

**A NOTE ON MULTIPLIERS
FOR INTEGRABLE BOEHMIANS**

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Abstract

The product of an entire function satisfying a growth condition at infinity and an integrable Bohmian is defined. Properties of this product are investigated.

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1. Introduction

Different aspects of the spaces of generalized functions known as Boehmians have been investigated by several authors. For example, the convergence structures for different spaces of Boehmians have been studied in [5] and [11]. Boehmians on the torus have been studied in [17], while Boehmians on the sphere were investigated in [12], [13], and [15]. Also, there have been several integral transforms extended to spaces of Boehmians ([1], [2], [4], [6], [8], [9], [10], [14], [16]).

However, defining a suitable product of a function and a Bohmian has been a problem ([18], [19]). In this note, we provide a definition for the product of an element from a class of entire functions and an integrable Bohmian. Then some properties of this product are established.

2. Integrable Boehmians

In this section, after presenting some notation, we discuss the space of integrable Boehmians. This includes a discussion of the convergence structure known as δ -convergence, and a brief discussion of the Fourier transform. For more results concerning integrable Boehmians see [1], [2], [4], [6], [8], [10], and [16].

Let $L^1(\mathbb{R})$ denote the space of complex-valued integrable functions on the real line \mathbb{R} . Let $C(\mathbb{R})$ denote the space of continuous functions on the real line and $C^\infty(\mathbb{R})$ denote all smooth functions on \mathbb{R} .

A sequence of integrable functions $\{f_n\}$ converges to $f \in L^1(\mathbb{R})$ provided that $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$, where $\|f\| = \int_{-\infty}^{\infty} |f(x)| dx$.

DEFINITION 2.1. A sequence $\{\delta_n\} \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ is called a *delta sequence*, provided

- (i) $\int_{-\infty}^{\infty} \delta_n(x) dx = 1$ for all $n \in \mathbb{N}$;
- (ii) $\int_{-\infty}^{\infty} |\delta_n(x)| dx \leq M$ for some constant M and all $n \in \mathbb{N}$;
- (iii) $\lim_{n \rightarrow \infty} \int_{|x| > \varepsilon} |\delta_n(x)| dx = 0$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$.

A pair of sequences (f_n, δ_n) is called a *quotient of sequences* if $f_n \in L^1(\mathbb{R})$ for $n \in \mathbb{N}$, $\{\delta_n\}$ is a delta sequence, and $f_k * \delta_m = f_m * \delta_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt. \quad (2.1)$$

Two quotients of sequences (f_n, δ_n) and (g_n, ψ_n) are said to be *equivalent* if $f_k * \psi_m = g_m * \delta_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called integrable Boehmians. The space of all integrable Boehmians will be denoted by β_{L^1} and a typical element of β_{L^1} will be written as $F = \left[\frac{f_n}{\delta_n} \right]$.

The space β_{L^1} is a convolution algebra with addition, scalar multiplication, and convolution as follows:

$$\left[\frac{f_n}{\delta_n} \right] + \left[\frac{g_n}{\varphi_n} \right] = \left[\frac{f_n * \varphi_n + g_n * \delta_n}{\delta_n * \varphi_n} \right], \quad (2.2)$$

$$\alpha \left[\frac{f_n}{\delta_n} \right] = \left[\frac{\alpha f_n}{\delta_n} \right], \text{ where } \alpha \in \mathbb{C}, \quad (2.3)$$

$$\left[\frac{f_n}{\delta_n} \right] * \left[\frac{g_n}{\varphi_n} \right] = \left[\frac{f_n * g_n}{\delta_n * \varphi_n} \right]. \quad (2.4)$$

The space $L^1(\mathbb{R})$ may be viewed as a subspace of β_{L^1} by identifying $f \in L^1(\mathbb{R})$ with $\left[\frac{f * \delta_n}{\delta_n} \right] \in \beta_{L^1}$, where $\{\delta_n\}$ is any delta sequence.

DEFINITION 2.2. A sequence $F_n \in \beta_{L^1}$ is said to be δ -convergent to $F \in \beta_{L^1}$, denoted $\delta\text{-}\lim_{n \rightarrow \infty} F_n = F$, if there exists a delta sequence $\{\delta_n\}$ such

that for all $n, k \in \mathbb{N}$, $F_n * \delta_k, F * \delta_k \in L^1(\mathbb{R})$ and, for each $k \in \mathbb{N}$, $F_n * \delta_k \rightarrow F * \delta_k$ in $L^1(\mathbb{R})$ as $n \rightarrow \infty$.

The Fourier transform of an $L^1(\mathbb{R})$ function is given by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt} dt. \tag{2.5}$$

The Fourier transform can be extended to the space β_{L^1} as follows.

DEFINITION 2.3. The Fourier transform for $F = \left[\frac{f_n}{\delta_n} \right] \in \beta_{L^1}$ is given by

$$\widehat{F}(x) = \lim_{n \rightarrow \infty} \widehat{f}_n(x). \tag{2.6}$$

The above limit exists, and is independent of the representative. Moreover, the Fourier transform of a Boehmian is a continuous function and satisfies the same basic properties as the classical Fourier transform of an L^1 function (see [10]).

The following will be needed in the proof of Theorem 2.5 and also in the following sections:

- (i) If $\{\delta_n\}$ is a delta sequence, then $\widehat{\delta}_n \rightarrow 1$ as $n \rightarrow \infty$, where the convergence is uniform on compact sets of \mathbb{R} .
- (ii) There exists a delta sequence $\{\delta_n\}$ such that $\widehat{\delta}_n \in C^\infty(\mathbb{R})$ and $\text{supp } \widehat{\delta}_n$ is compact, $n \in \mathbb{N}$. Let $\mathcal{S}(\mathbb{R})$ denote the space of rapidly decreasing smooth functions. Let $\psi \in \mathcal{S}(\mathbb{R})$ such that ψ has compact support and $\psi(0) = 1$. Since the Fourier transform maps the space $\mathcal{S}(\mathbb{R})$ onto itself, there exists a $\sigma \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\sigma} = \psi$. Now, put $\delta_n(x) = n\sigma(nx), n \in \mathbb{N}$. Then, $\{\delta_n\}$ is the desired delta sequence.

The proof of the following lemma is left to the reader. The space of all continuous functions which vanish at infinity is denoted by $C_0(\mathbb{R})$. Also, a function $f \in C^{(2)}(\mathbb{R})$ provided that it is twice differentiable and $f'' \in C(\mathbb{R})$.

LEMMA 2.4. Let $g \in C^{(2)}(\mathbb{R})$ such that $g^{(j)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ for $j = 0, 1, 2$. Then there exists $f \in L^1(\mathbb{R})$ such that $\widehat{f}(x) = g(x), x \in \mathbb{R}$.

The following theorem will be useful when investigating the product in the next section.

THEOREM 2.5. If $g \in C^{(2)}(\mathbb{R})$, then there exists a unique $F \in \beta_{L^1}$ such that $\widehat{F}(x) = g(x), x \in \mathbb{R}$.

P r o o f. Let $\{\delta_n\}$ be a delta sequence with $\text{supp } \widehat{\delta}_n$ bounded and $\widehat{\delta}_n \in C^\infty(\mathbb{R})$. Thus, $(g\delta_n)^{(j)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ for $j = 0, 1, 2$. By Lemma 2.4, there exists $f_n \in L^1(\mathbb{R})$ such that $\widehat{f}_n = g\widehat{\delta}_n$, $n \in \mathbb{N}$.

Hence, for all $k, n \in \mathbb{N}$,

$$(f_n * \delta_k)^\wedge = (g\widehat{\delta}_n)\widehat{\delta}_k = (g\widehat{\delta}_k)\widehat{\delta}_n = (f_k * \delta_n)^\wedge. \quad (2.7)$$

This yields, for all $k, n \in \mathbb{N}$,

$$f_n * \delta_k = f_k * \delta_n. \quad (2.8)$$

So, $F = \left[\frac{f_n}{\delta_n} \right] \in \beta_{L^1}$. Moreover, for each $x \in \mathbb{R}$,

$$\widehat{F}(x) = \lim_{n \rightarrow \infty} \widehat{f}_n(x) = \lim_{n \rightarrow \infty} g(x)\widehat{\delta}_n(x) = g(x). \quad (2.9)$$

The uniqueness follows from the fact that the Fourier transform is injective [10]. \blacksquare

3. The product

In this section, using the convergence structure, the definition of the product of an entire function satisfying a growth condition and an integrable Boehmian is presented. It is shown that many of the properties satisfied by the product of functions are also satisfied by the product of a function and a Boehmian.

Let

$$\mathcal{M} = \{\varphi : \varphi \text{ is entire, } \exists r > 0, \varepsilon > 0, \gamma > 0 \ni |\varphi(z)| \leq \frac{\gamma e^{r|\text{Im}z|}}{(1+|z|)^{3+\varepsilon}}, z \in \mathbb{C}\}.$$

REMARKS 3.1.

- (i) Elements of \mathcal{M} are called multipliers for β_{L^1} . We will see that each element of \mathcal{M} has a well-defined product with each element of β_{L^1} .
- (ii) $\mathcal{Z} \subset \mathcal{M}$ (\mathcal{Z} is the space of testing functions used to define the space of ultradistributions (see [20]). Let $\mathcal{D}(\mathbb{R})$ denote the space of all infinitely differentiable functions with compact supports. Then, \mathcal{Z} is the space of all functions whose Fourier transforms are elements of $\mathcal{D}(\mathbb{R})$.)
- (iii) If $\varphi \in \mathcal{M}$, then $\widehat{\varphi} \in C^{(2)}(\mathbb{R})$ and $\text{supp } \widehat{\varphi}$ is compact.
- (iv) \mathcal{M} is an algebra which is invariant under differentiation.

Let $\{\sigma_n\}$ be any fixed delta sequence such that $\widehat{\sigma}_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$. For $\varphi \in \mathcal{M}$ and $F \in \beta_{L^1}$, define

$$f_n^\varphi(x) = \frac{1}{2\pi} \int_{|t| \leq n} \left(1 - \frac{|t|}{n}\right) (\widehat{\varphi} * \widehat{F})(t) e^{ixt} dt \quad (3.1)$$

and

$$F_n^\varphi = \left[\frac{f_n^\varphi * \sigma_k}{\sigma_k} \right] \text{ for } n = 1, 2, \dots \quad (3.2)$$

DEFINITION 3.2. Let $\varphi \in \mathcal{M}$ and $F \in \beta_{L^1}$. The product of φ and F , denoted $\varphi \bullet F$, is given by

$$\varphi \bullet F = \delta\text{-}\lim_{n \rightarrow \infty} F_n^\varphi. \quad (3.3)$$

Let $\varphi \in \mathcal{M}$ and $F \in \beta_{L^1}$. Since $\widehat{\varphi} \in C^{(2)}(\mathbb{R})$ and $\text{supp } \widehat{\varphi}$ is compact, $\widehat{\varphi} * \widehat{F} \in C^{(2)}(\mathbb{R})$. So, by Theorem 2.5 there exists a unique $G \in \beta_{L^1}$ such that $\widehat{G} = \widehat{\varphi} * \widehat{F}$.

Also, by examining the proof of Theorem 2.5, $G = \left[\frac{g_n}{\sigma_n} \right]$ for some $g_n \in L^1(\mathbb{R})$, $n \in \mathbb{N}$.

Much of the following is similar to the proof of Theorem 4 in [10]. Using the above and Fubini's Theorem, for each $k, n \in \mathbb{N}$ we obtain

$$(f_n^\varphi * \sigma_k)(x) = \frac{1}{2\pi} \int_{|t| \leq n} \left(1 - \frac{|t|}{n} \right) \widehat{g}_k(t) e^{ixt} dt. \quad (3.4)$$

By another application of Fubini's Theorem,

$$((f_n^\varphi * \sigma_k) * \sigma_m)(x) = \frac{1}{2\pi} \int_{|t| \leq n} \left(1 - \frac{|t|}{n} \right) \widehat{g}_k(t) \widehat{\sigma}_m(t) e^{ixt} dt, \quad (3.5)$$

$$k, m, n \in \mathbb{N}.$$

Since $G = \left[\frac{g_n}{\sigma_n} \right] \in \beta_{L^1}$,

$$\widehat{g}_k \widehat{\sigma}_m = \widehat{g}_m \widehat{\sigma}_k, \quad k, m \in \mathbb{N}. \quad (3.6)$$

It follows from (3.5) and (3.6) that

$$(f_n^\varphi * \sigma_k) * \sigma_m = (f_n^\varphi * \sigma_m) * \sigma_k, \text{ for all } k, m, n \in \mathbb{N}. \quad (3.7)$$

Now,

$$f_n^\varphi * \sigma_k = \omega_n * g_k \quad (k, n \in \mathbb{N}), \quad (3.8)$$

where ω_n is Fejér's kernel [7]. Thus, $f_n^\varphi * \sigma_k \in L^1(\mathbb{R})$ ($k, n \in \mathbb{N}$), and for each $k \in \mathbb{N}$, $f_n^\varphi * \sigma_k \rightarrow g_k$ (in $L^1(\mathbb{R})$) as $n \rightarrow \infty$.

Thus, $F_n^\varphi = \left[\frac{f_n^\varphi * \sigma_k}{\sigma_k} \right] \in \beta_{L^1}$, $n \in \mathbb{N}$, and $\delta\text{-}\lim_{n \rightarrow \infty} F_n^\varphi = G$.

Therefore, the product is well-defined for any $\varphi \in \mathcal{M}$ and any $F \in \beta_{L^1}$.

We also obtain from above, the following

THEOREM 3.3. *Let $\varphi \in \mathcal{M}$ and $F \in \beta_{L^1}$. Then $(\varphi \bullet F)^\wedge = \widehat{\varphi} * \widehat{F}$.*

The product of a function and a Boehmian is consistent with multiplication of ordinary functions. More precisely, let $\varphi \in \mathcal{M}$ and $f \in L^1(\mathbb{R})$. Since $L^1(\mathbb{R}) \subset \beta_{L^1}$, f may be considered as an integrable Boehmian. Thus, $\varphi \bullet f$ has meaning. Moreover, it is not difficult to show that $\varphi \bullet f = \varphi f$.

THEOREM 3.4. *Let $\varphi, \psi \in \mathcal{M}$ and $F, G \in \beta_{L^1}$. Then:*

- (i) $\varphi \bullet (F + G) = \varphi \bullet F + \varphi \bullet G$,
- (ii) $(\varphi + \psi) \bullet F = \varphi \bullet F + \psi \bullet F$,
- (iii) $\alpha(\varphi \bullet F) = \alpha\varphi \bullet F = \varphi \bullet \alpha F$, $\alpha \in \mathbb{C}$,
- (iv) $\varphi \bullet (\psi \bullet F) = (\varphi \bullet \psi) \bullet F$.

P r o o f. The verification of (i), (ii), and (iii) are routine. For (iv),

$$\begin{aligned} (\varphi \bullet (\psi \bullet F))^\wedge &= \widehat{\varphi} * (\widehat{\psi \bullet F}) = \widehat{\varphi} * (\widehat{\psi} * \widehat{F}) = (\widehat{\varphi} * \widehat{\psi}) * \widehat{F} \\ &= (\varphi \bullet \psi)^\wedge * \widehat{F} = ((\varphi \bullet \psi) \bullet F)^\wedge. \end{aligned}$$

Thus, $\varphi \bullet (\psi \bullet F) = (\varphi \bullet \psi) \bullet F$. ■

Let $\mathcal{E}'(\mathbb{R})$ denote the space of distributions with compact supports. By identifying $f \in \mathcal{E}'(\mathbb{R})$ with $\left[\frac{f * \delta_n}{\delta_n} \right] \in \beta_{L^1}$, where $\{\delta_n\}$ is any delta sequence such that $\delta_n \in \mathcal{D}(\mathbb{R})$ ($n \in \mathbb{N}$), we may consider $\mathcal{E}'(\mathbb{R})$ a subspace of β_{L^1} .

EXAMPLE 3.5. Using Definition 3.2, the product $\varphi \bullet \delta^{(p)}$ is determined, where $\delta^{(p)}$ is the p^{th} derivative of the Dirac delta measure.

Let $\varphi \in \mathcal{M}$ and $\{\sigma_n\}$ be a delta sequence with $\widehat{\sigma}_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$,

$$\begin{aligned} (\widehat{\varphi} * (\delta^{(p)})^\wedge)(x) &= (\widehat{\varphi} * (it)^p)(x) = i^p \sum_{j=0}^p (-1)^j \binom{p}{j} x^{p-j} \int_{-\infty}^{\infty} t^j \widehat{\varphi}(t) dt. \end{aligned} \tag{3.9}$$

Thus, with $F = \delta^{(p)}$, for each $k \in \mathbb{N}$

$$\begin{aligned} & (f_n^\varphi * \sigma_k)(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \frac{|t|}{n}\right) \left(i^p \sum_{j=0}^p (-1)^j \binom{p}{j} t^{p-j} \int_{-\infty}^{\infty} u^j \widehat{\varphi}(u) du \right) e^{ixt} \widehat{\sigma}_k(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \frac{|t|}{n}\right) \left(i^p \sum_{j=0}^p (-1)^j \binom{p}{j} t^{p-j} (-i)^j \varphi^{(j)}(0) \right) e^{ixt} \widehat{\sigma}_k(t) dt \\ &= \sum_{j=0}^p (-1)^j \binom{p}{j} \varphi^{(j)}(0) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \frac{|t|}{n}\right) \widehat{\sigma}_k^{(p-j)}(t) e^{ixt} dt \right) \end{aligned} \tag{3.10}$$

$$\rightarrow \sum_{j=0}^p (-1)^j \binom{p}{j} \varphi^{(j)}(0) \sigma_k^{(p-j)}(x) \text{ as } n \rightarrow \infty.$$

(Here the convergence is in $L^1(\mathbb{R})$).

Thus,

$$\varphi \bullet \delta^{(p)} = \sum_{j=0}^p (-1)^j \binom{p}{j} \varphi^{(j)}(0) \left[\frac{\sigma_k^{(p-j)}}{\sigma_k} \right], \quad p = 0, 1, 2, \dots \quad (3.11)$$

This agrees with the product in the space of distributions. That is, if $\varphi \in \mathcal{M}$, then

$$\varphi \delta^{(p)} = \sum_{j=0}^p (-1)^j \binom{p}{j} \varphi^{(j)}(0) \delta^{(p-j)}, \quad p = 0, 1, 2, \dots \quad (3.12)$$

The previous is not an isolated example. If $\varphi \in \mathcal{M}$ and $F \in \mathcal{E}'(\mathbb{R})$, then the product given in Definition 3.2 gives the same result as the product φF considered in the theory of distributions. This will be made more precise in the next theorem.

By applying Theorem 6.11 in [3] and using the fact that the Fourier transform is injective on β_{L^1} , we obtain:

THEOREM 3.6. *Let $\varphi \in \mathcal{M}$ and $f \in \mathcal{E}'(\mathbb{R})$. Then, $\varphi f = \varphi \bullet f$. (φf denotes the product of a slowly increasing C^∞ function and a distribution with compact support.)*

Before the final result of this section, Leibniz formula, is given, a lemma and the definition of a generalized derivative are needed.

Let $\{\varphi_n\}$ be a delta sequence such that $\varphi_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$. For $F = \left[\frac{f_n}{\delta_n} \right] \in \beta_{L^1}$, the generalized n -th derivative of F , denoted $D^n F$, is given by

$$D^n F = \left[\frac{f_k * \varphi_k^{(n)}}{\delta_k * \varphi_k} \right]. \quad (3.13)$$

LEMMA 3.7. *Let $\alpha \in \mathbb{C}$ and $f, g \in C(\mathbb{R})$ where f has compact support. Then,*

$$(\alpha x)^n (f * g) = \sum_{j=0}^n \binom{n}{j} ((\alpha x)^{n-j} f * (\alpha x)^j g), \text{ for } n = 1, 2, \dots \quad (3.14)$$

P r o o f. The equality follows from

$$x(f * g) = x f * g + f * x g \quad (3.15)$$

by induction. ■

THEOREM 3.8. Let $\varphi \in \mathcal{M}$ and $F \in \beta_{L^1}$. Then,

$$D^n(\varphi \bullet F) = \sum_{j=0}^n \binom{n}{j} \left(\varphi^{(n-j)} \bullet D^j F \right), \quad n = 1, 2, \dots \quad (3.16)$$

P r o o f. For $n = 1, 2, \dots$

$$\begin{aligned} (D^n(\varphi \bullet F))^\wedge &= (ix)^n (\widehat{\varphi} * \widehat{F}) \quad (3.17) \\ &= \sum_{j=0}^n \binom{n}{j} \left((ix)^{n-j} \widehat{\varphi} * (ix)^j \widehat{F} \right) \quad (\text{by Lemma 3.7}) \\ &= \sum_{j=0}^n \binom{n}{j} \left((\varphi^{(n-j)})^\wedge * (D^j F)^\wedge \right) = \sum_{j=0}^n \binom{n}{j} \left(\varphi^{(n-j)} \bullet D^j F \right)^\wedge \\ &= \left(\sum_{j=0}^n \binom{n}{j} \left(\varphi^{(n-j)} \bullet D^j F \right) \right)^\wedge. \end{aligned}$$

Thus, $D^n(\varphi \bullet F) = \sum_{j=0}^n \binom{n}{j} \left(\varphi^{(n-j)} \bullet D^j F \right)$, for $n = 1, 2, \dots$ ■

4. Continuity of the product

In this section, the continuity of multiplication is established. This is made more precise in Theorem 4.2.

The space of square integrable functions on \mathbb{R} is denoted by $L^2(\mathbb{R})$. The topology for $L^2(\mathbb{R})$ is generated by the norm $\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}}$.

Let

$$\beta_A = \{F \in \beta_{L^1} : \widehat{F} \in C^{(2)}(\mathbb{R})\}.$$

Let $\{\sigma_n\}$ be a fixed delta sequence such that $\widehat{\sigma}_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$.

REMARKS 4.1.

- (i) β_A is a convolution algebra.
- (ii) For each $F \in \beta_A$, $F * \sigma_n \in L^2(\mathbb{R})$, $n \in \mathbb{N}$. This follows by observing that, for each $n \in \mathbb{N}$, $F * \sigma_n \in L^1(\mathbb{R})$ (see proof of Theorem 2.5), $(F * \sigma_n)^\wedge = \widehat{F} \widehat{\sigma}_n \in C^{(2)}(\mathbb{R})$, and $\text{supp}(F * \sigma_n)^\wedge$ is compact.

Now, we define a separating family of seminorms on β_A .

For $F \in \beta_A$,

$$\gamma_p(F) = \|F * \sigma_p\|_2, \quad p = 1, 2, \dots \quad (4.1)$$

The sequence of seminorms $\{\gamma_p\}$ generates a locally convex topology for β_A . A sequence $\{F_n\}$ in β_A converges to $F \in \beta_A$, denoted $\gamma\text{-}\lim_{n \rightarrow \infty} F_n = F$, if for each p , $\gamma_p(F_n - F) \rightarrow 0$ as $n \rightarrow \infty$.

We now show that multiplication by an element of \mathcal{M} is continuous.

THEOREM 4.2. *Let $\varphi \in \mathcal{M}$. Then $F \rightarrow \varphi \bullet F$ is a continuous map from β_{L^1} into β_A . (That is, if $\delta\text{-}\lim_{n \rightarrow \infty} F_n = F$, then $\gamma\text{-}\lim_{n \rightarrow \infty} \varphi \bullet F_n = \varphi \bullet F$.)*

P r o o f. First notice that for $F \in \beta_{L^1}$, $\varphi \bullet F \in \beta_A$. Indeed,

$$(\varphi \bullet F)^\wedge = \widehat{\varphi} * \widehat{F} \in C^{(2)}(\mathbb{R}). \tag{4.2}$$

Now, suppose $F_n, F \in \beta_{L^1}$ such that $\delta\text{-}\lim_{n \rightarrow \infty} F_n = F$. Then,

$$\widehat{F}_n \rightarrow \widehat{F} \text{ uniformly on compact sets as } n \rightarrow \infty \text{ (see [10]).}$$

$$((\varphi \bullet F_n) * \sigma_k)^\wedge, ((\varphi \bullet F) * \sigma_k)^\wedge \in C^{(2)}(\mathbb{R}), \quad (k, n \in \mathbb{N}), \tag{4.3}$$

and,

$$\text{supp}((\varphi \bullet F_n) * \sigma_k)^\wedge \cup \text{supp}((\varphi \bullet F) * \sigma_k)^\wedge \subseteq \text{supp } \widehat{\sigma}_k, \quad k, n \in \mathbb{N}. \tag{4.4}$$

Now, for each $k, n \in \mathbb{N}$ and all $x \in \mathbb{R}$,

$$\begin{aligned} & |((\varphi \bullet F_n) * \sigma_k)^\wedge(x) - ((\varphi \bullet F) * \sigma_k)^\wedge(x)| \\ &= |((\widehat{\varphi} * \widehat{F}_n) \widehat{\sigma}_k)(x) - ((\widehat{\varphi} * \widehat{F}) \widehat{\sigma}_k)(x)| \\ &\leq \left(\int_{\text{supp } \widehat{\varphi}} |\widehat{F}_n(x-t) - \widehat{F}(x-t)| |\widehat{\varphi}(t)| dt \right) |\widehat{\sigma}_k(x)|. \end{aligned} \tag{4.5}$$

By using the above, we obtain, for each $k \in \mathbb{N}$,

$$\|((\varphi \bullet F_n) * \sigma_k)^\wedge - ((\varphi \bullet F) * \sigma_k)^\wedge\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{4.6}$$

Plancherel's theorem yields,

$$\|(\varphi \bullet F_n) * \sigma_k - (\varphi \bullet F) * \sigma_k\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{4.7}$$

Thus, $\gamma\text{-}\lim_{n \rightarrow \infty} \varphi \bullet F_n = \varphi \bullet F$. ■

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