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INCLUSION PROPERTIES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING A FAMILY OF FRACTIONAL INTEGRAL OPERATORS

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Abstract

A known family of fractional integral operators is used here to define some new subclasses of analytic functions in the open unit disk $\mathbb{U}$. For each of these new function classes, several inclusion relationships are established.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk $\mathbb{U} = \{z \in \mathbb{C};\ |z| < 1\}$. If $f \in \mathcal{A}$ is given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $z \in \mathbb{U}$, then the *Hadamard product (or convolution)* of f and g is defined by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n$.

Let $P_k(\alpha)$ denote the class of functions $h(z)$ analytic in the unit disk $\mathbb{U}$ satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \Re\left(\frac{h(z)-\alpha}{1-\alpha}\right)\right| d\theta \ \leqq \ k\pi \qquad (z = re^{i\theta};\ 0 \leqq \alpha < 1;\ k \geqq 2). \tag{1.2}$$

This class $P_k(\alpha)$ has been introduced in [6]. Note that for $\alpha = 0$, we obtain the class P_k defined and studied in [7] and for $k = 2$, we have the class $P(\alpha)$ of functions with positive real part greater then α. In particular, $P(0)$ is the class P of functions with positive real part.

From (1.2), we can easily deduce that $h \in P_k(\alpha)$, if and only if

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \qquad h_1,\ h_2 \in P(\alpha). \tag{1.3}$$

Following a recent investigation by Noor [5], we define the following function classes:

$$R_k(\alpha) = \left\{ f \in \mathcal{A} \ : \ \frac{zf'(z)}{f(z)} \in P_k(\alpha), \quad z \in \mathbb{U} \right\}, \tag{1.4}$$

$$V_k(\alpha) = \left\{ f \in \mathcal{A} \ : \ \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), \quad z \in \mathbb{U} \right\}, \tag{1.5}$$

$$T_k(\beta,\alpha) = \left\{ f \in \mathcal{A} \ : \ g \in R_2(\alpha) \ \text{ and } \ \frac{zf'(z)}{g(z)} \in P_k(\beta), \ z \in \mathbb{U} \right\}, \tag{1.6}$$

$$T_k^*(\beta,\alpha) = \left\{ f \in \mathcal{A} \ : \ g \in V_2(\alpha) \ \text{ and } \ \frac{(zf'(z))'}{g'(z)} \in P_k(\beta), \ z \in \mathbb{U} \right\}. \tag{1.7}$$

We note that the class $R_2(\alpha) = \mathcal{S}^*(\alpha)$ and $V_2(\alpha) = \mathcal{K}(\alpha)$ are, respectively, the subclasses of $\mathcal{A}$ consisting of functions which are starlike of order α and *convex* of order α in $\mathbb{U}$. The class $T_2^*(\beta,\alpha) = C^*(\beta,\alpha)$ was considered by Noor [3] and $T_2^*(0,0) = C^*$ is the class of *quasi-convex univalent functions* which was first introduced and studied in [4]. It can be easily seen from the above definitions that

$$f(z) \in V_k(\alpha) \iff zf'(z) \in R_k(\alpha), \tag{1.8}$$

and

$$f(z) \in T_k^*(\beta,\alpha) \iff zf'(z) \in T_k(\beta,\alpha). \tag{1.9}$$

For $\lambda > 0$, $\mu, \eta \in \mathbb{R}$ and $\min\{\lambda+\eta, -\mu+\eta, -\mu\} > -2$, Srivastava *et al.* [10] introduced a family of *fractional integral operators*

$$J_{0,z}^{\lambda,\mu,\eta} f(z) : \mathcal{A} \to \mathcal{A},$$

defined by

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2-\mu)\,\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)}\, z^{\mu}\, I_{0,z}^{\lambda,\mu,\eta} f(z), \tag{1.10}$$

where $I_{0,z}^{\lambda,\mu,\eta}$ is the *hypergeometric fractional integral operator* due to Saigo [9] (nowadays known as *Saigo operator*, see also details in [1]):

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} \, {}_2F_1\left(\lambda+\mu, -\eta;\ \lambda;\ 1-\frac{t}{z}\right) f(t)\, dt. \tag{1.11}$$

Here the ${}_2F_1$-function in the kernel of (1.11) is the *Gauss hypergeometric function*, the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon}) \qquad (z \to 0;\ \varepsilon > \max\{0,\ \mu-\eta\} - 1),$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

If $f(z) \in \mathcal{A}$ is of the form (1.1), then fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$ has the form (see, [1, p.167, Eq.(25)]):

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = z + \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)} \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n+\lambda+\eta+1)}\, a_n z^n. \tag{1.12}$$

It is easily verified from (1.12) that

$$z\big(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\big)' = (\lambda+\eta+2) J_{0,z}^{\lambda,\mu,\eta} f(z) - (\lambda+\eta+1) J_{0,z}^{\lambda+1,\mu,\eta} f(z). \tag{1.13}$$

Let us note that the generalized fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$, contains such well known operators as the *Riemann-Liouville fractional integral operator*, the *Srivastava-Owa fractional integral operator*, the *Multiplier transformation operator* and the *Bernardi-Libera-Livengston operator*. One may refer to the papers [1] and [8] for further details and references on these operators.

Using the generalized fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$, we now define the following subclasses of $\mathcal{A}$:

Definition 1. Let $f(z) \in \mathcal{A}$. Then $f(z) \in R^{\lambda,\mu,\eta}(k,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in R_k(\alpha)$, for $z \in \mathbb{U}$.

Definition 2. Let $f(z) \in \mathcal{A}$. Then $f(z) \in V^{\lambda,\mu,\eta}(k,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in V_k(\alpha)$, for $z \in \mathbb{U}$.

Definition 3. Let $f(z) \in \mathcal{A}$. Then $f(z) \in T^{\lambda,\mu,\eta}(k,\beta,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in T_k(\beta,\alpha)$, for $z \in \mathbb{U}$.

Definition 4. Let $f(z) \in \mathcal{A}$. Then $f(z) \in T_*^{\lambda,\mu,\eta}(k,\beta,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in T_k^*(\beta,\alpha)$, for $z \in \mathbb{U}$.

In this paper we establish some inclusion relationships for the above-mentioned function classes.

2. Main inclusion relationships

We recall first the following necessary lemma.

LEMMA 1. (see [2]) *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let ϕ be a complex valued function satisfying the following conditions:*

(i) $\phi(u, v)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C}^2$,
(ii) $(1, 0) \in \mathbb{D}$ and $\phi(1, 0) > 0$,
(iii) $\Re(\phi(iu_2, v_1)) \leqq 0$, whenever $(iu_2, v_1) \in \mathbb{D}$ and $v_1 \leqq -\frac{1}{2}(1 + u_2^2)$.

If $h(z) = 1 + \sum_{m=2}^{\infty} b_m z^m$, is a function analytic in $\mathbb{U}$ such that $(h(z), zh'(z)) \in \mathbb{D}$ and $\Re(\phi(h(z),\ zh'(z))) > 0$ for $z \in \mathbb{U}$, then $\Re(h(z)) > 0$ for $z \in \mathbb{U}$.

Our first main inclusion relationship is given by the theorem below.

THEOREM 1. *Let $f \in \mathcal{A}$, $\lambda > 0$, $\lambda + \eta > -1$ and $\min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2$. Then*

$$R^{\lambda,\mu,\eta}(k, 0) \ \subset\ R^{\lambda+1,\mu,\eta}(k, \alpha), \tag{2.1}$$

where

$$\alpha \ =\ \frac{2}{(\beta + 3) + \sqrt{\beta^2 + 6\beta + 17}} \qquad \text{with} \qquad \beta \ =\ 2(\lambda + \eta). \tag{2.2}$$

P r o o f. Let $f \in R^{\lambda,\mu,\eta}(k, 0)$. Then, by setting

$$\frac{z\big(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\big)'}{J_{0,z}^{\lambda+1,\mu,\eta} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \ \ (z \in \mathbb{U}), \tag{2.3}$$

we see that the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1$ in $z \in \mathbb{U}$. Using identity (1.13) in (2.3) and differentiating with respect to z, we get

$$\frac{z\big(J_{0,z}^{\lambda,\mu,\eta} f(z)\big)'}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \ =\ \left(p(z) + \frac{zp'(z)}{\lambda + \eta + 1 + p(z)}\right) \ \in\ P_k \qquad (z \in \mathbb{U}).$$

Let

$$\phi(z) = \sum_{j=1}^{\infty} \frac{\lambda + \eta + 1 + j}{\lambda + \eta + 2}\, z^j,$$

then

$$\begin{aligned} p(z) * \frac{\phi(z)}{z} &= p(z) + \frac{zp'(z)}{p(z) + \lambda + \eta + 1} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\left(p_1(z) * \frac{\phi(z)}{z}\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(p_2(z) * \frac{\phi(z)}{z}\right) \end{aligned}$$

and this implies that

$$\left(p_i(z) + \frac{zp_i'(z)}{p_i(z) + \lambda + \eta + 1}\right) \in P \qquad (z \in \mathbb{U};\ i = 1, 2). \tag{2.4}$$

We want to show that $p_i(z) \in P(\alpha)$, where α is given by (2.2) and this will show that $p(z) \in P_k(\alpha)$ for $z \in \mathbb{U}$. Let

$$p_i(z) = (1-\alpha)h_i(z) + \alpha \qquad (z \in \mathbb{U};\ i = 1, 2). \tag{2.5}$$

Then in view of (2.4) and (2.5), we obtain for $z \in \mathbb{U}$, $i = 1, 2$:

$$\Re\left((1-\alpha)h_i(z) + \alpha + \frac{(1-\alpha)\, zh_i'(z)}{(1-\alpha)zh_i(z) + \alpha + \lambda + \eta + 1}\right) > 0. \tag{2.6}$$

We now form a function $\phi(u, v)$ by choosing $u = h_i(z)$ and $v = zh_1'(z)$ in (2.6). Thus

$$\phi(u, v) = (1-\alpha)u + \alpha + \frac{(1-\alpha)v}{(1-\alpha)u + \alpha + \lambda + \eta + 1}\,. \tag{2.7}$$

We can easily see that the first two conditions of Lemma 1, are easily satisfied as $\phi(u, v)$ is continuous in $\mathbb{D} = \mathbb{C} - \left(-\frac{\alpha+\lambda+\eta+1}{1-\alpha}\right) \times \mathbb{C}$, $(1, 0) \in \mathbb{D}$ and $\Re(\phi(1, 0)) > 0$. Now for $v_1 \leqq -\frac{1}{2}(1 + u_2^2)$, we obtain

$$\Re(\phi(iu_2, v_1)) = \alpha + \frac{(1-\alpha)(\alpha + \lambda + \eta + 1)v_1}{(\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 u_2^2}$$

$$\leqq \alpha - \frac{1}{2}\frac{(1-\alpha)(\alpha + \lambda + \eta + 1)(1 + u_2^2)}{(\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 u_2^2} = \frac{A + Bu_2^2}{2C},$$

where

$$\begin{aligned} A &= (\alpha + \lambda + \eta + 1)[2\alpha(\alpha + \lambda + \eta + 1) - (1-\alpha)],\\ B &= (1-\alpha)[2\alpha(1-\alpha) - (\alpha + \lambda + \eta + 1)],\\ C &= (\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 u_2^2 \ > \ 0\,. \end{aligned}$$

We note that $\Re(\phi(iu_2, v_1)) \leqq 0$ if and only if, $A \leqq 0$ and $B \leqq 0$. From $A \leqq 0$, we obtain α as given by (2.1) and $B \leqq 0$ gives us $0 \leqq \alpha < 1$. This completes the proof of Theorem 1. ■

Theorem 2. *Let $f \in \mathcal{A}$, $\lambda > 0$, $\lambda + \eta > -1$ and $\min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2$. Then*

$$V^{\lambda,\mu,\eta}(k, 0) \ \subset \ V^{\lambda,\mu,\eta}(k, \alpha), \tag{2.8}$$

where α is given by (2.2).

P r o o f. To prove the inclusion relationship, we observe (in view of Theorem 1) that

$$\begin{aligned} f(z) \in V^{\lambda,\mu,\eta}(k,0) &\iff zf'(z) \in R^{\lambda,\mu,\eta}(k,0) \\ &\implies zf'(z) \in R^{\lambda+1,\mu,\eta}(k,\alpha) \\ &\iff f(z) \in V^{\lambda+1,\mu,\eta}(k,\alpha), \end{aligned}$$

which establishes Theorem 2. ■

THEOREM 3. *Let* $f \in \mathcal{A}$, $\lambda > 0$, $\lambda + \eta > -1$ *and* $\min\{\lambda+\eta, -\mu+\eta, -\mu\} > -2$. *Then*

$$T^{\lambda,\mu,\eta}(k,\beta,0) \subset T^{\lambda+1,\mu,\eta}(k,\gamma,\alpha), \tag{2.9}$$

where α *is given by (2.2) and* $\gamma \leqq \beta$ *is defined in the proof.*

P r o o f. Let $f(z) \in T^{\lambda,\mu,\eta}(k,\beta,0)$. Then there exists $g(z) \in R^{\lambda,\mu,\eta}(2,0)$ such that

$$\frac{z\big(J_{0,z}^{\lambda,\mu,\eta} f(z)\big)'}{J_{0,z}^{\lambda,\mu,\eta} g(z)} \in P_k(\beta) \qquad (z \in \mathbb{U};\ 0 \leqq \beta < 1).$$

Let

$$\begin{aligned} \frac{z\big(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\big)'}{J_{0,z}^{\lambda+1,\mu,\eta} g(z)} &= (1-\gamma)p(z)+\gamma \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\gamma)p_1(z)+\gamma] - \left(\frac{k}{4}-\frac{1}{2}\right)[(1-\gamma)p_2(z)+\gamma], \end{aligned} \tag{2.10}$$

where $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1$. Using the identity (1.13) in (2.10), and after some computation, we obtain

$$\frac{z\big(J_{0,z}^{\lambda,\mu,\eta} f(z)\big)'}{J_{0,z}^{\lambda,\mu,\eta} g(z)} = [(1-\gamma)p(z)+\gamma] + \frac{(1-\gamma)\, zp'(z)}{\lambda+\eta+2}\, \frac{J_{0,z}^{\lambda+1,\mu,\eta} g(z)}{J_{0,z}^{\lambda,\mu,\eta} g(z)}. \tag{2.11}$$

Since $g(z) \in R^{\lambda,\mu,\eta}(2,0)$, by Theorem 1 we know that $g(z) \in R^{\lambda+1,\mu,\eta}(2,\alpha)$, where α is given by (2.2). Hance there exists an analytic function $q(z)$ with $q(0) = 1$, such that

$$\frac{z\big(J_{0,z}^{\lambda+1,\mu,\eta} g(z)\big)'}{J_{0,z}^{\lambda+1,\mu,\eta} g(z)} = (1-\alpha)\, q(z) + \alpha \qquad (z \in \mathbb{U}). \tag{2.12}$$

Then, by using identity (1.13) once again for the function g(z) in (2.12) and using (2.11) therein, we get

$$\frac{z\big(J_{0,z}^{\lambda,\mu,\eta} f(z)\big)'}{J_{0,z}^{\lambda,\mu,\eta} g(z)} - \beta \;=\; (1-\gamma)\,p(z) + (\gamma-\beta) + \frac{(1-\gamma)\,zp'(z)}{(1-\alpha)\,q(z)+\alpha+\lambda+\eta+1}$$

$$\in P_k. \tag{2.13}$$

We form a function $\phi(u,v)$ by taking $u = p_i(z),\; v = zp_i'(z)$ in (2.13) as

$$\phi(u,v) \;=\; (1-\gamma)\,u + (\gamma-\beta) + \frac{(1-\gamma)\,v}{(1-\alpha)\,q(z)+\alpha+\lambda+\eta+1}. \tag{2.14}$$

It can be easily seen that the function $\phi(u,v)$ defined by (2.14) satisfies the conditions (i) and (ii) of Lemma 1. We verify the condition (iii) as follows:

$$\begin{aligned}\Re\big(\phi(iu_2,v_1)\big) &= \gamma-\beta+\frac{(1-\gamma)[(1-\alpha)q_1+\alpha+\lambda+\eta+1]v_1}{[(1-\alpha)q_1+\alpha+\lambda+\eta+1]^2+(1-\alpha)^2\,q_2^2}\\ &\leqq \gamma-\beta-\frac{1}{2}\,\frac{(1-\gamma)[(1-\alpha)+\alpha+\lambda+\eta+1](1+u_2^2)}{[(1-\alpha)q_1+\alpha+\lambda+\eta+1]^2+(1-\alpha)^2\,q_2^2}\\ &\leqq 0 \qquad \text{for} \qquad \gamma \leqq \beta < 1.\end{aligned}$$

Therefore applying Lemma 1, $p_i \in P,\; i = 1,2,$ and consequently $p \in P_k$ and thus $f \in T^{\lambda+1,\mu,\eta}(k,\beta,\alpha)$. ■

Using the same techniques and relation (1.9) with Theorem 3, we have the following

THEOREM 4. *Let* $f \in \mathcal{A},\; \lambda > 0,\; \lambda+\eta > -1$ *and* $\min\{\lambda+\eta, -\mu+\eta, -\mu\} > -2$. *Then*

$$T_*^{\lambda,\mu,\eta}(k,\beta,0) \;\subset\; T_*^{\lambda+1,\mu,\eta}(k,\gamma,\alpha),$$

where γ *and* α *are as in Theorem 3.*

REMARK. Upon setting $\mu = 0$ and $\eta = \delta - 1$, Theorems 1 to 4 would yield the corresponding known results due to Noor [5]. Furthermore, for different choices of k, λ, μ and η, we can obtain several interesting special cases of Theorems 1 to 4.

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References

[1] V. Kiryakova, On two Saigo's fractional integral operators in the class of univalent functions. *Fract. Calc. Appl. Anal.* **9**, No 2 (2006), 159-176.

[2] S. S. Miller, Differential inequalities and caratheordary functions. *Bull. Amer. Math. Soc.* **81** (1975), 79-81.

[3] K.I. Noor, On quasi convex functions and related topics. *Int. J. Math. Math. Sci.* **10** (1987), 241-258.

[4] K.I. Noor, On close-to-convex and related functions. *Ph. D. Thesis,* University of Wales, U.K. (1972).

[5] K.I. Noor, On analytic function related to certain family of integral operators. *J. Inequal. Pure and Appl. Math.* **7**, No 2 (2006), Article 69, 1-6 (Electronic).

[6] K.S. Padamanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation. *Ann. Polon. Math.* **31** (1975), 311-323.

[7] B. Pinchuk, Functions with bounded boundary rotation. *Israel J. Math.* **10** (1971), 7-16.

[8] J.K. Prajapat, R.K. Raina and H.M. Srivastava, Some inclusion properties for certain subclasses of strongly starlike and strongly convex functions involving a family of fractional integral operators. *Integr. Transf. Spec. Funct.* **18**, No 9 (2007), 639-651.

[9] M. Saigo, A remark on integral operators involving Gauss hypergeometric functions. *Math. Rep. College General Ed. Kyushu Univ.* **11** (1978), 135-143.

[10] H.M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus. *J. Math. Anal. Appl.* **131** (1988), 412-420.

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