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INTEGRAL REPRESENTATIONS OF GENERALIZED MATHIEU SERIES VIA MITTAG-LEFFLER TYPE FUNCTIONS

Živorad Tomovski

Abstract

The main purpose of this paper is to present a number of potentially useful integral representations for the generalized Mathieu series as well as for its alternating versions via Mittag-Leffler type functions.

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1. Introduction

The following infinite series:

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \quad (r \in \mathbb{R}^+) \tag{1.1}$$

is known as named after *Emile Leonard Mathieu* (1835-1890), who investigated it in his 1890 work [11] on elasticity of solid bodies.

A remarkable useful integral representation for $S(r)$ in the elegant form:

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x \sin(xr)}{e^x - 1} dx \tag{1.2}$$

was given by Emersleben [6].

Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the *Mathieu series with a fractional power*:

$$\mathbb{S}_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^\mu} \quad (r \in \mathbb{R}^+;\ \mu > 1) \tag{1.3}$$

can be found in the recent works by Diananda [4], Tomovski and Trenčevski [21] and Cerone and Lenard [3]. Motivated essentially by the works of Cerone and Lenard [3] (and Qi [17]), we defined in [18] a *family of generalized Mathieu series*:

$$\mathbb{S}_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = \mathbb{S}_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu} \quad (r,\alpha,\beta,\mu \in \mathbb{R}^+), \tag{1.4}$$

where $\lim\limits_{k\to\infty} a_k = \infty$.

Comparing the definitions (1.1), (1.3) and (1.4), we see that $\mathbb{S}_2(r) = S(r)$ and $\mathbb{S}_\mu(r) = \mathbb{S}_\mu^{(2,1)}(r,\{k\}_{k=1}^{\infty})$.

Furthermore, the special cases

$$\mathbb{S}_2^{(2,1)}(r,\{a_k\}_{k=1}^{\infty}),\quad \mathbb{S}_\mu^{(2,1)}(r,\{k^\gamma\}_{k=1}^{\infty}) \text{ and } \mathbb{S}_\mu^{(\alpha,\alpha/2)}(r,\{k\}_{k=1}^{\infty})$$

were investigated by Qi [17], Tomovski [20] and Cerone and Lenard [3].

In [18] the following integral representations were proposed:

$$\mathbb{S}_\mu^{(\alpha,\beta)}(r;\{k^\gamma\}_{k=1}^{\infty}) = \frac{2}{\Gamma(\mu)} \int_0^\infty \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x-1} {}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2x^{\gamma\alpha}\right] dx \tag{1.5}$$

$(r,\alpha,\beta,\gamma \in \mathbb{R}^+; \gamma(\mu\alpha-\beta) > 1)$;

$$\mathbb{S}_\mu^{(\alpha,\beta)}(r;\{k^{q/\alpha}\}) = \frac{2}{\Gamma(q[\mu-\frac{\beta}{\alpha}])} \int_0^\infty \frac{x^{q[\mu-\frac{\beta}{\alpha}]-1}}{e^x-1} {}_1F_q\left(\mu;\triangle\left(q;q\left[\mu-\frac{\beta}{\alpha}\right]\right);-r^2\left(\frac{x}{q}\right)^q\right) dx \tag{1.6}$$

$\left(r,\alpha,\beta \in \mathbb{R}^+; \mu-\dfrac{\beta}{\alpha} > q^{-1};\ q \in \mathbb{N}\right)$,

where $\triangle(q;\lambda)$ is the $q-$tuple $\dfrac{\lambda}{q}, \dfrac{\lambda+1}{q}, \ldots, \dfrac{\lambda+q-1}{q}$;

$$\mathbb{S}_{\mu+1}^{(\alpha,\alpha/2)}(r;\{k^{2/\alpha}\}_{k=1}^{\infty}) = \mathbb{S}_{\mu+1}^{(2,1)}(r;\{k\}_{k=1}^{\infty}) = \mathbb{S}_{\mu+1}(r)$$

$$= \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^{\infty} \frac{x^{\mu+\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx)dx, \quad (r,\mu \in \mathbb{R}^+); \tag{1.7}$$

$$\mathbb{S}_{\mu}^{(\alpha,0)}(r;\{k^{2/\alpha}\}_{k=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{2}{(n^2+r^2)^{\mu}}$$

$$= \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)} \int_0^{\infty} \frac{x^{\mu-\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx)dx, \quad (r \in \mathbb{R}^+;\ \mu > \frac{1}{2}). \tag{1.8}$$

Here ${}_p\Psi_q$ denotes the *Fox-Wright generalization of the hypergeometric* ${}_pF_q$ *function* with p numerator and q denominator parameters (see for example [7], v.1; [19]), p.50, eq. 1.5 (21); [10], Appendix) and J_ν – the Bessel function of first kind.

In [14] we introduced an alternating variant of $S(r)$,

$$\tilde{S}(r) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{(n^2+r^2)^2} \quad (r \in \mathbb{R}^+), \tag{1.9}$$

called *alternative Mathieu series.*

Using the formulas given in [12]:

$$\sum_{k=1}^{\infty} \mathcal{L}'(k) = -\int_0^{\infty} \frac{tf(t)}{e^t-1}dt, \tag{1.10}$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\mathcal{L}'(k) = \int_0^{\infty} \frac{tf(t)}{e^t+1}dt, \tag{1.11}$$

where $\mathcal{L}(k)$ is the classical Laplace transform of $f(t)$, we obtained in [14] the integral representation of $\tilde{S}(r)$:

$$\tilde{S}(r) = \frac{1}{r}\int_0^{\infty} \frac{x\sin{(xr)}}{e^x+1}dx. \tag{1.12}$$

Let $\tilde{\mathbb{S}}_{\mu}^{(\alpha,\beta)}(r;\mathbf{a})$ be alternating variant of (1.4). Integral representations for the alternating variant of (1.8) are given in [14] by:

$$\tilde{\mathbb{S}}_\mu^{(\alpha,\beta)}(r;\{k^\gamma\}_{k=1}^\infty)$$
$$= \frac{2}{\Gamma(\mu)} \int_0^\infty \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x+1} {}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2x^{\gamma\alpha}\right]dx \tag{1.13}$$
$(r,\alpha,\beta,\gamma \in \mathbb{R}^+; \gamma(\mu\alpha-\beta)>1)$;

$$\tilde{\mathbb{S}}_{\mu+1}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x+1} J_{\mu-\frac{1}{2}}(rx)dx \quad (r,\mu\in\mathbb{R}^+); \tag{1.14}$$

$$\tilde{\mathbb{S}}_\mu^{(\alpha,0)}(r;\{k^{2/\alpha}\}_{k=1}^\infty) = \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-\frac{1}{2}}}{e^x+1} J_{\mu-\frac{1}{2}}(rx)dx \quad (r\in\mathbb{R}^+, \mu>\frac{1}{2}). \tag{1.15}$$

2. Mittag-Leffler type functions

The special functions of the form
$$E_\rho(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k+1)}, \quad E_{\rho,\mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k+\mu)} \tag{2.1}$$
with complex $\rho,\mu\in\mathbb{C}$ $(Re(\rho)>0, Re(\mu)>0)$ are known as *Mittag-Leffler (M-L) functions.*

The former was introduced by Mittag-Leffler [13] and named after him in [7], vol. 3; and the latter is due to Agarwal [1] and also studied in details by Dzrbashjan (see e.g. [5]) who denoted it as $\Phi_{1/\rho}(z;\mu)$.

Prabhakar [16] introduced the M-L type function $E_{\rho,\mu}^\delta(z)$ of the form
$$E_{\rho,\mu}^\delta(z) = \sum_{k=0}^\infty \frac{(\delta)_k}{\Gamma(\rho k+\mu)} \frac{z^k}{k!} \quad (\rho,\mu,\gamma\in\mathbb{C},\ Re(\rho)>0), \tag{2.2}$$
where $(\delta)_k$ is the Pochhammer symbol:
$$(\delta)_0 = 1, \quad (\delta)_k = \delta(\delta+1)\dots(\delta+k-1) \quad (k=1,2,\dots).$$
For $\delta=1$, we have
$$E_{\rho,\mu}^1(z) = E_{\rho,\mu}(z), \quad E_{\rho,1}^1(z) = E_\rho(z). \tag{2.3}$$
The function $E_{\rho,\mu}^\delta$ can be represented via the Fox-Wright Psi-function ${}_1\Psi_1$ and the generalized hypergeometric function ${}_1F_m$ as follows:

$$E^{\delta}_{\rho,\mu}(z) = \frac{1}{\Gamma(\delta)} {}_1\Psi_1[(\delta,1),(\mu,\rho);z], \tag{2.4}$$

$$E^{\delta}_{m,\nu}(z) = \frac{1}{\Gamma(\delta)} {}_1F_m(\delta;\triangle(m;\nu);\frac{z}{m^m}) \ \ (m \in \mathbb{N}, \delta, \nu \in \mathbb{C}). \tag{2.5}$$

Let $m > 1$ be an integer, $\rho_1, \rho_2, \ldots, \rho_m > 0$ and $\mu_1, \mu_2, \ldots, \mu_m$ be arbitrary real numbers. By means of the "multi-indices" $(\rho_i), (\mu_i)$, Kiryakova [9] introduced the so-called *multi-index ($m-$tuple, multiple) M-L functions*:

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_2 + k/\rho_2)\ldots\Gamma(\mu_m + k/\rho_m)}\,. \tag{2.6}$$

About generalized fractional calculus differential and integral operators and Laplace type integral transform, closely related to the functions $E_{(1/\rho_i),(\mu_i)}$, see also Al-Mussalam, Kiryakova and Vu Kim Tuan [2].

The special case $m = 2$ of (2.6) was introduced by Dzrbashjan [5]. He denoted it by

$$E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)} = \Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\Gamma(\mu_2 + k/\rho_2)}\,. \tag{2.7}$$

We mention here some special cases of (2.6), presented by Kiryakova in [9]:

$$E_{1/\rho,\mu}(z) = E_{(1/\rho,0),(\mu,1)}(z) = \Phi_{p,\infty}(z;\mu,1), \tag{2.8}$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} E_{(1,1),(\nu+1,1)}\left(-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^{\nu} \Phi_{1,1}\left(-\frac{z^2}{4};1,\nu+1\right), \tag{2.9}$$

$$\begin{aligned} E_{(1,1,\ldots 1),(\mu_{i+1})}(z) &= {}_1\Psi_m \left[\begin{array}{c|c} (1,1) & \\ (\mu_i,1)_1^m & z \end{array} \right] \\ &= \left(\prod_{i=1}^{m} \Gamma(\mu_i)\right)^{-1} \cdot {}_1F_m(1;\mu_1,\mu_2,\ldots,\mu_m;z). \end{aligned} \tag{2.10}$$

Using the relations (2.4) and (2.5) we give here some integral representations for the Mathieu type series $\mathbb{S}^{(\alpha,\beta)}_{\mu}(r,\mathbf{a})$ and $\tilde{\mathbb{S}}^{(\alpha,\beta)}_{\mu}(r,\mathbf{a})$ via (M-L) type functions:

$$\mathbb{S}^{(\alpha,\beta)}_{\mu}(r;\{k^\gamma\}_{k=1}^{\infty}) = 2\int_0^{\infty} \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x - 1} E^{\mu}_{\gamma\alpha,\gamma(\mu\alpha-\beta)}(-r^2 x^{\gamma\alpha})dx; \tag{2.11}$$

$$\tilde{\mathbb{S}}_{\mu}^{(\alpha,\beta)}(r;\{k^{\gamma}\}_{k=1}^{\infty}) = 2\int\limits_{0}^{\infty} \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x+1} E^{\mu}_{\gamma\alpha,\gamma(\mu\alpha-\beta)}(-r^2x^{\gamma\alpha})dx$$
$$(r,\alpha,\beta,\gamma\in\mathbb{R}^+;\ \ \gamma(\mu\alpha-\beta)>1); \tag{2.12}$$

$$\mathbb{S}_{\mu}^{(\alpha,\beta)}(r;\{k^{q/\alpha}\}_{k=1}^{\infty}) = 2\int\limits_{0}^{\infty} \frac{x^{q[\mu-\beta/\alpha]-1}}{e^x-1} E^{\mu}_{q,q(\mu-\beta/\alpha]}(-r^2x^{q})dx$$
$$(r,\alpha,\beta\in\mathbb{R}^+;\ \ \mu-\beta/\alpha>q^{-1};\ q\in\mathbb{N}). \tag{2.13}$$

From the relation (see [16])

$$\mathcal{L}\left[t^{\mu-1}E^{\delta}_{\rho,\mu}(wt^{\rho})\right](s) = s^{-\mu}(1-ws^{-\rho})^{-\delta}$$
$$s\in\mathbb{C},\ \ Res>0,\ \ |w/s^{\rho}|<1, \tag{2.14}$$

and (1.11), upon setting $f(t)=t^{q[\mu-\beta/\alpha]-2}E^{\mu}_{q,q[\mu-\beta/\alpha]}(-r^2t^q)$, it easily follows the integral representation

$$\tilde{\mathbb{S}}_{\mu}^{(\alpha,\beta)}(r;\{k^{q/\alpha}\}_{k=1}^{\infty}) = 2\int\limits_{0}^{\infty} \frac{x^{q[\mu-\beta/\alpha]-1}}{e^x+1} E^{\mu}_{q,q[\mu-\beta/\alpha]}(-r^2x^{q})dx. \tag{2.15}$$

Next, using the relation (2.7), we obtain also the integral representations:

$$\begin{aligned}\mathbb{S}_{\mu+1}^{(\alpha,\alpha/2)}(r;\{k^{2/\alpha}\}_{k=1}^{\infty}) &= \mathbb{S}_{\mu+1}^{(2,1)}(r;\{k\}_{k=1}^{\infty}) = \mathbb{S}_{\mu+1}(r)\\ &= \frac{\sqrt{\pi}}{2^{2\mu}\Gamma(\mu+1)}\int_0^{\infty}\frac{x^{2\mu}}{e^x-1}E_{(1,1),(\mu+\frac{1}{2},1)}\left(-\frac{r^2x^2}{4}\right)dx;\end{aligned} \tag{2.16}$$

$$\tilde{\mathbb{S}}_{\mu+1}^{(\alpha,\alpha/2)}(r;\{k^{2/\alpha}\}_{k=1}^{\infty}) = \frac{\sqrt{\pi}}{2^{2\mu}\Gamma(\mu+1)}\int_0^{\infty}\frac{x^{2\mu}}{e^x+1}E_{(1,1),(\mu+\frac{1}{2},1)}\left(-\frac{r^2x^2}{4}\right)dx$$
$$(r,\mu\in\mathbb{R}^+); \tag{2.17}$$

$$\mathbb{S}_{\mu}^{(\alpha,0)}(r;\{k^{2/\alpha}\}_{k=1}^{\infty}) = \frac{\sqrt{\pi}}{2^{2\mu-1}\Gamma(\mu)}\int_0^{\infty}\frac{x^{2\mu-1}}{e^x-1}E_{(1,1),(\mu+\frac{1}{2},1)}\left(-\frac{r^2x^2}{4}\right)dx; \tag{2.18}$$

$$\tilde{\mathbb{S}}_{\mu}^{(\alpha,0)}(r;\{k^{2/\alpha}\}_{k=1}^{\infty}) = \frac{\sqrt{\pi}}{2^{2\mu-1}\Gamma(\mu)}\int_0^{\infty}\frac{x^{2\mu-1}}{e^x+1}E_{(1,1),(\mu+\frac{1}{2},1)}\left(-\frac{r^2x^2}{4}\right)dx$$
$$(r,\in\mathbb{R}^+,\ \ \mu>\frac{1}{2}). \tag{2.19}$$

3. Mathieu-type series whose terms contain multi-index M-L functions

In the recent articles ([14], [15], [22]) Pogany, Srivastava and Tomovski considered special kind of Matieu-type series and their alternating variants whose terms contain the Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric ${}_pF_q$ and the Meijer G functions. The derived results concern, among others, closed integral form expressions for the considered series and bilateral bounding inequalities. Here we are interested in giving the integral expression for the Mathieu-type series (and its alternating variants) whose terms contain the multi-index M-L functions from [9] and [2].

Definition. By a Fox's $H-$function we mean a generalized hypergeometric function, defined by means of the Mellin-Barnes-type contour integral

$$H_{p,q}^{m,n} = \left[\sigma \left| \begin{array}{c} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{\prod_{k=1}^{m} \Gamma(b_k - B_k s) \prod_{j=1}^{n} (1 - a_j + sA_j)}{\prod_{k=m+1}^{q} \Gamma(1 - b_k + sB_k) \prod_{j=n+1}^{p} (a_j - sA_j)} \sigma^s ds, \tag{3.1}$$

where $\mathcal{L}'$ is a suitable contour in $\mathbb{C}$, the orders (m, n, p, q) are integers, $0 \le m \le q$, $0 \le n \le p$ and the parameters $a_j \in \mathbb{R}$, $A_j > 0$, $j = 1, 2, \ldots, p$, $b_k \in \mathbb{R}$, $B_k > 0$, $k = 1, 2, \ldots, q$ are such that $A_j(b_k + l) \ne B_k(a_j - l' - 1), l, l' = 0, 1, 2, \ldots$.

For various type of contours and conditions for existence and analyticity of function (3.1) in disks $\subset \mathbb{C}$ whose radii are $\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{B_k} > 0$, one can see, for example [10], Appendix.

We define the Mathieu-type series $\mu_{(1/\rho_i),(\mu_i)}$ and its alternating variant $\tilde{\mu}_{(1/\rho_i),(\mu_i)}$ as follows:

$$\mu_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \sum_{n=1}^{\infty} \frac{E_{(1/\rho_i),(\mu_i)}\left(-\frac{r^2}{a_n}\right)}{a_n^{\lambda}(a_n + r^2)^{\eta}}, \tag{3.2}$$

$$\tilde{\mu}_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{E_{(1/\rho_i),(\mu_i)}\left(-\frac{r^2}{a_n}\right)}{a_n^{\lambda}(a_n + r^2)^{\eta}} \quad (\lambda, \eta, r \in \mathbb{R}^+), \tag{3.3}$$

where we make the convention that the real sequence $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}}$ increases and tends to ∞.

We will need the Laplace transform of $x^{\lambda-1} H_{0,m+1}^{1,0}(wx|\cdot)$. Using the definition (3.1), for real w it follows easily

$$\int_0^\infty e^{-Ax} x^{\lambda-1} H^{1,0}_{0,m+1}\left(wx^\rho \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m} \right) dx = A^{-\lambda} E_{(1/\rho_i),(\mu_i)}\left(\frac{w}{A^p}\right),$$

$$\text{where } m>0, \quad Re\{A\}, \quad Re\{\lambda\}+\rho, \quad \min_{1\le j\le m} Re\left\{\frac{1-\mu_j}{1/\rho_j}\right\}>0. \qquad (3.4)$$

THEOREM. *Let $\lambda > 0, \eta > 0, r > 0$ and let the real sequence $\mathbf{a}$ increase and tend to ∞. Then,*

$$\mu_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \eta I(\lambda, \eta+1) + I(\lambda+1, \eta), \qquad (3.5)$$

$$\tilde{\mu}_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \eta \tilde{I}(\lambda, \eta+1) + \tilde{I}(\lambda+1, \eta), \qquad (3.6)$$

where

$$I(\lambda,\eta) = \int_{a_1}^\infty E_{(1/\rho_i),(\mu_i)}\left(-\frac{r^2}{x}\right) \frac{[a^{-1}(x)]}{x^\lambda (r^2+x)^\eta} dx, \qquad (3.7)$$

$$\tilde{I}(\lambda,\eta) = \int_{a_1}^\infty E_{(1/\rho_i),(\mu_i)}\left(-\frac{r^2}{x}\right) \frac{\sin^2(\frac{\pi}{2}[a^{-1}(x)]}{x^\lambda (r^2+x)^\eta} dx, \qquad (3.8)$$

and $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is an increasing function such that $a(x)|_{x\in\mathbb{N}} = \mathbf{a}$, $a^{-1}(x)$ denotes the inverse of $a(x)$, $[a^{-1}(x)]$ stands for the integer part of the quantity $a^{-1}(x)$.

P r o o f. Taking $\xi = a_n + r^2$ in the formula

$$\Gamma(\eta)\xi^{-\eta} = \int_0^\infty e^{-\xi t} t^{\eta-1} dt \quad (Re\,\xi > 0\,,\; Re\,\eta > 0)$$

and $A = a_n$, $\rho = 1$, $w = -r^2$ in (3.4), we get

$$\mu_{(1/\rho_i),(\mu_i)}(\mathbf{a}; r) = \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty H^{1,0}_{0,m+1}\left(-r^2 t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m} \right) \mathcal{D}_{\mathbf{a}}(s+t)$$

$$\times\; e^{-r^2 s} t^{\lambda-1} s^{\eta-1} ds dt,$$

where

$$\mathcal{D}_{\mathbf{a}}(s+t) = \sum_{n=1}^\infty e^{a_n(s+t)} = (s+t)\int_{a_1}^\infty e^{-(s+t)x}[a^{-1}(x)] dx$$

is the integral representation of the Dirichlet series (see [8], [14], [15]). Hence,

$$\mu_{(1/\rho_i),(\mu_i)}(\mathbf{a};r) = \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0}\left(-r^2t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m}\right)$$
$$\times\ e^{-(r^2+x)s-tx}s^{\eta-1}t^{\lambda-1}(s+t)[a^{-1}(x)]dsdxdt = I_s + I_t,$$

where

$$I_s = \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0}\left(-r^2t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m}\right)$$
$$\times\ e^{-(r^2+x)s-tx}t^{\lambda-1}s^{\eta}[a^{-1}(x)]dsdxdt$$
$$= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0}\left(-r^2t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m}\right)$$
$$\times\ e^{-tx}t^{\lambda-1}[a^{-1}(x)]\left(\int_0^\infty e^{-(r^2+x)s}s^{\eta}ds\right)dxdt$$
$$= \eta \int_{a_1}^\infty \frac{[a^{-1}(x)]}{(r^2+x)^{\eta+1}}\left(\int_0^\infty H_{0,m+1}^{1,0}\left(-r^2t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m}\right)t^{\lambda-1}e^{-tx}dt\right)dx$$
$$= \eta \int_{a_1}^\infty \frac{[a^{-1}(x)]}{(r^2+x)^{\eta+1}x^{\lambda}}E_{(1/\rho_i),(\mu_i)}\left(-\frac{r^2}{x}\right)dx;$$

$$I_t = \frac{1}{\Gamma(\eta)} \int_0^\infty \int_0^\infty \int_{a_1}^\infty H_{0,m+1}^{1,0}\left(-r^2t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m}\right)$$
$$\times\ e^{-(r^2+x)s-tx}s^{\eta-1}t^{\lambda}[a^{-1}(x)]dsdxdt$$

$$= \frac{1}{\Gamma(\eta)} \int_0^\infty \int_{a_1}^\infty e^{-(r^2+x)s}s^{\eta-1}[a^{-1}(x)]$$
$$\times\ \left(\int_0^\infty H_{0,m+1}^{1,0}\left(-r^2t \middle| \overline{(0,1),(1-\mu_i,\frac{1}{\rho_i})_{i=1}^m}\right)t^{\lambda}e^{-tx}dt\right)dsdx$$

$$= \frac{1}{\Gamma(\eta)} \int\limits_{a_1}^{\infty} E_{(1/\rho_i),(\mu_i)} \left(-\frac{r^2}{x} \right) \frac{[a^{-1}(x)]}{x^{\lambda+1}} \left(\int\limits_{0}^{\infty} e^{-(r^2+x)s} s^{\eta-1} ds \right) dx$$

$$= \int\limits_{a_1}^{\infty} E_{(1/\rho_i),(\mu_i)} \left(-\frac{r^2}{x} \right) \frac{[a^{-1}(x)]}{x^{\lambda+1}(r^2+x)^{\eta}} dx.$$

The derivation of (3.6) is similar to the previous proof; as the alternating Dirichlet series $\tilde{\mathcal{D}}_{\mathbf{a}}(x)$ equals to (see [14]):

$$\tilde{\mathcal{D}}_{\mathbf{a}}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-a_n x} = x \int\limits_{a_1}^{\infty} e^{-xt} \tilde{A}(t) dt,$$

where the so-called continuing function $\tilde{A}(t)$ has been found easily in the following manner:

$$\tilde{A}(t) = \sum_{n: a_n \leq t} (-1)^{n-1} = \frac{1-(-1)^{[a^{-1}(t)]}}{2} = \sin^2 \left(\frac{\pi}{2} [a^{-1}(t)] \right).$$

Hence $\tilde{\mathcal{D}}_{\mathbf{a}}(s+t) = (s+t) \int\limits_{a_1}^{\infty} e^{-x(s+t)} \sin^2 \left(\frac{\pi}{2} [a^{-1}(x)] \right) dx$, i.e. we get (3.6). ■

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University "St. Cyril and Methodius"
Faculty of Natural Sciences and Mathematics
Institute of Mathematics, P.O. Box 162
Skopje - 1000, Republic of MACEDONIA

e-mail: tomovski@iunona.pmf.ukim.edu.mk

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