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POLYNOMIAL EXPANSIONS FOR SOLUTIONS OF HIGHER-ORDER BESSEL HEAT EQUATION IN QUANTUM CALCULUS

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Abstract

In this paper we give the q-analogue of the higher-order Bessel operators studied by I. Dimovski [3],[4], I. Dimovski and V. Kiryakova [5],[6], M. I. Klyuchantsev [17], V. Kiryakova [15], [16], A. Fitouhi, N. H. Mahmoud and S. A. Ould Ahmed Mahmoud [8], and recently by many other authors.

Our objective is twofold. First, using the q-Jackson integral and the q-derivative, we aim at establishing some properties of this function with proofs similar to the classical case. Second, our goal is to construct the associated q-Fourier transform and the q-analogue of the theory of the heat polynomials introduced by P. C. Rosenbloom and D. V. Widder [22]. For some value of the vector index, our operator generalizes the q-j_α Bessel operator of the second order in [9] and a q-Third operator in [12].

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Key Words and Phrases: q-analysis, q-Fourier transform, q-heat equation, q-Laguerre polynomials, q-heat polynomials

1. Introduction

The Bessel operator of r-order is defined on $(0, \infty)$ by

$$B_r u = u^{(r)} + \frac{a_1}{x} u^{(r-1)} + ... + \frac{a_{r-1}}{x^{r-1}} u^{(1)}, \tag{1}$$

where the coefficients a_k depend on the components α_k,

$$\alpha_k \geq -1 + \frac{k}{r} \quad , \quad k = 1, ..., r-1, \tag{2}$$

and

$$a_{r-k} = \frac{1}{(k-1)!} \sum_{j=1}^{k} (-1)^{k-j} \begin{pmatrix} j-1 \\ k-1 \end{pmatrix} \prod_{i=1}^{r-1} (r\alpha_i + j), \tag{3}$$

where r is positive integer and $\alpha = (\alpha_1, ..., \alpha_{r-1})$ a vector having $(r-1)$ components with $|\alpha| = \alpha_1 + ... + \alpha_{r-1}$.

The higher-order Bessel differential operators, called recently as hyper-Bessel operators, have been introduced by I. Dimovski [3],[4] and studied by I. Dimovski and V. Kiryakova [5],[6], M. I. Klyuchantsev [17], V. Kiryakova [15, Ch. 3], [16], A. Fitouhi, N. H. Mahmoud and S. A. Ould Ahmed Mahmoud [8], and by many other authors (see references in [15], [16].

When $r = 2$, we obtain the classical Bessel operator of the second order

$$B_2 u = u'' + \frac{2\alpha + 1}{x} u', \tag{4}$$

and for $r = 3$, $\alpha_1 = -2/3$, $\alpha_2 = \nu - 1/3$, we obtain the operator $B_3 u$, studied in [1] and in [10]

$$B_3 u = \frac{d^3}{dx^3} + \frac{3\,\nu}{x} \frac{d^2}{dx^2} - \frac{3\,\nu}{x^2} \frac{d}{dx}, \qquad \nu > 0. \tag{5}$$

For λ being a complex number, let us now consider the system

$$\begin{aligned} B_r u(x) &= -\lambda^r u(x), \\ u(0) &= 1, \\ u^k(0) &= 0, \quad k = 1, ..., r-1. \end{aligned}$$

The use of the Frobenius method leads us to conclude that (6) has a unique solution which is r-even and given by

$$j_\alpha(\lambda x) = \sum_{m \geq 0} (-1)^m \frac{1}{m!} \prod_{i=1}^{r-1} \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + m + 1)} \left(\frac{\lambda x}{r}\right)^{rm}. \tag{6}$$

In this paper we are concerned with the q-analogue of the j_α higher-order Bessel function (6). This choice is motivated in particular by the context of [8], [9], [12].

The reader will notice that the definition (39) derives from that given in [8] with minor changes. With the help of the q-integral representation we establish the q-integral representation of the Mehler and Sonine types. Moreover, we define the higher-order q-Bessel translation and the higher-order q-Bessel Fourier transform and establish some of their properties. Finally, we study the higher-order q-Bessel heat equation.

2. Notation and preliminary results

Let q be a fixed real number $0 < q < 1$. We use the following notation:

$$(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b), \text{ if } n = 0,1,2,...,\infty, \tag{7}$$

$$(1+a)_q^t = \frac{(1+a)_q^\infty}{(1+q^t a)_q^\infty}, \quad \text{if} \quad t \in C, \tag{8}$$

and put

- $R_q = \{\pm q^k,\ k \in Z\} \cup \{0\}, \quad R_{q,+} = \{q^k,\ k \in Z\}$ and $(n)_2 = \frac{n(n-1)}{2}$.

Note that for $\lambda \in R$, $n = 0,1,2,...,$

$$\begin{aligned}
(a;\,q)_n &= (1-a)(1-aq)...(1-aq^{n-1}), \qquad (\lambda)_q = \frac{1-q^\lambda}{1-q},\\
(\lambda)_n^q &= \frac{(q^\lambda;\,q)}{(1-q)^n}, \quad [n]_q! = \frac{(q;\,q)_n}{(1-q)^n}, \quad \frac{(\lambda)_n^q}{[n]_q!} = \frac{(q^\lambda;\,q)_n}{(q;\,q)_n},\\
\frac{(\lambda)_n^q}{(\lambda+n-1)_q} &= (\lambda)_{n-1}^q, \quad \frac{(1)_n^q}{(1)_{n-k}^q} = (-1)^k(-n)_k^q q^{nk-(k)_2}.
\end{aligned}$$

and then, the q-Binomial formula is:

$$(ab;\,q)_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q b^k (a;\,q)_k (b;\,q)_{n-k}, \quad \text{with} \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q;\,q)_n}{(q;\,q)_{n-k}(q;\,q)_k}. \tag{9}$$

Further we denote by D_q the q–derivative of a function by:

$$D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x} \ . \tag{10}$$

$$D_q^n f(x) = \frac{q^{-(n)_2}}{x^n(1-q)^n}\sum_{k=0}^{n}(-1)^k \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{(n-k)(n-k)/2} f(q^k x),\ n = 0,1,2,.... \tag{11}$$

$$D_q^n[f(x)g(x)] = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q (D_q^{n-k} f)(q^k x)(D_q^k g)(x),\ n = 0,1,2,..., \tag{12}$$

and define the q-shift operators by:

$$(\Lambda_q f)(x) = f(qx) \qquad \text{and} \qquad (\Lambda_q^{-1} f)(x) = f(q^{-1}x),$$

noting that $(\Lambda_{q^\delta}^{-1} f)(x) = f(q^{-\delta}x)$.

The q-Jackson integrals (introduced by Thomae and Jackson [13]) from 0 to a and from aq to ∞ are defined by

$$\int_0^a f(x)d_qx = (1-q)\sum_{j=0}^{\infty} aq^j f(aq^j) \text{ and } \int_{aq}^{\infty} f(t)d_qt = (1-q)\sum_{k=0}^{+\infty} aq^{-k} f(aq^{-k}). \tag{13}$$

Notice that the last series are guaranteed to be convergent, see [9].

We define the Jackson integral in a generic interval $[a, b]$ by [13]:

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \ .$$

This is a special case of the following more general change of variable formula, [14, p 107]. If $u(x) = \alpha x^\beta$, then

$$\int_{u(a)}^{u(b)} f(u)d_qu = \int_a^b f(u(x))D_{q^{1/\beta}}u(x)d_{q^{1/\beta}}x \ .$$

Using the q-Jackson integrals from 0 to 1, we define the q-integral $\int_0^1 ... \int_0^1 f(t_1, ..., t_n)d_qt_1...d_qt_n$ by:

$$\int_0^1 ... \int_0^1 f(t_1, ..., t_n)\, d_qt_1...d_qt_n = (1-q)^n \sum_{i_1,...,i_n=0}^{\infty} q^{i_1+...+i_n}\ f(q^{i_1+...+i_n}), \tag{14}$$

provided the sums converge absolutely.

We present two q–analogues exponential function:

$$E_q(x) = \sum_{n=0}^{\infty} q^{(n)} \frac{x^n}{[n]_q!} = (1 + (1-q)x)_q^{\infty} \ , \tag{15}$$

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}} \ . \tag{16}$$

Notice that for $q \in (0, 1)$ the series expansion of $e_q(x)$ has radius of convergence $1/(1-q)$. On the contrary, the series expansion of $E_q(x)$ converges for every x. Both product expansions (15) and (16) converge for all x.

We define the q^δ-basic hypergeometric series ${}_r\phi_s^\delta$ by

$${}_r\phi_s^\delta \left(\begin{matrix} q^{a_1}, ..., q^{a_r} \\ q^{b_1}, ..., q^{b_s} \end{matrix} \middle| q; (q-1)^{1+s-r}z \right) = \sum_{k=0}^{\infty} (q^\delta)^{(k)_2} \frac{(a_1; q)_k^q ... (a_r; q)_k^q}{(b_1; q)_k^q ... (b_s; q)_k^q} \frac{z^k}{[k]_q!}, \tag{17}$$

$$\lim_{q\uparrow 1} {}_r\phi_s^\delta \left(\begin{array}{c} q^{a_1}, ..., q^{a_r} \\ q^{b_1}, ..., q^{b_s} \end{array} \middle| q; (q-1)^{1+s-r} z \right) = {}_r F_s \left[\begin{array}{c} a_1, ...a_r \\ b_1, ...b_s \end{array} \middle| z \right]. \tag{18}$$

Here $\delta > 0$ and $r < s+1$, thus the expansion converges for all values of z.

For $\delta = 1 + s - r$, we obtain the classical basic hypergeometric series ${}_r\phi_s$, [18, p 11,12].

Note that for $\delta > 0$ by $e_q(x, \delta) = \sum_{n=0}^{\infty} q^{\delta (n)_2} \frac{x^n}{[n]_q!}$, this expansion converges for all values of x.

The q–gamma function $\Gamma_q(t)$, a q–analogue of Euler's gamma function, was introduced by Thomae and later by Jackson as the infinite product

$$\Gamma_q(t) = \frac{(1-q)_q^{t-1}}{(1-q)^{t-1}} , \quad t > 0 . \tag{19}$$

The q–Beta function defined by the usual formula

$$\beta_q(t, s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)} , \tag{20}$$

has a q–integral representation, which is a q–analogue of Euler's formula:

$$\beta_q(t, s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x , \quad t, s > 0 . \tag{21}$$

The q-duplication formula holds:

$$\prod_{i=1}^{r-1} \Gamma_{q^r}(n + \frac{i}{r}) \frac{1}{[rn]_q!} = \prod_{i=1}^{r-1} \Gamma_{q^r}(\frac{i}{r}) \frac{1}{[n]_{q^r}!} \frac{1}{((r)_q)^{rn}}, \tag{22}$$

and

$$((r)_q)^{rn} (1)_n^{q^r} \prod_{i=1}^{r-1} (\frac{i}{r})_n^{q^r} = [rn]_q!. \tag{23}$$

We also denote, $\prod(\alpha_i + 1)_n^{q^r} = \prod_{i=1}^{r-1} (\alpha_i + 1)_n^{q^r}$.

3. q-Trigonometric function of r-order

The $r - q^\delta$-cosinus is defined for $\delta > 0$ by

$$\cos_r(x, q^r; \delta) = {}_0\phi_{r-1}^\delta \left(\begin{array}{c} - \\ (q^r)^{1/r}, ..., (q^r)^{(r-1)/r} \end{array} \middle| q^r; -\frac{(q^r-1)^r x^r}{(1+q+...+q^{r-1})^r} \right) \tag{24}$$

$$= \sum_{m \geq 0} (-1)^m b_{rm}(x, q^r; \delta), \tag{25}$$

where

$$b_{rm}(x,\, q^r;\delta) = (q^\delta)^{r(m)_2}\frac{x^{rm}}{[rm]_q!} = (q^r)^{\delta(m)_2}\frac{x^{rm}}{\alpha_{rm,q}}. \qquad (26)$$

For every $\lambda \in C$, the function $\cos_r(x,\, q^r;\delta)$ is a unique solution of the system

$$\begin{cases} \Lambda_{q\delta}^{-1}D_q^r u(x) & = & -\lambda^r u(x), \\ u(0) & = & 1, \\ D_q^k u(0) & = & 0, \quad k = 1,...,r-1. \end{cases}$$

We note $r - q^\delta$-sinus of order (r,l) , $l = 1,...,r-1$ by

$$\sin_{r,l}(x,\, q^r;\delta) = \sum_{m\geq 0}(-1)^m(q^\delta)^{r(m)_2}\frac{x^{rm+r-l}}{[rm+r-l]_q!}. \qquad (27)$$

Let $\mu = e^{i\pi/r}$ and $w_k = e^{2i\pi(k-1)/r}$, $k = 1,2,...,r$. Since

$$\sum_{k=1}^{r}(w_k)^m = \begin{cases} r & \text{for integers } m \text{ divisible by } r \\ 0 & \text{for integers } m \text{ not divisible by } r \end{cases} \qquad (28)$$

and expanding the q-exponential function in series, we obtain

$$\cos_r(x,\, q^r;r\delta) = \frac{1}{r}\sum_{k=1}^{r} e_{q^r}\big(\frac{\mu w_k x}{q^{(r-1)/2}}\ ,\, \delta\big). \qquad (29)$$

When $r = 3$, $\delta = 1$, we obtain the result in [12].

Definition 3.1. Let $x \in R$ and $w_k = e^{2i\pi(k-1)/r}$, $k = 1,2,...,r$, a function $f(x)$ is called r-even, if

$$f(w_k x) = f(x) \qquad k = 1,...,r, \qquad (30)$$

and r-odd of l order, if

$$f(x) = w_k^l f(w_k x), \qquad k = 1,...,r. \qquad (31)$$

Proposition 3.1. *The functions* $\cos_r$ *and* $\sin_{r,l}$ $(l = 1,...,r-1)$ *are, respectively,* r*-even and* r*-odd of order* l*. From (24) and (27) we obtain the following* q*-derivative formulas:*

$$\begin{aligned} D_q^l\cos_r(x,\, q^r;\delta) &= -\, q^{-\delta(r-l)}\sin_{r,l}(q^\delta x,\, q^r;\delta), \\ D_q^r\cos_r(x,\, q^r;\delta) &= -\cos_r(q^\delta x,\, q^r;\delta,), \\ D_q^{l-m}\sin_{r,m}(x,\, q^r;\delta) &= \sin_{r,l}(x,\, q^r;\delta), \\ D_q^{r-m}\sin_{r,m}(x,\, q^r;\delta) &= \cos_r(x,\, q^r;\delta). \end{aligned}$$

Proposition 3.2. *The function* $\cos_r(x,\, q^r;1)$ *is* r*-even and satisfies, in particular*

$$\cos_r(xt,\, q^r;1) = (-1)^n q^{r(n(n+1)/2)} \frac{1}{x^{rn}} D^{rn}_{q,t}(\cos_r(xtq^{-n},\, q^r;1)). \tag{32}$$

Proposition 3.3. *Let* $x \in R$ *for* $n \geq 1$, *the function* $b_{rn}(x,\, q^r;1)$ *verifies the following properties*

$$b_0(x,\, q^r;1) = 1,\ b_{rn}(0,\, q^r;1) = 0 \text{ and } \ \Lambda_q^{-1} D_q^r b_{rn}(x,\, q^r;1) = b_{r(n-1)}(x,\, q^r;1).$$

Furthermore,

$$\mid b_{rn}(x,\, q^r;\delta) \mid \leq \mid b_{rn}(x,\, q^r;1) \mid \leq \frac{q^{-(r)_2} x^{rn}}{(rn)!},\ \delta \geq 1. \tag{33}$$

P r o o f. When we put $q = e^{-t}, \quad t > 0$. The coefficients $b_{rn}(x,\, q^r;1)$ defined by (24) can be written as

$$b_{rn}(x,\, q^r;1) = \prod_{j=0}^{n-1}\prod_{i=0}^{r-1} \frac{q^j - q^{j+1}}{1 - q^{rj+1+i}} = \prod_{j=0}^{n-1}\prod_{i=0}^{r-1} \frac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-(rj+1+i)t}}.$$

Preceding like in [20], we can deduce the result and we have when $|x| \uparrow \infty$, we have
$|\cos_r(x,\, q^r;\delta)| \leq q^{-(r)_2} |\cos_r(x)| \leq q^{-(r)_2}\, e^{(r-2)|x|}, \quad \delta \geq 1,$ see [9]. ■

4. q^δ-product formula

We set now the product formula for q^δ-cosinus function. We note by

$$P = \cos_r(x,\, q^r;\delta)\cos_r(y,\, q^r;\delta).$$

Proposition 4.1. *Let* x *and* y *be complex numbers, with* $y \neq 0$, *we have:*

$$P = \sum_{k\geq 0} \frac{(-1)^k (q^\delta)^{rk^2}}{(1-q)^{rk}[rk]_q!} q^{-(rk)_2} \left(\frac{x}{y}\right)^{rk} \sum_{s=0}^{rk} (-1)^s q^{(s)_2} \begin{bmatrix} rk \\ s \end{bmatrix}_q \Lambda_{q^\delta}^{-k} \cos_r(yq^{rk-s},\, q^r;\delta).$$

P r o o f. For $y \neq 0$,

$$P = \sum_{k\geq 0} \frac{(q^\delta)^{rk^2}}{[rk]_q!} \left(\frac{x}{y}\right)^{rk} \sum_{n\geq 0} (-1)^n \frac{(q^{r\delta})^{(n)_2}}{[r(n-k)]_q!} (q^\delta)^{-rnk} y^{rn}.$$

Moreover, if we use the previous relation

$$\frac{[rn]_q!}{[r(n-k)]_q!}(1-q)^{rk} = (-1)^{rk} q^{-(rk)_2 + r^2 nk} \sum_{s=0}^{rk} (-1)^s q^{(s)_2} \left[\begin{array}{c} rk \\ s \end{array} \right]_q q^{-rns},$$

we obtain that

$$P = \sum_{k\geq 0} \frac{(-1)^k q^{-(rk)_2} (q^\delta)^{rk^2}}{(1-q)^{rk}[rk]_q!} \left(\frac{x}{y}\right)^{rk} \sum_{s=0}^{rk} (-1)^s q^{(s)_2} \left[\begin{array}{c} rk \\ s \end{array} \right]_q \cos_r(yq^{k(r-\delta)}q^{-s},\, q^r; \delta).$$

■

5. The q-Bessel operator of r-order

We suppose now that the components of the vector $\alpha = (\alpha_1, ..., \alpha_{r-1})$, where α_k is a real number, satisfy $\alpha_k \geq -1 + \frac{k}{r}$, $\quad k = 1, ..., r-1$ and $\delta > 0$.

The q-Bessel operator of r-order is defined by:

$$B_{r,\delta} u = \Lambda_{q^\delta}^{-1} \Big(\frac{1}{x^{r-1}} \prod_{i=1}^{r-1} \big(q^{r\alpha_i + 1} x D_q + (r\alpha_i + 1)_q \, \big) D_q u \Big). \tag{34}$$

Remark 5.1. For $r = 2$, we obtain the q-Bessel operator $B_{2,\delta}$ of the second order studied in [9] for $\delta = 1$:

$$B_{2,\delta} u = \Lambda_{q^\delta}^{-1} \big(q^{2\alpha+1} D_q^2 u + \frac{(2\alpha+1)_q}{x} D_q u \big) \tag{35}$$

and for $r = 3$, $\alpha_1 = -2/3$, $\alpha_2 = \nu - 1/3$, we obtain the operator $B_{3,\delta}$ studied in [12]:

$$B_{3,\delta} u = \Lambda_{q^\delta}^{-1} \big(q^{3\nu} D_q^3 u + \frac{1}{q} \frac{(3\,\nu)_q}{x} D_q^2 u - \frac{1}{q} \frac{(3\,\nu)_q}{x^2} D_q u \big). \tag{36}$$

Proposition 5.1. *For λ in C, the function $j_\alpha(\lambda x, q^r, \delta)$*

$$j_\alpha(\lambda x, q^r, \delta) = {}_0\phi_{r-1}^{\delta} \left(\begin{array}{c} - \\ (q^r)^{\alpha_1+1}, ..., (q^r)^{\alpha_{r-1}+1} \end{array} \middle| q^r; -\frac{\lambda^r (q^r-1)^r x^r}{(1+q+...+q^{r-1})^r} \right) \tag{37}$$

is a solution of the q-problem

$$\left\{ \begin{array}{lcl} B_{r,\delta} u(x) & = & -\lambda^r u(x) \\ u(0) & = & 1, \\ D_q^k u(0) & = & 0, \quad k = 1, ..., r-1\,. \end{array} \right. \tag{38}$$

Furthermore, $j_\alpha(\lambda x, q^r, \delta)$ has the following representation

$$j_\alpha(\lambda x, q^r, \delta) = \sum_{n=0}^{\infty} (-1)^n b_{rn,\alpha}(x, q^r, \delta) \lambda^{rn}, \tag{39}$$

where

$$b_{rn,\alpha}(x, q^r, \delta) = \frac{(q^r)^{\delta (n)_2} x^{rn}}{((r)_q)^{rn} (1)_n^{q^r} \prod (\alpha_i + 1)_n^{q^r}} = \frac{(q^r)^{\delta (n)_2} x^{rn}}{\alpha_{rn,\alpha,q}}, \tag{40}$$

and

$$\alpha_{rn,\alpha,q} = (1 + q + ... + q^{r-1})^{rn} \, [n]_{q^r}! \prod_{i=1}^{r-1} \frac{\Gamma_{q^r}(\alpha_i + n + 1)}{\Gamma_{q^r}(\alpha_i + 1)}. \tag{41}$$

For $\delta = r$, we obtain the q-hypergeometric function ${}_0\phi_{r-1}$.

Let now $| \alpha |= \alpha_1 + ... + \alpha_{r-1} = \alpha_0 + ... + \alpha_{r-1}$ with $\alpha_0 = 0$,

$$b_{rn,\alpha}(1, \, q^r; \delta) \leq b_{rn,\alpha}(1, q^r, 1) = \frac{(q^r)^{(n)_2}}{((r)_q)^{rn} (1)_n^{q^r} \prod (\alpha_i + 1)_n^{q^r}}, \quad \delta \geq 1, \tag{42}$$

the right term can be written by

$$\frac{\left((q^r)^{-|\alpha|/r}\right)^n}{((r)_q)^{rn}} \prod_{j=0}^{n-1} \prod_{i=0}^{r-1} \frac{(q^r)^{(\alpha_i + j)/r} - (q^r)^{1 + (\alpha_i + j)/r}}{1 - (q^r)^{1 + (\alpha_i + j)}}. \tag{43}$$

Now, by [19], Lemma A.1, [20] and Proposition A.2, we see that the general terms of product increases to $(j + \alpha_i + 1)^{-1}$ if $q \uparrow 1$. Using Stirling's formula, we find that, for some constant C,

$$b_{rn,\alpha}(1, \, q^r; \delta) \leq \frac{\left((q^r)^{-|\alpha|/r}\right)^n}{((r)_q)^{rn} \prod (\alpha_i + 1)_n} \leq \; C \, \left((q^r)^{-|\alpha|/r}\right)^n \left(\frac{e}{n(r)_q}\right)^{rn + |\alpha|}, \tag{44}$$

this inequality generalizes the inequality in [12].

Proposition 5.2. *For* $\alpha_i \geq -1 + \frac{i}{r}$, $i = 1, ..., r-1$, *and* $n = 0, 1, 2...$,

$$D_q j_\alpha(., q^r, \delta)(x) = -\left(\frac{x}{(r)_q}\right)^{r-1} \frac{1}{\prod (\alpha_i + 1)_{q^r}} j_{\alpha+1}(q^\delta x, q^r, \delta), \tag{45}$$

and

$$\{\frac{1}{x^{r-1}} D_q\}^n j_\alpha(x, q^r, \delta) = \left((\frac{1}{(r)_q})^{r-1} \right)^n \frac{(-1)^n (q^\delta)^{(n)_2}}{\prod (\alpha_i + 1)_n^{q^r}} j_{\alpha+n}(q^{n\delta} x, q^r, \delta). \tag{46}$$

By the q-duplication formula of Γ_q (22), we have in particular

$$j_{(-1/r, -2/r, ..., -(r-1)/r)}(x, q^r, \delta) = \cos_r(x, \, q^r; \delta). \tag{47}$$

6. q-integral representations

In this section, we give two q-integral representations of the q-j_α function (39) involving the q-Jackson integral. We denote by W_α the function

$$W_\alpha(t_1,...,t_{r-1};q^r) \;=\; \prod_{i=1}^{r-1}\frac{(t_i^r q^r;\,q^r)_\infty}{(t_i^r q^{\alpha_i-\frac{i}{r}+1};\,q^r)_\infty}t_i^{i-1} = \prod_{i=1}^{r-1}(t_i^r q^r;\,q^r)_{\alpha_i-\frac{i}{r}}t_i^{i-1}$$

which tends to $\prod(1-t_i^r)^{\alpha_i-\frac{i}{r}}\,t_i^{i-1}$ as $q\longrightarrow 1^-$.

THEOREM 6.1. *For $\alpha_i\geq -1+\frac{i}{r},\ i=1,...,r-1,$ the function j_α has the following q-integral representation of Mehler type*

$$j_\alpha(z,q^r,\delta)=C_{r,\alpha}\int_0^1...\int_0^1 W_\alpha(t_1,..t_{r-1};q^r)\cos_r(zt_1,...t_{r-1};q^r,\delta)\,d_qt_1..d_qt_{r-1}, \tag{48}$$

where

$$C_{r,\alpha}=((r)_q)^{r-1}\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+1)}{\Gamma_{q^r}(\frac{i}{r})\Gamma_{q^r}(\alpha_i-\frac{i}{r}+1)}. \tag{49}$$

P r o o f. This formula can be proved by expanding $\cos_r(zt,q^r;\delta)$ in a series of power of t and then there arise q-integrals of the form

$$\int_0^1 t_i^{rm}(1-q^rt_i^r)_{q^r}^{\alpha_i-\frac{i}{r}}t_i^{i-1}\,d_qt_i=\frac{\Gamma_{q^r}(m+\frac{i}{r})\Gamma_{q^r}(\alpha_i-\frac{i}{r}+1)}{(r)_q\Gamma_{q^r}(\alpha_i+m+1)}. \tag{50}$$

Based on the q-duplication formula for the Γ_q function (22), the formula is proved. ■

PROPOSITION 6.1. *For $\alpha_i\geq -1+\frac{i}{r},\ i=1,...,r-1,$ and $n=0,1,2...,$*

$$\left|D_q^n\big[j_\alpha(x,q^r,\delta)\big]\right|\leq\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+1)\Gamma_{q^r}(\frac{n+i}{r})}{\Gamma_{q^r}(\frac{i}{r})\Gamma_{q^r}(\alpha_i+1+\frac{n}{r})}\left|\big[D_{q,x}^n\cos_r(x,\,q^r;\delta)\big]\right|, \tag{51}$$

in particular

$$\big|j_\alpha(x,q^r,\delta)\big|\leq q^{-(r)_2}\,e^{(r-2)|x|}. \tag{52}$$

THEOREM 6.2. *For $\alpha_i\geq -1+\frac{i}{r},\ i=1,...,r-1$ and $p_i\geq 1,$ the function $j_{\alpha+p}$ has the following q-integral representation of Sonine type*

$$j_{\alpha+p}(z,q^r,\delta)=c_{r,\alpha}^p\int_0^1...\int_0^1 V_p(t_1,..t_{r-1};q^r)\,j_\alpha(zt_1,..t_{r-1},q^r;\delta)\,d_qt_1...d_qt_{r-1}, \tag{53}$$

where

$$c^p_{r,\alpha} = ((r)_q)^{r-1} \prod_{i=1}^{r-1} \frac{\Gamma_{q^r}(\alpha_i + p_i + 1)}{\Gamma_{q^r}(p_i)\Gamma_{q^r}(\alpha_i + 1)} \tag{54}$$

$$V_p(t_1, ..., t_{r-1}; q^r) = \prod_{i=1}^{r-1} (1 - q^r t_i^r)_{q^r}^{p_i} \, t_i^{r(\alpha_i - \frac{i}{r} + 1)} \, t_i^{i-1} \, . \tag{55}$$

P r o o f. This formula can be proved by expanding j_α in a series of power of t_i, there arise q-integrals of the form

$$\int_0^1 t_i^{r(\alpha_i - \frac{i}{r} + 1 + m)} (1 - q^r t_i^r)_{q^r}^{p_i - 1} t_i^{i-1} \, d_q t_i = \frac{\Gamma_{q^r}(\alpha_i + m + 1)\Gamma_{q^r}(p_i)}{(r)_q \Gamma_{q^r}(m + \alpha_i + p_i + 1)}.$$

■

7. q-Fourier transform

NOTATIONS: Some q-functional spaces will be used to establish our result.

We design by $\mathcal{E}_{*,q}(R)$ (resp. $\mathcal{E}_{*,q}(R_q)$) the space of r-even functions defined on R (resp R_q) infinitely q-derivative, and by $\mathcal{D}_{*,q}(R)$ (resp $\mathcal{D}_{*,q}(R_q)$) the space of r-even functions defined on R (resp. R_q) infinitely q-derivative with compact support.

In this section we introduce the space $\mathcal{L}^1_{\alpha,q^\delta}(R_{q,+}, \, d_q x)$ of functions f satisfying

$$\int_0^\infty |f(x) j_\alpha(\lambda x, \, q^r; \delta)| \, d_q x < \infty, \qquad \lambda \in R_q.$$

DEFINITION 7.1. The Fourier transform related with $B_{r,\delta}$ of $f \in \mathcal{L}^1_{\alpha,\,q^\delta}(R_{q,+}, \, d_q x)$ is the function $\mathcal{F}_{q^\delta}(f)$ defined by

$$\mathcal{F}_{q^\delta}(f)(\lambda) = \int_0^\infty f(t) \, j_\alpha(\lambda t, \, q^r; \delta) \, d_q t, \quad \lambda \in R_q. \tag{56}$$

We define also the Fourier transform $\mathcal{F}_{0,q^\delta}$ by

$$\mathcal{F}_{0,q^\delta}(f)(\lambda) = \int_0^\infty f(t) \cos_r(\lambda t, \, q^r; \delta) \, d_q t, \quad \lambda \in R_q. \tag{57}$$

8. q-translation and q-convolution

In this section we study the generalized translation operator associated with the operator $B_{r,\delta}$. We give the following definition related to $\Lambda^{-1}_{q^\delta} D^r_q$.

Definition 8.1. The translation operator $\tau_{x,\,q^\delta}$, $x \in R$ (resp R_q) associated with the r-order derivative operator $\Lambda_{q\delta}^{-1} D_q^r$ is defined for f in $\mathcal{E}_{*,q}(R)$ (resp. $\mathcal{E}_{*,\,q}(R_q)$) and $y \in R$ (resp. R_q) by

$$\tau_{x,\,q^\delta}(f)(y) = \sum_{n=0}^{\infty} b_{rn}(y,\, q^r;\delta)\,(\Lambda_{q\delta}^{-1} D_q^r)^{(n)}\, f(x), \tag{58}$$

the functions $b_{rn}(y,\, q^r;\delta)$ are given by (24).

Proposition 8.1. *The operators $\tau_{x,\,q^\delta}$ satisfy:*
1- the product formula:

$$\cos_r(\lambda x,\, q^r;\delta).\cos_r(\lambda y,\, q^r;\delta) = \tau_{x,\,q^\delta}\cos_r(\lambda y,\, q^r;\delta) = \tau_{y,\,q^\delta}\cos_r(\lambda x,\, q^r;\delta).$$

2- For $x \in R$, $\tau_{x,\,q^\delta}$ belong in $\mathcal{L}(\mathcal{E}_{*,q}(R),\, \mathcal{E}_{*,q}(R))$.
3- The map $x \longrightarrow \tau_{x,\,q^\delta}$ is infinitely q-derivative, r-even.

Lemma 8.1. *For $f \in D_{*,q}(R)$, $n \in N$, we have:*

$$\left(\Lambda_{q^\delta}^{-1} D_q^r\right)^n f(x) = \frac{q^{-(rn)_2}}{(1-q)^{rn}(q^{-\delta n})^{rn} b_{rn}(x,\, q^r;\delta)} \sum_{k=0}^{rn} \frac{(-1)^k q^{(rn-k)_2}}{[rn-k]_q![k]_q!} \Lambda_{q^\delta}^{-n} f(q^k x).$$

P r o o f. For $\delta > 0$, by [21] and relation (11),

$$D_{q,x}^{rn} f(x) = \frac{q^{-(rn)_2}}{(1-q)^{rn} x^{rn}} \sum_{k=0}^{rn} (-1)^k \begin{bmatrix} rn \\ k \end{bmatrix}_q q^{(rn-k)_2} f(q^k x),$$

using the fact that $\left(\Lambda_{q^\delta}^{-1} D_q^r\right)^n = \left((q^\delta)^r\right)^{-(n)_2} \Lambda_{q^\delta}^{-n} D_q^{rn}$, then we obtain

$$\left(\Lambda_{q^\delta}^{-1} D_q^r\right)^n f(x) = \frac{q^{-(rn)_2}\left((q^\delta)^r\right)^{-(n)_2}}{(1-q)^{rn}(q^{-\delta n} x)^{rn}} \sum_{k=0}^{rn} (-1)^k \begin{bmatrix} rn \\ k \end{bmatrix}_q q^{(rn-k)_2} \Lambda_{q^\delta}^{-n} f(q^k x).$$

this leads to the result. ∎

Remark 8.1. We obtain for $\delta > 0$

$$\tau_{y,q^\delta} f(x) = \sum_{n=0}^{\infty} \frac{b_{rn}(1,q^r;\delta) q^{-(rn)_2}}{(1-q)^{rn}(q^{-\delta n})^{rn}} \left(\frac{y}{x}\right)^{rn} \sum_{k=0}^{rn} (-1)^k \begin{bmatrix} rn \\ k \end{bmatrix}_q q^{(rn-k)_2} \Lambda_{q^\delta}^{-n} f(q^k x).$$

Proposition 8.2. *For $f \in \mathcal{D}_{*,q}(R_q)$ we have:*

$$\mathcal{F}_{0,q^\delta}({}^t\tau_{x,\,q^\delta} f)(\lambda) = \cos_r(\lambda x, q^3;\delta)\mathcal{F}_{0,q^\delta}(f)(\lambda), \tag{59}$$

the convolution product of two functions f and g of $\mathcal{D}_{*,q}(R_q)$ is defined in $\mathcal{D}_{*,\,q}(R_q)$ by:

$$f \star_{q^\delta}\; g(x) = \int_0^\infty {}^t\tau_{x,\,q^\delta} f(y)g(y)\, d_q y = \int_0^\infty f(y)\tau_{x,\,q^\delta} g(y)\, d_q y, \tag{60}$$

and we have:

$$\mathcal{F}_{0,\,q^\delta}(f \star_{q^\delta}\; g)(\lambda) = \mathcal{F}_{0,\,q^\delta}(f)(\lambda).\,\mathcal{F}_{0,\,q^\delta}(g)(\lambda). \tag{61}$$

Definition 8.2. We call generalized translation operators associated with $B_{r,\delta}$, the operators $\mathrm{T}^\alpha_{x,\,q^\delta}$, $x \in R$ (resp. R_q), defined on $\mathcal{E}_{*,\,q}(R)$ (resp. $\mathcal{E}_{*,q}(R_q)$) by:

$$\mathrm{T}^\alpha_{x,\,q^\delta}(f)(y) = \sum_{n=0}^\infty b_{rn,\,\alpha}(y,\, q^r; \delta)\, B^n_{r,\delta}\,(f)(y), \qquad y \in R \text{ (resp } R_q), \tag{62}$$

where the functions $b_{rn,\alpha}(y, q^r, \delta)$ is given by (39).

Proposition 8.3. *The operators $\mathrm{T}^\alpha_{x,\,q^\delta}$ satisfy:*

1. *For $x \in R$, $\mathrm{T}^\alpha_{x,q^\delta}$ in $\mathcal{L}(\mathcal{E}_{*,q}(R), \mathcal{E}_{*,q}(R))$.*
2. *The map $x \longrightarrow \mathrm{T}^\alpha_{x,q^\delta}$ are infinitely q-derivative and r-even.*
3. *For all functions f in $\mathcal{E}_{*,q}(R)$:*
   – $\mathrm{T}^\alpha_{x,q^\delta} f(y) = \mathrm{T}^\alpha_{y,q^\delta} f(x)$
   – $\mathrm{T}^\alpha_{0,q^\delta} f(y) = f(y)$.
4. *For given f in $\mathcal{E}_{*,q}(R)$, we put: $u(x;y) = \mathrm{T}^\alpha_{x,q^\delta}\, f(y)$,*

then the function u is solution of the Cauchy problem:

$$(I) \begin{cases} B_{x,r,\delta} u(x,\, y) &= B_{y,r,\delta} u(x,\, y), \\ u(x,\, 0) &= f(x); D_{q,\,y} u(x,\, 0) = 0 \\ D^k_{q,y} u(x,\, 0) &= 0 \quad, k = 0.1.2...r-1 \end{cases}$$

and we have

$$\mathrm{T}^\alpha_{x,\,q^\delta} j_\alpha(\lambda y, q^r, \delta) = j_\alpha(\lambda x, q^r, \delta) j_\alpha(\lambda y, q^r, \delta) = \mathrm{T}^\alpha_{y,\,q^\delta} j_\alpha(\lambda x, q^r, \delta). \tag{63}$$

Now we are able to define the convolution product related to the operator $B_{r,\delta}$.

Definition 8.3. The convolution product associated with $B_{r,\delta}$ of two functions f and g in $\mathcal{D}_{*,q}(R_q)$ is the function $f \star_{\alpha, q^\delta} g$ defined by:

$$f \star_{\alpha, q^\delta} g(y) = \int_0^\infty f(x) \mathrm{T}^\alpha_{y, q^\delta} g(x)\, d_q x = \int_0^\infty {}^t\mathrm{T}^\alpha_{y, q^\delta} f(x)\, g(x)\, d_q x. \quad (64)$$

9. Higher-order q-Bessel heat polynomials

We recall that the function $e_{q^r}(-z^r t) j_\alpha(xz;\, q^r;\, \delta)$ is analytic in z^r. We thus have, for $t \in R$ and $\delta \geq 1$,

$$e_{q^r}(-z^r t) j_\alpha(xz;\, q^r;\, \delta) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{z^{rn}}{\alpha_{rn,\alpha,q}} p_n^\alpha(x,\, t,\, q^r;\, \delta), \quad (65)$$

then

$$p_n^\alpha(x,\, t,\, q^r;\, \delta) = \sum_{k=0}^{n} (q^r)^{\delta(n-k)_2} \frac{(x^r)^{n-k} t^k}{[k]_{q^r}!} \frac{\alpha_{rn,\alpha,q}}{\alpha_{r(n-k),\alpha,q}} \quad (66)$$

$$= \frac{\prod(\alpha_i+1)_n^{q^r}}{(1+q+\ldots+q^{r-1})^{-rn}} t^n \sum_{k=0}^{\infty} \frac{(-1)^k (-n)_k^{q^r} (q^r)^{(\delta-1)(k)_2} (q^r)^{nk} (x^r)^k t^{-k}}{\prod(\alpha_i+1)_k^{q^r} (1+q+\ldots+q^{r-1})^{rk} [k]_{q^r}!}$$

$$= \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} t^n\, {}_1\phi_{r-1}^{\delta-1} \left(\begin{array}{c} (q^r)^{-n} \\ (q^r)^{\alpha_1+1}, .., (q^r)^{\alpha_{r-1}+1} \end{array} \middle| q^r; \frac{(q^r-1)^{r-1}(-x^r (q^r)^n)}{(1+q+..+q^{r-1})^r t} \right).$$

10. Application: q-heat equation

We give an applications of the Fourier transform related with $B_{r,\delta}$. We begin by recalling that

$$\int_0^\infty e_{q^r}(-cx^r)\, (c^n x^{rn}) x^{r\alpha_k + (r-1)}\, d_q x = \frac{(q^r)^{-n(\alpha_k+1)-(n)_2} (\alpha_k+1)_n^{q^r}}{c^{\alpha_k+1}(1+q+\ldots+q^{r-1})} I_{(\alpha_k,+1,q^r)} \quad (67)$$

where

$$I(\alpha_k+1;\, q^r) = \int_0^\infty e_{q^r}(-x)\, x^{\alpha_k}\, d_{q^r} x \quad \text{and} \quad H_{q^r}(\alpha_k+1) = \frac{I(\alpha_k+1;\, q^r)}{\Gamma_{q^r}(\alpha_k+1)}. \quad (68)$$

We note by $d\eta_{q,\alpha_k}(y) = \frac{y^{r\alpha_k+(r-1)}}{(1+q+...+q^{r-1})^{\alpha_k}\Gamma_{q^r}(\alpha_k+1)}\, d_qy$ and we define for $\delta > 1$ the fundamental solution $\mathcal{K}_{\alpha_k}(x, t,\, q^r; \delta)$ by

$$\begin{aligned}
\mathcal{K}_{\alpha_k}(x,t,q^r;\delta) &= \int_0^\infty e_{q^r}(-ty^r) j_\alpha(xy,q^r;\delta) d\eta_{q,\alpha_k}(y),\\
&= \tfrac{H_{q^r}(\alpha_k+1)}{(t(1+q+...+q^{r-1}))^{\alpha_k+1}} \sum_{n=0}^{\infty} \frac{(-1)^n (q^\delta)^{r(n)_2} \times (q^r)^{-(\alpha_k+1)n-(n)_2} x^{rn} t^{-n}}{(1+q+...+q^{r-1})^{rn}[n]_{q^r}! \prod_{i\neq k}(\alpha_i+1)_n^{q^r}},\\
&= \tfrac{H_{q^r}(\alpha_k+1)}{(t(1+q+..q^{r-1}))^{\alpha_k+1}}\\
&\times\ {}_0\phi_{r-2}^{\delta-1}\left(\begin{matrix} - \\ (q^r)^{\alpha_1+1}, .., (q^r)^{\alpha_{r-1}+1} \end{matrix} \,\middle|\, q^r; (\tfrac{-x^r (q^r)^{-(\alpha_k+1)}(q^r-1)^{r-1}}{(1+q+..q^{r-1})^r t}) \right).
\end{aligned}$$

For $\delta = r$, we obtain the basic hypergeometric series.

We consider the q-problem for $t, x \geq 0$

$$(II) \begin{cases} B_{r,\delta} u(x,\, t) &= D_{q^r,\, t} u(x,\, t) \\ D_q^k u(0,\, t) &= 0, \quad k = 1, ..., r-1 \\ u(w_k\, x, t) &= u(x,\, t), \quad k = 1, ..., r-1 \\ u(x, 0) &= f(x). \end{cases}$$

THEOREM 10.1. *Let $f \in L^1_{\alpha,\, q^\delta}(R_{q,+},\, d_qx)$, the function*

$$u(x,\, t) = \int_0^\infty \mathrm{T}^\alpha_{y,\, q^\delta} \mathcal{K}_{\alpha_k}(x, t,\, q^r; \delta)\, f(y)\, d_qy = \big(f \star_{\alpha, q^\delta}\ \mathcal{K}_{\alpha_k}(.\,,\, t,\, q^r; \delta)\big)(x), \tag{69}$$

is a solution of the equation (II) for $\alpha_k \geq -1 + \frac{k}{r}, k = 1, ..., r-1,\ t, x \in R_{q,+}$.

11. Analytic Cauchy problem related to the r-order q-Bessel operator $B_{r,\delta}$

We say that a function $u(x, t)$ in $\mathcal{H}_\alpha([0, a] \times [0, \sigma])$ if

$$B_{r,\delta}\, u(x, t) = D_{q^r,t}\, u(x, t). \tag{70}$$

The diffusion polynomials $p_n^\alpha(x, t)$ satisfy the q-equation (70). Hence we expect to obtain infinite series expansions $u(x, t) = \sum_{m\geq 0} a_m p_m^\alpha(x,\, t,\, q^r;\, \delta)$ with possible convergence in a strip $|t| < \sigma$.

Let $\delta \geq 1$, we note

$$\begin{aligned}
\mathcal{R}^{\delta}_{\alpha,q}(x) &= \sum_{n=0}^{\infty} \frac{(q^r)^{(\delta-1)(n)_2} x^{rn}}{(1+q+...+q^{r-1})^{rn} \prod(\alpha_i+1)^{q^r}_n} \\
&= {}_1\varphi^{\delta-1}_{r-1}\left(\begin{array}{c} (q^r)^1 \\ (q^r)^{\alpha_1+1}, ..., (q^r)^{\alpha_{r-1}+1} \end{array} \middle| q^r; \frac{(q^r-1)^{r-1}x^r}{(1+q+..+q^{r-1})^r} \right).
\end{aligned}$$

LEMMA 10.1. *Let* $s > 0$ *and* $\delta \geq 1$

$$\frac{p^{\alpha}_n(|x|,|t|,q^r,\delta)}{\alpha_{rn,\alpha,q}} \leq \frac{s^n}{[n]_{q^r}!}(1+\frac{|t|}{s})^n \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}).$$

P r o o f. We have

$$\begin{aligned}
\frac{p^{\alpha}_n(|x|,|t|,q^r,\delta)}{\alpha_{rn,\alpha,q}} &\leq \frac{s^n}{[n]_{q^r}!} \sum_{k=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right]_{q^r} \left(\frac{|t|}{s}\right)^{n-k} \frac{(q^r)^{\delta(k)_2} \frac{|x|^{rk}}{s^k}}{((r)_q)^{rk} \prod(\alpha_i+1)^{q^r}_n} \\
&\leq \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}) \frac{s^n}{[n]_{q^r}!} \sum_{k=0}^{\infty} (q^r)^{(k)_2} \left[\begin{array}{c} n \\ k \end{array} \right]_{q^r} \left(\frac{|t|}{s}\right)^{n-k} \\
&= \frac{s^n}{[n]_{q^r}!}(1+\frac{|t|}{s})^n_{q^r} \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}) \\
&\leq \frac{s^n}{[n]_{q^r}!}(1+\frac{|t|}{s})^n \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}),
\end{aligned}$$

since, $\frac{(q^r)^{\delta-1(k)_2} \frac{|x|^{rk}}{s^k}}{((r)_q)^{rk} \prod(\alpha_i+1)^{q^r}_n} < \mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}}).$ ∎

LEMMA 10.2. *For* $t, x > 0$, $\delta > 0$

$$p^{\alpha}_n(x,t,q^r,\delta) \geq \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} t^n.$$

P r o o f. Since the coefficients of p^{α}_n are positive, it follows that

$$p^{\alpha}_n(x,t,q^r,\delta) \geq p^{\alpha}_n(0,t,q^r,\delta) = \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} t^n.$$

∎

THEOREM 10.2. *If the series* $\sum_{n=0}^{\infty} a_n p_n^{\alpha}(x_0, t_0, q^r, \delta)$ *converges for* $t_0 > 0$ *and* $x_0 > 0$, *the series* $\sum_{n=0}^{\infty} a_n p_n^{\alpha}(x_0, t_0, q^r, \delta)$ *and* $\sum_{n=0}^{\infty} d_{rn,\alpha,q} a_n p_{n-1}^{\alpha}(x, t, q^r, \delta)$ *converge absolutely and locally uniformly in the strip* $|t| < t_0$ *and the series* $\sum_{n=0}^{\infty} a_n p_n^{\alpha}(x_0, t_0, q^r, \delta)$ *is in* $\mathcal{H}_{\alpha}(R_+)$ *for* $|t| < t_0$.

P r o o f. We note by $d_{rn,\alpha,q} = \alpha_{rn,\alpha,q}/\alpha_{r(n-1),\alpha,q}$. Since the general term of a convergent series must go to zero, $\lim_{n \longrightarrow \infty} a_n p_n^{\alpha}(x, t, q^r, \delta) = 0$. By Lemma 10.2, it therefore follows that $a_n = O\left(\frac{[n]_{q^r}!}{\alpha_{rn,\alpha,q} t_0^n}\right)$. Using Lemma 10.1, we get for $s > 0$ and $\delta \geq 1$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n d_{rn,\alpha,q} p_{n-1}^{\alpha}(x, t, q^r, \delta) &\leq M \sum_{n=1}^{\infty} \frac{[n]_{q^r}!}{\alpha_{rn,\alpha,q} t_0^n} \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} (s + |t|)^n \mathcal{R}_{\alpha,q}^{\delta}\left(\frac{|x|}{s^{1/r}}\right) \\ &\leq M \mathcal{R}_{\alpha,q}^{\delta}\left(\frac{|x|}{s^{1/r}}\right) \sum_{n=0}^{\infty} \left(\frac{s + |t|}{t_0}\right)^n \end{aligned}$$

which converges for $s + |t| < t_0$. Since $s > 0$ is arbitrary it converges for $(s + |t|) < t_0$, and as before for $|t| < t_0$. ∎

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be an entire function of order ρ, $\rho > 0$, and of type $0 < \sigma < \infty$. The type is determined by $\limsup_{n \to \infty} \frac{rn}{e\rho} |a_n|^{\frac{\rho}{rn}} = \sigma$. Therefore,

$$|a_n| \leq M \left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho}. \tag{71}$$

THEOREM 10.3. *If* $f(z)$ *is an entire function of order* ρ *with* $0 < \rho < r/r - 1$ *and of type* σ, $0 < \sigma < \infty$, *then*

$$u(x, t) = \sum_{n=0}^{\infty} a_n p_n^{\alpha}(x, t, q^r, \delta) \tag{72}$$

is in $\mathcal{H}_{\alpha}(R)$ *in the strip* $|t| < 1/(\sigma\rho)^{r/\rho}$ *and* $u(x, 0) = f(x)$.

P r o o f. Using (71) and Lemma 10.1, for $s > 0$ we obtain

$$\sum_{n=0}^{\infty} a_n p_n^{\alpha}(x, t, q^r, \delta) \leq M \sum_{n=0}^{\infty} \left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho} \frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!} (s + |t|)^n \mathcal{R}^{\delta}{}_{\alpha,q}\left(\frac{|x|}{s^{1/r}}\right). \tag{73}$$

Since, $\left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho}\frac{\alpha_{rn,\alpha,q}}{[n]_{q^r}!}\leq\left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho}r^{rn}\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+n+1)}{\Gamma_{q^r}(\alpha_i+1)}$, or for $n\uparrow\infty$, we have $\prod_{i=1}^{r-1}\frac{\Gamma_{q^r}(\alpha_i+n+1)}{\Gamma_{q^r}(\alpha_i+1)}\sim\prod_{i=1}^{r-1}\Gamma_{q^r}(\alpha_i+n+1)$, by [19, p. 53] , for $n\uparrow\infty$ $\prod_{i=1}^{r-1}\Gamma_{q^r}(\alpha_i+n+1)\leq\prod_{i=1}^{r-1}\Gamma(\alpha_i+n+1)$. Using Stirling's formula, we get

$$\left(\frac{e\sigma\rho}{rn}\right)^{rn/\rho}r^{rn}\prod_{i=1}^{r-1}\Gamma(\alpha_i+n+1)\sim\left[\frac{e^{1-\frac{r-1}{r}\rho}\,r^{\rho-1}}{n^{1-\frac{r-1}{r}\rho+(\sum\alpha_i+\frac{r-1}{2})\rho/rn}}\right]^{rn/\rho}(2\pi)^{\frac{r-1}{2}}(\sigma\rho)^{rn/\rho}$$

for $0<\rho<\frac{r}{r-1}$. Thus the series in (73) is dominated by

$$M_{t,q}\mathcal{R}^{\delta}_{\alpha,q}(\frac{|x|}{s^{1/r}})\sum_{n=0}^{\infty}\{(\sigma\rho)^{r/\rho}(s+|t|)\}^n\,,$$

which converges for $(\sigma\rho)^{3/\rho}(s+|t|)<1$. Since $s>0$ is arbitrary, we get absolute and local uniform convergence for $|t|<\frac{1}{(\sigma\rho)^{3/\rho}}$. Since the order and type of entire function is not changed by taking derivatives, a similar type argument shows that the derived series $\sum_{n\geq1}a_nd_{rn,\alpha,q}p^{\alpha}_{n-1}(x,t,q^r,\delta)$, also converges absolutely and locally uniformly for $|t|<\frac{1}{(\sigma\rho)^{r/\rho}}$. It follows that $u(x,t)$ given by (72) is in $\mathcal{H}_\alpha$ in the stated strip. ∎

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