ʃractional Calculus & ʃApplied Analysis
An International Journal for Theory and Applications
VOLUME 10, NUMBER 2 (2007) ISSN 1311-0454

ON THE EQUIVALENCE OF THE RIEMANN-LIOUVILLE AND THE CAPUTO FRACTIONAL ORDER DERIVATIVES IN MODELING OF LINEAR VISCOELASTIC MATERIALS

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Dedicated to Prof. Michele Caputo
on the occasion of his 80th birthday

Abstract

In the process of constructing empirical mathematical models of physical phenomena using the fractional calculus, investigators are usually faced with the choice of which definition of the fractional derivative to use, the Riemann-Liouville definition or the Caputo definition. This investigation presents the case that, with some minimal restrictions, the two definitions produce completely equivalent mathematical models of the linear viscoelastic phenomenon.

Mathematics Subject Classification: 26A33

Key Words and Phrases: Riemann-Liouville and Caputo fractional derivatives, fractional calculus, linear viscoelastic materials

1. Introduction

When modeling linear viscoelastic materials using fractional order derivatives, one must choose between the classical Riemann-Liouville definition [1], and the now recognized Caputo definition [2]. This investigation establishes that, with some restrictions, the two definitions are equivalent models for linear viscoelastic materials. The Riemann-Liouville derivative is the first derivative of $1-\beta$ order integral of the function of the time $\varepsilon(t)$:

[1]The author will be always grateful to Michele Caputo, an honored mentor and friend.

$$D^{\beta}_{RL}[\varepsilon(t)] := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{\varepsilon(t-\tau)}{\tau^{\beta}} d\tau \,. \tag{1}$$

Here $\varepsilon(t)$ represents the one-dimensional, time-dependent strain in the material.

Note that the β-order derivative operator (1) is subscripted by "RL" to denote the Riemann-Liouville definition.

Using the Leibnitz rule to differentiate the integral, breaks this definition into two parts: a singular term containing the initial strain and an integral term:

$$D^{\beta}_{RL}[\varepsilon(t)] = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{\varepsilon}(t-\tau)}{\tau^{\beta}} d\tau + \frac{\varepsilon(0)}{\Gamma(1-\beta)t^{\beta}} \,. \tag{2}$$

The integral term is the $1-\beta$ order integral of the strain rate. This integral is the Caputo definition, subscripted by "C", for the fractional order derivative of the function $\varepsilon(t)$:

$$D^{\beta}_{C}[\varepsilon(t)] := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{\varepsilon}(t-\tau)}{\tau^{\beta}} d\tau \,. \tag{3}$$

Thus the difference between the two definitions is a singular term that contains the initial value of the function $\varepsilon(t)$,

$$D^{\beta}_{RL}[\varepsilon(t)] = D^{\beta}_{C}[\varepsilon(t)] + \frac{\varepsilon(0)}{\Gamma(1-\beta)t^{\beta}}. \tag{4}$$

This relationship between the two definitions is one of the essential features that leads to their equivalence in the modeling of linear viscoelastic materials.

2. The viscoelastic models

A four-parameters viscoelastic model, relating one dimensional stress $\sigma(t)$ to one dimensional strain $\varepsilon(t)$ is constructed using the Riemann-Liouville definition. Based on two restrictions, this model is shown to be equivalent to a similar four-parameters viscoelastic model based on the Caputo definition.

The four parameters are the glassy modulus E_0, the rubbery modulus E_{∞}, the time-temperature equivalence parameter α, and the order of the derivatives β. Here the orders of differentiation on stress and strain are the same. This feature is a consequence of the thermodynamic considerations that ensure the real and imaginary parts of this model's complex modulus, given in (11) below, are positive for all positive frequencies of motion [3],

$$\sigma(t) + \alpha^{\beta} D^{\beta}_{RL}[\sigma(t)] = E_{\infty}\varepsilon(t) + E_0 \alpha^{\beta} D^{\beta}_{RL}[\varepsilon(t)]. \tag{5}$$

It will be demonstrated that this Riemann-Liouville (RL) model, subject to one more restriction, is equivalent to a model of the same mathematical structure using Caputo (C) derivatives.

The Leibnitz rule is applied again, as in (2) above, to separate the singular and integral terms in the Riemann-Liouville derivative. The result is:

$$\sigma(t) + \alpha^{\beta}\left\{ D^{\beta}_{C}[\sigma(t)] + \frac{\sigma(0)}{\Gamma(1-\beta)t^{\beta}} \right\} = E_{\infty}\varepsilon(t) + E_0\alpha^{\beta}\left\{ D^{\beta}_{C}[\varepsilon(t)] + \frac{\varepsilon(0)}{\Gamma(1-\beta)t^{\beta}} \right\}. \tag{6}$$

Notice that if the initial stress $\sigma(0)$ is equal to the glassy modulus E_0 multiplied by the initial strain $\varepsilon(0)$:

$$\sigma(0) = E_0\, \varepsilon(0), \tag{7}$$

then the singular terms on both sides of (6) add out. The result (8) is the original viscoelastic model (5), where the Riemann-Liouville derivatives have been replaced by Caputo derivatives:

$$\sigma(t) + \alpha^{\beta} D^{\beta}_{C}[\sigma(t)] = E_{\infty} + E_0 \alpha^{\beta} D^{\beta}_{C}[\varepsilon(t)]. \tag{8}$$

In this case, the Caputo derivative and the Riemann-Liouville derivative produce equivalent mathematical descriptions of a linear viscoelastic material.

This equivalence can also be seen in the frequency domain. Taking the Laplace transform

$$\mathcal{L}[\varepsilon(t)] = \bar{\varepsilon}(s) := \int_0^{\infty} \varepsilon(t)\, e^{-st}\, dt \tag{9}$$

of (5) or (8), and applying the initial condition in (7), produces the same mathematical relationship between the transform of stress and the transform of strain:

$$\bar{\sigma}(s) = \frac{E_{\infty} + E_0(\alpha s)^{\beta}}{1 + (\alpha s)^{\beta}}\, \bar{\varepsilon}(s)\,. \tag{10}$$

Dividing both sides of (10) by the transform of the strain history $\bar{\varepsilon}(s)$ and replacing the Laplace parameter s with the frequency parameter $i\omega$, where $i = \sqrt{-1}$, produces the expression for the model's frequency-dependent complex modulus,

$$E(i\omega) = \frac{\bar{\sigma}(i\omega)}{\bar{\varepsilon}(i\omega)} \; . \tag{11}$$

As expected, the equivalence of the two viscoelastic models in time domain extends into the frequency domain.

3. Conclusion

This equivalence of the Caputo and Riemann-Liouville derivatives is based on two propositions. The first is a thermodynamic consideration, and the second is that singular terms in stress-strain relationships should balance out. The thermodynamic considerations ensure that the material's complex modulus will have positive real and imaginary parts over all positive frequencies of motion. The balancing of the singular terms ensures that the initial stress is a function only of the initial strain. With these minimally restrictive conditions applied, the two derivatives produced identical models of a linear viscoelastic material.

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Received: 18 July 2007