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PSEUDO-DIFFERENTIAL OPERATORS IN A WAVE DIFFRACTION PROBLEM WITH IMPEDANCE CONDITIONS *

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Abstract

We consider an impedance boundary-value problem for the Helmholtz equation which models a wave diffraction problem with imperfect conductivity on a strip. Pseudo-differential operators are used to deal with this wave diffraction problem. Therefore, single and double layer potentials allow a reformulation of the problem into a system of integral equations. By using operator theoretical methods, the well-posedness of the problem is obtained for a set of impedance parameters, and in a framework of Bessel potential spaces.

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1. Problem formulation

Consider a time-harmonic electromagnetic plane wave incident on a layer of some inhomogeneous, isotropic, dielectric material sitting on a strip in

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$\mathbb{R}^3$. Adopting Cartesian axes $Oxyz$ with the y-axis vertically upwards, perpendicular to the strip, we assume throughout that the material is invariant in the z-direction. Thus, the problem geometry is two dimensional and the strip will be therefore represented by $\Sigma :=]0, \tau[$ (for $0 < \tau < \infty$).

The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (time dependence $e^{-\omega t}$ with frequency $\omega > 0$):

$$\nabla \times E - i\omega\mu H = 0 \tag{1}$$

and

$$\nabla \times H + (i\omega\epsilon - \lambda)E = 0, \tag{2}$$

where ϵ denotes the electric permittivity of the medium, μ is the magnetic permeability, λ is the electric conductivity, and E and H represent the electric and magnetic fields, respectively. Then it follows from the Maxwell equations that the total field u satisfies the two-dimensional Helmholtz equation (for both cases of complex or pure real wave numbers $k \neq 0$):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u = 0 \qquad \text{in} \qquad \Omega\,, \tag{3}$$

with $\Omega := \mathbb{R}^2 \backslash \left\{(x,0) \in \mathbb{R}^2 \,|\, x \in \overline{\Sigma}\right\}$, and

$$k^2 = \omega^2\mu\epsilon[1 + i\lambda/(\omega\epsilon)]. \tag{4}$$

Additionally, the following impedance boundary functional condition is assumed:

$$\left\{ \begin{array}{rcl} u_1^+ - i\varrho^+ u_0^+ & = & h^+ \\ u_1^- + i\varrho^- u_0^- & = & h^- \end{array} \right. \qquad \text{on} \qquad \Sigma\,, \tag{5}$$

where $\varrho^\pm \in \mathbb{C}$ refer to the given (possibly different) surface impedances, and the Dirichlet and Neumann traces are denoted by $u_0^\pm = u_{|y=\pm 0}$ and $u_1^\pm = -(\partial u/\partial y)_{|y=\pm 0}$, respectively. Here $h^\pm$ are also given elements in appropriate corresponding Sobolev spaces to be specified later.

In fact, from the mathematical point of view, one of the main interests in such kind of problems consists in investigate the largest set of possible spaces where it is possible to show the existence of a unique solution, and continuous dependence on the known data. In view of this, in the present paper we provide new results on the possible smoothness orders of Bessel potential spaces for the well-posedness of the announced problem. Thus, the present work improves the known corresponding results of [2] where only

Hilbert Bessel potential spaces were considered. We also refer e.g. to [1, 3, 6] for additional descriptions of corresponding wave diffraction problems with impedance boundary conditions.

We will now fix the general notation (already necessary for presenting the complete mathematical formulation of the problem). Let m be a positive integer number. As usual, $\mathscr{S}(\mathbb{R}^m)$ denotes the Schwartz space of all rapidly vanishing functions at infinity and $\mathscr{S}'(\mathbb{R}^m)$ the dual space of tempered distributions on $\mathbb{R}^m$. For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space $H_p^s(\mathbb{R}^m)$ is formed by the elements $\varphi \in \mathscr{S}'(\mathbb{R}^m)$ such that $\|\varphi\|_{H_p^s(\mathbb{R}^m)} = \|\mathscr{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathscr{F}\varphi\|_{L_p(\mathbb{R}^m)} < \infty$. The function $\|\cdot\|_{H_p^s(\mathbb{R}^m)}$ provides a norm for the space $H_p^s(\mathbb{R}^m)$ which makes it a Banach space. Here, $\mathscr{F} = \mathscr{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^m$. For a given domain, $\mathscr{D}$, on $\mathbb{R}^m$ we denote by $\widetilde{H}_p^s(\mathscr{D})$ the closed subspace of $H_p^s(\mathbb{R}^m)$ whose elements have supports in $\overline{\mathscr{D}}$, and $H_p^s(\mathscr{D})$ denotes the space of generalized functions on $\mathscr{D}$ which have extensions into $\mathbb{R}^m$ that belong to $H_p^s(\mathbb{R}^m)$. The space $\widetilde{H}_p^s(\mathscr{D})$ is endowed with the subspace topology, and on $H_p^s(\mathscr{D})$ we put the norm of the quotient space $H_p^s(\mathbb{R}^m)/\widetilde{H}_p^s(\mathbb{R}^m \backslash \overline{\mathscr{D}})$. We will often use the restriction operator $r_\Sigma : H_p^s(\mathbb{R}) \to H_p^s(\Sigma)$. Recall also that $H^1_{2,\mathrm{loc}}(\Omega)$ denotes the Bessel potential space of vector-functions u in Ω such that

$$(u,u)_K = \left(\int_K (|u(x)|^2 + |\nabla u(x)|^2) dx \right)^{1/2} < \infty, \tag{6}$$

for any compact set $K \subset \Omega$.

Within the just introduced Bessel potential spaces, we are interested to study the problem of existence and uniqueness of an element $u \in L_2(\mathbb{R}^2)$, with $u_{|\Omega} \in H^1_{2,\mathrm{loc}}(\Omega)$, satisfying (3)–(5) for given elements $h^\pm \in H_2^{-1/2}(\Sigma)$. Later on we will also consider this problem in Bessel potential spaces with larger ranges of smoothness and integrability.

We recall that we are dealing with both situations of a dissipative and a non-dissipative medium reflected by the conditions $\Im\mathrm{m}\, k \neq 0$ and $\Im\mathrm{m}\, k = 0$, respectively. Thus, for the real wave number case ($k \in \mathbb{R}$) it is natural to require that the eventual solution of (3)–(5) should also satisfy the *Sommerfeld radiation condition at infinity*, $u \in \mathrm{Som}(\Omega)$:

$$\frac{\partial}{\partial |x|} u(x) - i|k|u(x) = \mathscr{O}\left(|x|^{-\frac{3}{2}}\right) \quad \text{for} \quad |x| \to \infty \tag{7}$$

see, e.g., [4]. Therefore, in what follows it is assumed that for a real k we have a radiating u (7), while for a complex k it follows that u exponentially decays at infinity. Additionally, conditions (3)–(5) lead to

$$\left\{ \begin{array}{lcl} u_0^+ - u_0^- & = & 0 \\ u_1^+ - u_1^- & = & 0 \end{array} \right. \qquad \text{on} \qquad \mathbb{R} \setminus \overline{\Sigma}\,. \tag{8}$$

Thus, when computing

$$h^+ - h^- = r_\Sigma(u_1^+ - u_1^-) - r_\Sigma(i\varrho^+ u_0^+ + i\varrho^- u_0^-), \tag{9}$$

we conclude that we necessarily need to have a compatibility condition between the data:

$$h^+ - h^- \in r_\Sigma \widetilde{H}_2^{-1/2}(\Sigma)\,. \tag{10}$$

This occurs because from (8) and (5) it follows $u_1^+ - u_1^- \in \widetilde{H}_2^{-1/2}(\Sigma)$, and also due to

$$i\varrho^+ r_\Sigma u_0^+ + i\varrho^- r_\Sigma u_0^- \in H_2^{1/2}(\Sigma) \tag{11}$$

(which does not change the space in (10) because $H_2^{1/2}(\Sigma) \hookrightarrow r_\Sigma \widetilde{H}_2^{-1/2}(\Sigma)$ is continuously embedded).

From now on, we will refer to *Problem* $\mathscr{P}_{\mathrm{Imp}}$ as the one characterized by (3), (5), (7) and (10) in the case of $k \in \mathbb{R}$, and by (3), (5) and (10) in the case of $k \notin \mathbb{R}$ (within the above explained space setting).

2. Uniqueness of solution for the homogeneous problem in finite energy norm spaces

Let us assume that Σ is a part of some smooth and simple curve S which separates the space $\mathbb{R}^2$ into two disjoint domains Ω^+ and $\Omega^- = \mathbb{R}^2 \backslash \overline{\Omega^+}$, such that Ω^+ is a bounded domain and $S = \partial\Omega^\pm$. In this case, we will denote by $n(z) = (n_1(z), n_2(z))$ the outward unit normal vector at the point $z \in S = \partial\Omega^+$.

Theorem 2.1.

(i) *If* $\Im\mathrm{m}\, k \neq 0$, *and one of the following situations holds:* (A) $(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) > 0$, $\Re\mathrm{e}\, \varrho^\pm \geq 0$; (B) $(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) < 0$, $\Re\mathrm{e}\, \varrho^\pm \leq 0$; (C) $|\Im\mathrm{m}\, k| \geq |\Re\mathrm{e}\, k|$, $\Im\mathrm{m}\, \varrho^\pm \geq 0$; (D) $\Im\mathrm{m}\, \varrho^- \geq (\Im\mathrm{m}\, \varrho^+)(\Re\mathrm{e}\, \varrho^-)/(\Re\mathrm{e}\, \varrho^+)$, $\Re\mathrm{e}\, k = 0$, $\Re\mathrm{e}\, \varrho^+ \neq 0$; (E) $\Im\mathrm{m}\, \varrho^+ \geq (\Im\mathrm{m}\, \varrho^-)(\Re\mathrm{e}\, \varrho^+)/(\Re\mathrm{e}\, \varrho^-)$, $\Re\mathrm{e}\, k = 0$, $\Re\mathrm{e}\, \varrho^- \neq 0$; (F) $(\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 - 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)(\Im\mathrm{m}\, \varrho^+)/(\Re\mathrm{e}\, \varrho^+) \geq 0$, $\Re\mathrm{e}\, k \neq 0$, $\varrho^- = 0$, $\Re\mathrm{e}\, \varrho^+ \neq 0$, $\Im\mathrm{m}\, \varrho^+ \neq 0$; (G) $\Re\mathrm{e}\, k \neq 0$, $\varrho^+ = 0$, $\Re\mathrm{e}\, \varrho^- \neq 0$, $\Im\mathrm{m}\, \varrho^- \neq 0$, $(\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 - 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)(\Im\mathrm{m}\, \varrho^-)/(\Re\mathrm{e}\, \varrho^-) \geq 0$, *then the homogeneous Problem* $\mathscr{P}_{\mathrm{Imp}}$ *has only the trivial solution* $u = 0$ *in the space* $H_2^1(\Omega)$.

(ii) *If* $\Im\mathrm{m}\, k = 0$, $\Re\mathrm{e}\, \varrho^{\pm} \geq 0$, *then the homogeneous Problem* $\mathscr{P}_{\mathrm{Imp}}$ *has only the trivial solution* $u = 0$ *in the space* $H^1_{2,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$.

P r o o f. Let us first consider the case of $\Im\mathrm{m}\, k = 0$ and $\Re\mathrm{e}\, \varrho^{\pm} \geq 0$. Take R to be a sufficiently large positive number and let $B(R)$ be the disk centered at the origin with radius R, such that $\overline{\Omega_+} \subset B(R)$. We define $\Omega^-_R := \Omega^- \cap B(R)$, and consider u to be a solution of the homogeneous problem $\mathscr{P}_{\mathrm{Imp}}$. Then, Green's formula for u and its complex conjugate $\bar{u}$ (in the domains Ω^+ and Ω^-_R) yields

$$\int_{\Omega^+} \Big[|\nabla u|^2 - k^2|u|^2\Big] dx = \Big\langle [\partial_n u]^+_S, [u]^+_S \Big\rangle_S , \tag{12}$$

$$\int_{\Omega^-_R} \Big[|\nabla u|^2 - k^2|u|^2\Big] dx = -\Big\langle [\partial_n u]^-_S, [u]^-_S \Big\rangle_S + \int_{\partial B(R)} \partial_n u\, \overline{u} dS, \tag{13}$$

where the symbols $[\cdot]^{\pm}$ denote the non-tangential limit values on S from $\Omega^{\pm}$ and $\langle\cdot,\cdot\rangle_S$, $\langle\cdot,\cdot\rangle_\Sigma$ denote the duality brackets between the dual spaces $H_2^{-\frac{1}{2}}(S)$ and $H_2^{\frac{1}{2}}(S)$, or $\widetilde{H}_2^{-\frac{1}{2}}(\Sigma)$ and $H_2^{\frac{1}{2}}(\Sigma)$, or $H_2^{-\frac{1}{2}}(\Sigma)$ and $\widetilde{H}_2^{\frac{1}{2}}(\Sigma)$. For regular functions, e.g., $f, g \in L_2(\mathscr{M})$, we have

$$\langle f, g\rangle_{\mathscr{M}} = \int_{\mathscr{M}} f\overline{g}\, d\mathscr{M}, \tag{14}$$

with $\mathscr{M} = S$ or $\mathscr{M} = \Sigma$.

Note that the interior regularity in Ω of solutions of the Helmholtz equation (3) gives us $[u]^+_{S\setminus\overline{\Sigma}} = [u]^-_{S\setminus\overline{\Sigma}}$ and $[\partial_n u]^+_{S\setminus\overline{\Sigma}} = [\partial_n u]^-_{S\setminus\overline{\Sigma}}$. Then, by summing up (12) and (13), we obtain

$$\begin{aligned} \int_{\Omega^+\cup\Omega^-_R} \Big[|\nabla u|^2 - k^2|u|^2\Big] dx &= \langle u_1^+, u_0^+\rangle_\Sigma - \langle u_1^-, u_0^-\rangle_\Sigma + \int_{\partial B(R)} \partial_n u\, \overline{u}\, dS \\ &= i\varrho^+\langle u_0^+, u_0^+\rangle_\Sigma + i\varrho^-\langle u_0^-, u_0^-\rangle_\Sigma \\ &\quad + \int_{\partial B(R)} \partial_n u\, \overline{u}\, dS\,. \end{aligned} \tag{15}$$

Now, since we are assuming R to be sufficiently large, we can apply the Sommerfeld radiation condition on the circle $\partial B(R)$. Let us now separate the imaginary part of equation (15), and use the fact that $u \in \mathrm{Som}(\Omega)$ implies $u(x) = \mathscr{O}(|x|^{-\frac{1}{2}})$ as $|x| \to \infty$. We therefore obtain

$$\Re\mathrm{e}\, \varrho^+ \int_\Sigma |u_0^+|^2 d\Sigma + \Re\mathrm{e}\, \varrho^- \int_\Sigma |u_0^-|^2 d\Sigma + |k| \int_{\partial B(R)} |u|^2\, dS = \mathscr{O}(R^{-1}), \tag{16}$$

which yields

$$\lim_{R\to\infty} \int_{\partial B(R)} |u|^2 \, dS = 0, \tag{17}$$

due to the conditions $\Re e\, \varrho^{\pm} \geq 0$. Consequently, from the *Rellich–Vekua Theorem*, it follows that $u = 0$ in Ω.

For the conditions in proposition (i), we can repeat the same reasoning as in the previous case up to the step of formula (15). Additionally, since $\Im m\, k \neq 0$, the solution $u \in H_2^1(\Omega)$ of the Helmholtz equation exponentially decays at infinity and so it follows

$$\int_{\Omega} [|\nabla u|^2 - k^2 |u|^2] dx = i\varrho^+ \langle u_0^+, u_0^+ \rangle_\Sigma + i\varrho^- \langle u_0^-, u_0^- \rangle_\Sigma \tag{18}$$

when passing (15) to the limit as $R \to \infty$. From the real and imaginary parts of the last identity, we obtain

$$\int_{\Omega} [|\nabla u|^2 + ((\Im m\, k)^2 - (\Re e\, k)^2)|u|^2] dx = -\Im m\, \varrho^+ \langle u_0^+, u_0^+ \rangle_\Sigma - \Im m\, \varrho^- \langle u_0^-, u_0^- \rangle_\Sigma \tag{19}$$

and

$$-2(\Re e\, k)(\Im m\, k) \int_{\Omega} |u|^2 \, dx = \Re e\, \varrho^+ \langle u_0^+, u_0^+ \rangle_\Sigma + \Re e\, \varrho^- \langle u_0^-, u_0^- \rangle_\Sigma. \tag{20}$$

Therefore, for each of the conditions (A)–(G), it follows from the last two identities that $u = 0$ in Ω. ■

3. Auxiliary pseudo-differential operators

In the present section, we will introduce auxiliary pseudo-differential operators in view of a representation of the solutions of the original problem. Without lost of generality, we will assume that $k > 0$ when in the case of a real wave number (the case of $k < 0$ runs with obvious changes).

We will denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\Gamma(x, k) := -\frac{i}{4} \mathscr{H}_0^{(1)}(k|x|), \tag{21}$$

where $\mathscr{H}_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero. Recall that the fundamental function $\Gamma(x, k)$ satisfies the Sommerfeld radiation condition (for a real positive k), and it has the following logarithmic singularity in the neighbourhood of the origin

$$\Gamma(x,k) = -\frac{1}{2\pi}\ln\frac{1}{|x|} + \mathscr{O}(|x|^2 \ln|x|), \quad |x| < \frac{1}{2} \tag{22}$$

(cf. [4, §3.4]). Then the corresponding single and double layer potentials are of the form $V(\psi)(x) = \int_\Sigma \Gamma(x-y,k)\psi(y)dS$, with $x \notin \Sigma$, and $W(\varphi)(x) = \int_\Sigma [\partial_{n(y)}\Gamma(x-y,k)]\varphi(y)dS$, with $x \notin \Sigma$, where ψ and φ are density functions.

Note that by the standard arguments of the Green identities we obtain the following integral representation of a radiating solution $u \in H^1_{2,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$ of the homogeneous Helmholtz equation

$$\pm\int_S \Big\{ \left[\partial_{n(y)}\Gamma(x-y,k)\right][u(y)]^\pm - \Gamma(x-y,k)\left[\partial_{n(y)}u(y)\right]^\pm \Big\}\, dS = \begin{cases} u(x)\,, & x \in \Omega^\pm \\ 0\,, & x \in \Omega^\mp \end{cases}. \tag{23}$$

In addition, by summing up and taking into account the continuity relations $[u]^+_{S\backslash\overline{\Sigma}} = [u]^-_{S\backslash\overline{\Sigma}}$ and $[\partial_n u]^+_{S\backslash\overline{\Sigma}} = [\partial_n u]^-_{S\backslash\overline{\Sigma}}$, we derive that $u(x) = \int_\Sigma \{[\partial_{n(y)}\Gamma(x-y,k)]\ ([u(y)]^+ - [u(y)]^-) - \Gamma(x-y,k)([\partial_{n(y)}u(y)]^+ - [\partial_{n(y)}u(y)]^-)\}dS = W(u_0^+ - u_0^-)(x) - V(u_1^+ - u_1^-)(x)$, $x \in \Omega$, which holds for a solution $u \in H^1_{2,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$ when k is a positive real number, as well as for a solution $u \in H^1_2(\Omega)$ of the Helmholtz equation when k is a complex (non-real) wave number.

Let us now recall some mapping properties of the above single and double layer potentials, in Bessel potential spaces with indices $s \in \mathbb{R}$, and $1 < p < \infty$:

$$\begin{aligned} V &: H^s_2(S) \to H^{s+\frac{3}{2}}_{2,\mathrm{loc}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) \qquad [H^s_2(S) \to H_2^{s+\frac{3}{2}}(\Omega^+)]\,, \\ W &: H^s_2(S) \to H^{s+\frac{1}{2}}_{2,\mathrm{loc}}(\Omega^-) \cap \mathrm{Som}(\Omega^-) \qquad [H^s_2(S) \to H_2^{s+\frac{1}{2}}(\Omega^+)] \end{aligned} \tag{24}$$

for a real positive wave number k, and

$$V : H^s_2(S) \to H_2^{s+\frac{3}{2}}(\Omega^\pm)\,, \qquad W : H^s_2(S) \to H_2^{s+\frac{1}{2}}(\Omega^\pm) \tag{25}$$

for a complex (non-real) k.

For $\psi \in H_2^{-\frac{1}{2}}(S)$ and $\varphi \in H_2^{\frac{1}{2}}(S)$, the following jump relations are also well-known

$$\begin{aligned} &[V(\psi)]^+_S = [V(\psi)]^-_S =: \mathscr{H}(\psi), \qquad [\partial_n V(\psi)]^\pm_S =: \left[\mp\tfrac{1}{2}I + \mathscr{K}\right](\psi), \\ &[W(\varphi)]^\pm_S =: \left[\pm\tfrac{1}{2}I + \mathscr{K}^*\right](\varphi), \qquad [\partial_n W(\varphi)]^+_S = [\partial_n W(\varphi)]^-_S =: \mathscr{L}(\varphi), \end{aligned} \tag{26}$$

where I denotes the identity operator, and

$$\mathscr{H}(\psi)(z) := \int_S \Gamma(z-y,k)\psi(y)dS, \quad z \in S, \tag{27}$$

$$\mathscr{K}(\psi)(z) := \int_S [\partial_{n(z)}\Gamma(z-y,k)]\psi(y)dS, \quad z \in S, \tag{28}$$

$$\mathscr{K}^*(\varphi)(z) := \int_S [\partial_{n(y)}\Gamma(y-z,k)]\varphi(y)dS, \quad z \in S, \tag{29}$$

$$\mathscr{L}(\varphi)(z) := \lim_{x\to z\in S} \partial_{n(x)} \int_S [\partial_{n(y)}\Gamma(y-x,k)]\varphi(y)dS, \quad z \in S. \tag{30}$$

THEOREM 3.1. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. The operators* (27)–(30) *considered now on Σ are pseudo-differential operators of order -1, 0, 0, and 1 (respectively) which can be extended/restricted to the following bounded mappings:*

$$r_\Sigma\mathscr{H} : \widetilde{H}^s_p(\Sigma) \to H^{s+1}_p(\Sigma)\,, \quad r_\Sigma\mathscr{L} : \widetilde{H}^{s+1}_p(\Sigma) \to H^s_p(\Sigma) \tag{31}$$

$$r_\Sigma\mathscr{K} : \widetilde{H}^s_p(\Sigma) \to H^s_p(\Sigma)\,, \quad r_\Sigma\mathscr{K}^* : \widetilde{H}^s_p(\Sigma) \to H^s_p(\Sigma) \tag{32}$$

In addition:

(i) *The operator $r_\Sigma\mathscr{H} : \widetilde{H}^s_p(\Sigma) \to H^{s+1}_p(\Sigma)$ is Fredholm if and only if*

$$(1/p) - (3/2) < s < (1/p) - (1/2)\,. \tag{33}$$

(ii) *The operator $r_\Sigma\mathscr{L} : \widetilde{H}^{s+1}_p(\Sigma) \to H^s_p(\Sigma)$ is Fredholm if and only if* (33) *holds true.*

(iii) *The operators in* (31) *are invertible provided that* (33) *holds true.*

The last result was derived using the methods detailed presented in §5 of [5].

4. Existence and regularity of solutions on Bessel potential spaces with additional smoothness

In this final section, for the spaces H^s_p, with $1 \le s = 1 + \varepsilon < 2$ and $1 < p < \infty$, we will analyze the existence of a solution of the corresponding Problem $\mathscr{P}_{Imp}$, in the above indicated form

$$u(x) = W(\varphi)(x) - V(\psi)(x)\,, \quad x \in \Omega\,, \tag{34}$$

where the unknown densities φ and ψ are related to the source u and its normal derivative by the equations $\varphi = u_0^+ - u_0^-$, and $\psi = u_1^+ - u_1^-$.

The boundary conditions (5) (together with (10)) can be equivalently rewritten in the form

$$\left\{\begin{array}{lcl} u_1^+ - u_1^- - i\varrho^+ u_0^+ - i\varrho^- u_0^- & = & f_0 \\ u_1^+ - i\varrho^+ u_0^+ & = & f_1 \end{array}\right. \quad \text{on} \quad \Sigma\,, \tag{35}$$

where

$$f_0 \;:=\; h^+ - h^- \in r_\Sigma \widetilde{H}_p^{\varepsilon - \frac{1}{p}}(\Sigma) \tag{36}$$

$$f_1 \;:=\; h^+ \in H_p^{\varepsilon - \frac{1}{p}}(\Sigma), \tag{37}$$

with $0 \le \varepsilon < 1$. Then, the representation formula (34) together with the jump relations (26) and the boundary conditions (35) lead to the following system of pseudo-differential equations on Σ with unknown φ and ψ:

$$\left\{\begin{array}{lcl} r_\Sigma\{\psi I - i\varrho^+[(\frac{1}{2}I + \mathscr{K}^*)\varphi - \mathscr{H}\psi] - i\varrho^-[(-\frac{1}{2}I + \mathscr{K}^*)\varphi - \mathscr{H}\psi]\} & = & f_0, \\ r_\Sigma\{\mathscr{L}\varphi - (-\frac{1}{2}I + \mathscr{K})\psi - i\varrho^+[(\frac{1}{2}I + \mathscr{K}^*)\varphi - \mathscr{H}\psi]\} & = & f_1. \end{array}\right. \tag{38}$$

For the matter of notation abbreviation, let us introduce the operator

$$\mathscr{A} := \begin{pmatrix} I + i\varrho^+\mathscr{H} + i\varrho^-\mathscr{H} & -i\varrho^+(\frac{1}{2}I + \mathscr{K}^*) - i\varrho^-(-\frac{1}{2}I + \mathscr{K}^*) \\ \frac{1}{2}I - \mathscr{K} + i\varrho^+\mathscr{H} & \mathscr{L} - i\varrho^+(\frac{1}{2}I + \mathscr{K}^*) \end{pmatrix} \tag{39}$$

and $\Phi := (\psi, \varphi)^\top$, $F := (f_0, f_1)^\top$. Then, from (38), we have

$$r_\Sigma \mathscr{A}\Phi = F \quad \text{on} \quad \Sigma\,, \tag{40}$$

where $\Phi \in \widetilde{H}_p^{\varepsilon - \frac{1}{p}}(\Sigma) \times \widetilde{H}_p^{\varepsilon + 1 - \frac{1}{p}}(\Sigma)$ and $F \in (r_\Sigma \widetilde{H}_p^{\varepsilon - \frac{1}{p}}(\Sigma)) \times H_p^{\varepsilon - \frac{1}{p}}(\Sigma)$.

Theorem 4.1. *Let $\varepsilon = 0$ and $2 \le p < 4$.*

(i) *If $\Im\mathrm{m}\, k = 0$ and $\Re\mathrm{e}\, \varrho^\pm \ge 0$, then Problem $\mathscr{P}_{\mathrm{Imp}}$ has a unique solution in the space $H^1_{p,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$, which is representable in the form (34) with the densities φ and ψ defined by the uniquely solvable pseudo-differential equation (40).*

(ii) *If* $\Im\mathrm{m}\, k \neq 0$ *and one of the conditions* (A)–(G) *of Theorem 2.1 is satisfied, then Problem* $\mathscr{P}_{\mathrm{Imp}}$ *has a unique solution in the space* $H^1_p(\Omega)$, *which is representable in the form* (34) *with the densities* φ *and* ψ *defined by the uniquely solvable pseudo-differential equation* (40).

P r o o f. We will analyse the invertibility of the matrix operator

$$r_\Sigma \mathscr{A} \quad : \quad \widetilde{H}_p^{-\frac{1}{p}}(\Sigma) \times \widetilde{H}_p^{1-\frac{1}{p}}(\Sigma) \to [r_\Sigma \widetilde{H}_p^{-\frac{1}{p}}(\Sigma)] \times H_p^{-\frac{1}{p}}(\Sigma). \tag{41}$$

Due to the mapping properties (31)–(32), and the compact embeddings

$$\widetilde{H}_p^{1-1/p}(\Sigma) \hookrightarrow \widetilde{H}_p^{-1/p}(\Sigma)\,, \qquad H_p^{1-1/p}(\Sigma) \hookrightarrow r_\Sigma \widetilde{H}_p^{-1/p}(\Sigma), \tag{42}$$

the operators

$$r_\Sigma I\,,\; r_\Sigma \mathscr{K}^* \quad : \quad \widetilde{H}_p^{1-\frac{1}{p}}(\Sigma) \to r_\Sigma \widetilde{H}_p^{-\frac{1}{p}}(\Sigma), \tag{43}$$

$$r_\Sigma I\,,\; r_\Sigma \mathscr{K}^* \quad : \quad \widetilde{H}_p^{1-\frac{1}{p}}(\Sigma) \to H_p^{-\frac{1}{p}}(\Sigma), \tag{44}$$

$$r_\Sigma \mathscr{H} \quad : \quad \widetilde{H}_p^{-\frac{1}{p}}(\Sigma) \to r_\Sigma \widetilde{H}_p^{-\frac{1}{p}}(\Sigma) \tag{45}$$

are compact. In addition, the operators in (32) are also compact, since their kernel have a weak singularity. Therefore, (41) is a compact perturbation of a triangular matrix operator with the invertible operators $r_\Sigma I$ and $r_\Sigma \mathscr{L}$ in the main diagonal (cf. Theorem 3.1 (iii) and (50) below). Consequently, (41) is a Fredholm operator with zero index. Then, Theorem 2.1 implies that

$$\operatorname{Ker} r_\Sigma \mathscr{A} = \{0\}, \tag{46}$$

and therefore equation (40) is uniquely solvable for arbitrary $F \in [r_\Sigma \widetilde{H}_p^{-\frac{1}{p}}(\Sigma)] \times H_p^{-\frac{1}{p}}(\Sigma)$. ■

Theorem 4.2. *Let* (i) $\Im\mathrm{m}\, k = 0$ *and* $\Re\mathrm{e}\, \varrho^{\pm} \geq 0$, *or* (ii) $\Im\mathrm{m}\, k \neq 0$ *and one of the conditions* (A)–(G) *of Theorem 2.1 be satisfied. If the boundary data satisfy the conditions*

$$(h^+, h^-) \quad \in \quad H_p^{-\frac{1}{p}+\varepsilon}(\Sigma) \times H_p^{-\frac{1}{p}+\varepsilon}(\Sigma), \tag{47}$$

$$h^+ - h^- \quad \in \quad r_\Sigma \widetilde{H}_p^{-\frac{1}{p}+\varepsilon}(\Sigma) \tag{48}$$

for $0 \leq \varepsilon < (2/p) - (1/2)$ *and* $2 \leq p < 4$, *then the solution* u *of the corresponding Problem* $\mathscr{P}_{\mathrm{Imp}}$ *possesses the following regularity* $u \in H^{1+\varepsilon}_{p,\mathrm{loc}}(\Omega) \cap \mathrm{Som}(\Omega)$ *in the case* (i), *and* $u \in H_p^{1+\varepsilon}(\Omega)$ *in the case* (ii).

P r o o f. Theorem 4.1 yields the solvability result. As for the regularity result, due to the continuity results in (24)–(25), and the representation formula, it is sufficient to show that

$$(\psi, \varphi) \in \widetilde{H}_p^{-\frac{1}{p}+\varepsilon}(\Sigma) \times \widetilde{H}_p^{1-\frac{1}{p}+\varepsilon}(\Sigma). \tag{49}$$

The operator $\mathscr{A}$ can be written as $\mathscr{A} = \mathscr{B} + T$, where

$$\mathscr{B} := \begin{pmatrix} I & 0 \\ \frac{1}{2}I & \mathscr{L} \end{pmatrix} \tag{50}$$

and

$$T = (T_{jl})_{j,l=1,2} := \begin{pmatrix} i(\varrho^+ + \varrho^-)\mathscr{H} & \frac{i}{2}(\varrho^- - \varrho^+)I - i(\varrho^+ + \varrho^-)\mathscr{K}^* \\ -\mathscr{K} + i\varrho^+\mathscr{H} & -i\varrho^+(\frac{1}{2}I + \mathscr{K}^*) \end{pmatrix}. \tag{51}$$

Due to Theorem 3.1, we have that

$$r_\Sigma\mathscr{B} : \widetilde{H}_p^{-\frac{1}{p}+\varepsilon}(\Sigma) \times \widetilde{H}_p^{1-\frac{1}{p}+\varepsilon}(\Sigma) \to \left[r_\Sigma\widetilde{H}_p^{-\frac{1}{p}+\varepsilon}(\Sigma)\right] \times H_p^{-\frac{1}{p}+\varepsilon}(\Sigma) \tag{52}$$

is an invertible operator for every $\varepsilon \in (\frac{2}{p} - \frac{3}{2}, \frac{2}{p} - \frac{1}{2})$, and $2 \le p < 4$.

We will now study the operator T. When $\varrho^- + \varrho^+ = 0$ and/or $\varrho^+ = 0$, some of the entries of the matrix operator T are zero, and therefore the below considerations are even simpler. We will therefore consider in detail in here the most usual case when all entries are different from zero: Due to the mapping properties of Theorem 3.1, and to the compact embeddings

$$H_p^{1-1/p+\varepsilon}(\Sigma) \hookrightarrow r_\Sigma\widetilde{H}_p^{\varepsilon-1/p}(\Sigma), \tag{53}$$

$$H_p^{1-1/p+\varepsilon}(\Sigma) \hookrightarrow H_p^{\varepsilon-1/p}(\Sigma), \tag{54}$$

$$r_\Sigma\widetilde{H}_p^{1-1/p+\varepsilon}(\Sigma) \hookrightarrow H_p^{\varepsilon-1/p}(\Sigma), \tag{55}$$

the operator $r_\Sigma T$ is a compact operator between the spaces $\widetilde{H}_p^{-\frac{1}{p}}(\Sigma) \times \widetilde{H}_p^{1-\frac{1}{p}}(\Sigma)$ and $(r_\Sigma\widetilde{H}_p^{-\frac{1}{p}}(\Sigma)) \times H_p^{-\frac{1}{p}}(\Sigma)$. Moreover, the entries T_{11}, T_{12}, T_{21}, and T_{22} are pseudo-differential operators of orders -1, 0, 0 and 0, respectively (cf. Theorem 3.1). Therefore, from $r_\Sigma\mathscr{A}\Phi = F$, we obtain

$$\Phi = (r_\Sigma\mathscr{B})^{-1}F - (r_\Sigma\mathscr{B})^{-1}r_\Sigma T\Phi, \tag{56}$$

i.e.,

$$\begin{aligned}
\psi &= \ell_0 f_0 - \ell_0 \left(r_\Sigma T_{11}\psi - r_\Sigma T_{12}\varphi \right), && (57)\\
\varphi &= (r_\Sigma \mathscr{L})^{-1}(-\frac{1}{2}f_0 + f_1) - (r_\Sigma \mathscr{L})^{-1}(r_\Sigma T_{21} - \frac{1}{2} r_\Sigma T_{11})\psi \\
&\quad -(r_\Sigma \mathscr{L})^{-1}(r_\Sigma T_{22} - \frac{1}{2} r_\Sigma T_{12})\varphi. && (58)
\end{aligned}$$

The above formula of ψ together with $(\psi, \varphi) \in \widetilde{H}_p^{-\frac{1}{p}}(\Sigma) \times \widetilde{H}_p^{1-\frac{1}{p}}(\Sigma)$, and the continuity properties of $r_\Sigma T_{11}$ and $r_\Sigma T_{12}$, imply $\psi \in \widetilde{H}_p^{-\frac{1}{p}+\varepsilon}(\Sigma)$. Taking into account this result in the last formula of φ, and noting that $\mathscr{L}^{-1}$ is a pseudo-differential operator of order -1, we obtain $\varphi \in \widetilde{H}_p^{1-\frac{1}{p}+\varepsilon}(\Sigma)$ and this concludes the proof. ■

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