. Thus (4.5) is proved. To prove (4.6), define $f_0 := f\,\chi_{\{|f|\le 1\}}$ and $f_1 := f\,\chi_{\{|f|>1\}}$ so that $f = f_0 + f_1$. We have

$$\begin{aligned} |f_0(x)|^{p(x)} + |f_0(x)|^{p_+} &\le 2\,|f_0(x)|^{p(x)}, \\ |f_1(x)|^{p(x)} + |f_0(x)|^{p_-} &\le 2\,|f_1(x)|^{p(x)}. \end{aligned}$$

Thus

$$L^{p(\cdot)} \hookrightarrow (L^{p(\cdot)} \cap L^{p_+}) + (L^{p(\cdot)} \cap L^{p_-})$$

and then by (4.4)-(4.5)

$$L^{p(\cdot)} \hookrightarrow (L^{p(\infty)} \cap L^{p_+}) + (L^{p(\infty)} + L^{p_-}) \hookrightarrow L^{p(\infty)} + L^{p_-}.$$

■

THEOREM 4.6. *Let p and q be bounded exponents on $\mathbb{R}^n$ with $1 \leq p_- \leq p_+ < \infty$, $1 \leq q_- \leq q_+ < \infty$ and $q(\infty) \geq p(\infty)$, which satisfy either* (2.1) *or its weaker form* (4.1). *Let $r_0, s_0 \in [1, \infty)$ be defined by*

$$\frac{1}{r_0} = 1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}, \quad \frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{q_+}, \qquad r_0 \leq s_0. \tag{4.7}$$

Then the convolution operator $*$ satisfies

$$* : L^{p(\cdot)} \times (L^r \cap L^s) \to L^{q(\cdot)} \cap L^{q^+}, \quad r \leq r_0 \leq s_0 \leq s.$$

In particular, under the choice $q(x) \equiv p(x)$ and $r = 1$,

$$* : L^{p(\cdot)} \times (L^1 \cap L^{s_0}) \to L^{p(\cdot)} \cap L^{p^+}. \tag{4.8}$$

P r o o f. Besides (4.7), define

$$\frac{1}{r_1} = 1 - \frac{1}{p_-} + \frac{1}{q(\infty)}, \qquad \frac{1}{s_1} = 1 - \frac{1}{p(\infty)} + \frac{1}{q_+}.$$

Then

$$1 \leq r \leq r_0 \leq \min\{r_1, s_1\} \leq \max\{r_1, s_1\} \leq s_0 \leq s \leq \infty. \tag{4.9}$$

By classical Young's inequality the convolution operator $*$ satisfies

$$\begin{aligned} * &: L^{p^-} \times L^{s_0} \to L^{q_+}, \\ * &: L^{p^-} \times L^{r_1} \to L^{q(\infty)}, \\ * &: L^{p(\infty)} \times L^{s_1} \to L^{q_+}, \\ * &: L^{p(\infty)} \times L^{r_0} \to L^{q(\infty)}. \end{aligned}$$

Therefore,

$$\begin{aligned} * &: L^{p^-} \times (L^{r_1} \cap L^{s_0}) \to L^{q(\infty)} \cap L^{q_+}, \\ * &: L^{p(\infty)} \times (L^{r_0} \cap L^{s_1}) \to L^{q(\infty)} \cap L^{q_+}. \end{aligned} \tag{4.10}$$

Moreover, from (4.9) we deduce

$$L^r \cap L^s \hookrightarrow L^{r_1} \cap L^{s_0}, \qquad L^r \cap L^s \hookrightarrow L^{r_0} \cap L^{s_1}.$$

This and (4.10) imply

$$* : (L^{p-} + L^{p(\infty)}) \times (L^r \cap L^s) \to L^{q(\infty)} \cap L^{q^+}.$$

Thus embeddings (4.6) and (4.4) imply

$$* : L^{p(\cdot)} \times (L^r \cap L^s) \to L^{q(\cdot)} \cap L^{q^+}.$$

This proves the theorem. ■

COROLLARY 4.7. *Let $k(y)$ satisfy the estimate*

$$|k(y)| \le \frac{C}{(1+|y|)^\nu}, \qquad y \in \mathbb{R}^n \tag{4.11}$$

for some $\nu > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$. Then the convolution operator

$$Af(x) = \int\limits_{\mathbb{R}^n} k(y) f(x-y) dy$$

is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n)$ to the space $L^{q(\cdot)}(\mathbb{R}^n)$ under the only assumption that $p, q \in \mathcal{P}_\infty(\mathbb{R}^n)$ and $q(\infty) \ge p(\infty)$.

P r o o f. The assumption on ν implies $k \in \cup_{r \ge r_0} L^r$ with r_0 given by (4.7). Now, use Theorem 4.6. ■

As an important immediate application of the above boundedness of convolution operators, consider the *Bessel potential operator*

$$\mathcal{B}^\alpha f(x) = \int_{\mathbb{R}^n} G_\alpha(x-y)\, f(y)\, dy, \qquad \alpha > 0, \tag{4.12}$$

well known in the theory of Sobolev type spaces of fractional smoothness. In (4.12)

$$G_\alpha(x) = c(\alpha, n)\, |x|^{\frac{\alpha-n}{2}}\ K_{\frac{n-\alpha}{2}}(|x|),$$

and $K_{\frac{n-\alpha}{2}}(r)$ is the Bessel-MacDonald function of order $\frac{n-\alpha}{2}$, so that $G_\alpha(x)$ exponentially vanishes at infinity and $G_\alpha(x) \sim c|x|^{\alpha-n}$ as $|x| \to 0$. According to Theorem 4.6 in the case

$$\frac{1}{p_-} - \frac{1}{p_+} < \frac{\alpha}{n}, \tag{4.13}$$

this operator is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$ under the only assumption that $p(\cdot)$ satisfies the decay condition, that is, $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. The boundedness of the operator $\mathcal{B}^\alpha$ in the space $L^{p(\cdot)}(\mathbb{R}^n)$ was earlier shown in [2], [1] without the condition (4.13) on the variation of $\frac{1}{p(x)}$, but under the assumption that besides the decay condition, p satisfies everywhere the local log-condition.

5. Proof of Theorem 3.1

We will follow the idea of reducing operators with homogeneous kernels to convolutions, presented in (2.5)-(2.6), and introduce the mapping

$$(W_p f)(t) = e^{-\frac{t}{p(0)}} f(e^{-t}) \,, \quad t \in \mathbb{R}^1. \tag{5.1}$$

5.1. Auxiliary lemmas

We need the following simple lemmas.

LEMMA 5.1. *Let* $p(x) \in \mathcal{P}(\mathbb{R}^1_+)$ *and* $p(0) = p(\infty)$. *Then the operator* W_p *maps isomorphically the space* $L^{p(\cdot)}(\mathbb{R}^1_+)$ *onto the space* $L^{p_*(\cdot)}(\mathbb{R}^1)$, *where* $p_*(t) = p\left(e^{-t}\right)$ *and* $p_*(-\infty) = p_*(+\infty)$ *and*

$$|p_*(t) - p_*(\infty)| \le \frac{A}{|t|+1}, \quad t \in \mathbb{R}^1. \tag{5.2}$$

P r o o f. We have

$$\int\limits_{\mathbb{R}^1} \left|\frac{W_p f(t)}{\lambda}\right|^{p_*(t)} dt = \int\limits_{\mathbb{R}^1} \left|\frac{e^{-\frac{t}{p(0)}} f\left(e^{-t}\right)}{\lambda}\right|^{p_*(t)} dt = \int\limits_{\mathbb{R}^1_+} x^{\frac{p(x)}{p(0)}-1} \left|\frac{f\left(x\right)}{\lambda}\right|^{p(x)} dx. \tag{5.3}$$

By (2.2) we have the equivalence

$$x^{\frac{1}{p(x)}} \sim x^{\frac{1}{p(0)}} \tag{5.4}$$

on the whole half-axis $\mathbb{R}^1_+$. Therefore, the boundedness

$$\|W_p f\|_{L^{p_*(\cdot)}(\mathbb{R}^1)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}$$

follows from (5.3) in view of equivalence (5.4). Similarly, the boundedness

$$\|W_p^{-1}\psi\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \le C\|\psi\|_{L^{p_*(\cdot)}(\mathbb{R}^1)}$$

no of the inverse operator

$$W_p^{-1}\psi(x) = x^{-\frac{1}{p(0)}}\psi(-\ln x)$$

is checked. ■

LEMMA 5.2. *For the Hardy operators H^α and $\mathcal{H}_\beta$ the following relations are valid*

$$(W_p H^\alpha W_p^{-1})\psi(t) = \int\limits_{\mathbb{R}^1} h_-(t-\tau)\psi(\tau)d\tau, \tag{5.5}$$

and

$$(W_p \mathcal{H}_\beta W_p^{-1}\psi(t) = \int\limits_{\mathbb{R}^1} h_+(t-\tau)\psi(\tau)d\tau, \tag{5.6}$$

where

$$h_-(t) = e^{\left(\frac{1}{p'(0)}-\alpha\right)t}\theta_-(t) \quad \textit{and} \quad h_+(t) = e^{-\left(\frac{1}{p(0)}+\beta\right)t}\theta_+(t) \tag{5.7}$$

and $\theta_-(t) = 1-\theta_+(t)$.

P r o o f. The proof is a matter of direct verification. ■

5.2. The proof of Theorem 3.1 itself

I. Sufficiency

1^0. The case $p(0) = p(\infty)$.

We first assume that

$$p(0) = p(\infty). \tag{5.8}$$

By Lemmas 5.1 and 5.2 the boundedness of the Hardy operators H^α and $\mathcal{H}_\beta$ in the space $L^{p(\cdot)}(\mathbb{R}^1_+)$ is equivalent to that of the convolution operators with the exponential kernels $h_-(t)$ and $h_+(t)$, respectively, in the space $L^{p_*(\cdot)}(\mathbb{R}^1)$. Under the conditions $\frac{1}{p'(0)} - \alpha > 0$ and $\frac{1}{p(0)} + \beta > 0$, the convolutions $h_- * \psi$ and $h_+ * \psi$ are bounded operators in $L^{p_*(\cdot)}(\mathbb{R}^1)$ according to Corollary 4.7, and consequently, the Hardy operators H^α and $\mathcal{H}_\beta$ are bounded in the space $L^{p(\cdot)}(\mathbb{R}^1_+)$

2^0. The case $p(0) \neq p(\infty)$.

Let $0 < \delta < N < \infty$ and let $\chi_E(x)$ denote the characteristic function of a set $E \subset \mathbb{R}^1_+$. (Note that the idea of the following splitting and extension of the exponent $p(\cdot)$ was suggested by E. Shargorodsky in the discussion of the proof of Hardy inequalities with the second author). We have :

$$\begin{aligned} H^\alpha f(x) &= \left(\chi_{[0,\delta]} + \chi_{[\delta,N]} + \chi_{[N,\infty)}\right) H^\alpha \left(\chi_{[0,\delta]} + \chi_{[\delta,N]} + \chi_{[N,\infty)}\right) f(x) \\ &= \chi_{[0,\delta]}(x) \left(H^\alpha \chi_{[0,\delta]} f\right)(x) + \chi_{[\delta,\infty)}(x) \left(H^\alpha \chi_{[0,N]} f\right)(x) \\ &+ \chi_{[N,\infty)}(x) \left(H^\alpha \chi_{[N,\infty)} f\right)(x) =: V_1(x) + V_2(x) + V_3(x). \end{aligned} \tag{5.9}$$

It suffices to estimate separately the modulars $I_p(V_k), k = 1, 2, 3$, supposing that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \leq 1$.

For $I_p(V_1)$ we obtain

$$I_p(V_1) = \int_0^\delta \left| \int_0^x \frac{x^{\alpha-1}}{y^\beta} f(y)dy \right|^{p(x)} dx$$

$$\leq \int_0^\infty \left(\int_0^x \frac{x^{\alpha-1}}{y^\beta} |f(y)|dy \right)^{p_1(x)} dx = I_{p_1}(H^\alpha f), \tag{5.10}$$

where $p_1(x)$ is any extension of $p(x)$ from $[0, \delta]$ to the whole half-axis $\mathbb{R}^1_+$ which satisfies conditions (1.3)-(1.5) and for which $p_1(0) = p_1(\infty)$. Such an extension is always possible, see Appendix in Section 7. Then from (5.10) we obtain

$$I_p(V_1) \leq C < \infty \quad \text{whenever} \quad \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \leq 1$$

according to the item $\mathbf{1}^0$ of the proof.

The estimation of $I_p(V_3)$ is quite similar to that of $I_p(V_1)$ with the only change that the corresponding extension of $p(x)$ must be made from $[N, \infty)$ to $\mathbb{R}^1_+$.

Finally, the estimation of the term $V_2(x)$ is evident:

$$I_p(V_2) \leq \int_\delta^\infty \left| x^{\alpha-1} \int_0^N \frac{f(y)}{y^\alpha} dy \right|^{p(x)} dx, \tag{5.11}$$

where it suffices to apply the Hölder inequality in $L^{p(\cdot)}(\mathbb{R}^1_+)$ when we integrate in y with $\alpha < \frac{1}{p'(0)}$ taken into account, and make use of the fact that $\alpha < \frac{1}{p'(\infty)}$ when we integrate in x.

Similarly the case of the operator $\mathcal{H}_\beta$ is considered (or alternatively, one can use the duality arguments, but the latter should be modified by considering separately the spaces on $[0, \delta]$ and $[N, \infty)$, because we admit $p(x) = 1$ in between).

II. Necessity

The necessity of condition (1.6) follows from the simple arguments: the boundedness of the operator H^α in $L^{p(\cdot)}(\mathbb{R}^1_+)$ implies that H^α is well defined on all the functions in $L^{p(\cdot)}(\mathbb{R}^1_+)$, in particular, on the function

$$f_0(x) = \frac{\chi_{[0,\frac{1}{2}]}(x)}{x^{\frac{1}{p(0)}} \ln \frac{1}{x}} \in L^{p(\cdot)}(\mathbb{R}^1_+) \tag{5.12}$$

so that the existence the integral

$$H^\alpha f_0(x) = x^{\alpha-1}\int_0^x \frac{dy}{y^{\alpha+\frac{1}{p(0)}}\ln\frac{1}{y}}dy, \quad 0<x<\frac{1}{2}$$

implies the condition $\alpha < \frac{1}{p'(0)}$.

To show the necessity of the condition $\alpha < \frac{1}{p'(\infty)}$, it suffices to choose

$$f_\infty(x) = \frac{\chi_{[2,\infty)}(x)}{x^\lambda} \in L^{p(\cdot)}(\mathbb{R}^1_+), \quad \lambda > \max(1, 1-\alpha). \tag{5.13}$$

For $x \geq 3$ we have $H^\alpha f_\infty(x) = x^{\alpha-1}\int_2^x \frac{dy}{y^{\alpha+\lambda}} \geq x^{\alpha-1}\int_2^3 \frac{dy}{y^{\alpha+\lambda}} = cx^{\alpha-1}$ which belongs to $L^{p(\cdot)}(\mathbb{R}^1_+)$ only if $(1-\alpha)p(\infty) > 1$, that is, $\alpha < \frac{1}{p'(\infty)}$.

Similarly the necessity of condition (1.7) is considered.

6. Proof of Theorem 3.3

6.1. Preliminaries

As in Section 5, we shall consider first the case where $p(0) = p(\infty)$ and suppose that $\mu(0) = \mu(\infty)$ as well.

Similarly to (5.1), we use the mappings

$$(W_p f)(t) = e^{-\frac{t}{p(0)}} f(e^{-t}) \quad \text{and} \quad (W_q f)(t) = e^{-\frac{t}{q(0)}} f(e^{-t}), \qquad t \in \mathbb{R}^1, \tag{6.1}$$

where $\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0)$. As a generalization of Lemma 5.2, we have the following statement for the operators

$$H^{\alpha,\mu} f(x) = x^{\alpha+\mu(0)-1}\int_0^x \frac{f(y)\,dy}{y^\alpha}, \quad \mathcal{H}_{\beta,\mu} f(x) = x^{\beta+\mu(0)}\int_x^\infty \frac{f(y)\,dy}{y^{\beta+1}}. \tag{6.2}$$

LEMMA 6.1. *For the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ the following relations hold*

$$(W_q H^{\alpha,\mu} W_p^{-1})\psi(t) = \int_{\mathbb{R}^1} h_-(t-\tau)\psi(\tau)d\tau, \tag{6.3}$$

and

$$(W_q \mathcal{H}_{\beta,\mu} W_p^{-1}\psi(t) = \int_{\mathbb{R}^1} h_+(t-\tau)\psi(\tau)d\tau, \tag{6.4}$$

where $\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0)$ and $h_-(t)$ and $h_+(t)$ are the same kernels as in (5.7).

P r o o f. The proof is direct. ■

6.2. The proof of Theorem 3.3 itself

1^0. The case $p(0) = p(\infty)$ and $\mu(0) = \mu(\infty)$.

By Lemma 5.1 we have

$$\|W_p f\|_{L^{p_*(\cdot)}(\mathbb{R}^1)} \sim \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \quad \text{and} \quad \|W_q^{-1}\psi\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \sim \|\psi\|_{L^{q_*(\cdot)}(\mathbb{R}^1)}, \tag{6.5}$$

where $p_*(t) = p(e^{-t}), q_*(t) = q(e^{-t})$. Therefore, the $L^{p(\cdot)}(\mathbb{R}^1) \to L^{q(\cdot)}(\mathbb{R}^1)$ boundedness of the operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ follows from the $L^{p_*(\cdot)}(\mathbb{R}^1) \to L^{q_*(\cdot)}(\mathbb{R}^1)$ boundedness of the convolution operators on $\mathbb{R}^1$ with the kernels $h_+(t)$ and $h_-(t)$, respectively.

Since $\frac{1}{p'(0)} - \alpha > 0$ and $\frac{1}{p(0)} + \beta > 0$, the convolutions $h_- * \psi$ and $h_+ * \psi$ are bounded operators from $L^{p_*(\cdot)}(\mathbb{R}^1)$ to $L^{q_*(\cdot)}(\mathbb{R}^1)$ in view of Corollary 4.7. Consequently, the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ are bounded from $L^{p(\cdot)}(\mathbb{R}^1_+)$ to $L^{q(\cdot)}(\mathbb{R}^1_+)$. Then we get at (3.5)-(3.6) since

$$x^{\mu(x)} \sim x^{\mu(0)} \quad \text{on} \quad \mathbb{R}^1_+$$

by (2.2).

2^0. The general case.

The proof follows the same lines as in (5.9)-(5.11). For example, instead of (5.10), we will have

$$I_q(V_1) = \int_0^\delta \left| \int_0^x \frac{x^{\alpha+\mu(x)-1}}{y^\beta} f(y)dy \right|^{q(x)} dx \le \int_0^\infty \left(\int_0^x \frac{x^{\alpha+\mu_1(x)-1}}{y^\beta} |f(y)|dy \right)^{q_1(x)} dx, \tag{6.6}$$

and analogously for $I_q(V_3)$. That is, we have to arrange the corresponding extension of $p(x), q(x)$ and $\mu(x)$, which is possible, see Lemma 7.1.

Similarly, the estimation

$$I_q(V_2) \le C \int_\delta^\infty \left| x^{\alpha+\mu(\infty)-1} \int_0^N \frac{f(y)}{y^\alpha} dy \right|^{q(x)} dx \le C < \infty$$

is easily obtained.

Finally, it suffices to observe that the counterexamples for the necessity in Theorem 3.3 are the same as in (5.12) and (5.13.)

7. Appendix

LEMMA 7.1. *Given a function* $p(x) \in \mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$*, there exists its extension* $p_1(x) \in \mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ *from an interval* $[0,\delta], \delta > 0,$ *to* $\mathbb{R}^1_+$ *such that*

$i)$ $p_1(x) \equiv p(x), \;\; x \in [0,\delta],$ and

$ii)$ $p_1(0) = p_1(\infty).$

P r o o f. The possibility of such an extension is obvious, but we give a direct constriction for the completeness of the presentation.

A function $p_1(x)$ may be taken in the form

$$p_1(x) = \omega(x)\, p(x) + (1 - \omega(x))\, p(\infty),$$

where $\omega \in C^\infty([0,\infty))$ has compact support and $\omega(x) = 1$ for $x \in [0,\delta]$. Then both log-conditions in (1.4)-(1.5) are satisfied for $p_1(x)$. Moreover,

$$1 \leq \min\{(p_0)_-, p(\infty)\} \leq (p_1)_- \quad \text{and} \quad (p_1)_+ \leq \max\{(p_0)_+, p(\infty)\} < \infty.$$

■

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