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ON A DIFFERENTIAL EQUATION WITH LEFT AND RIGHT FRACTIONAL DERIVATIVES

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Abstract

We treat the fractional order differential equation that contains the left and right Riemann-Liouville fractional derivatives. Such equations arise as the Euler-Lagrange equation in variational principles with fractional derivatives. We reduce the problem to a Fredholm integral equation and construct a solution in the space of continuous functions. Two competing approaches in formulating differential equations of fractional order in Mechanics and Physics are compared in a specific example. It is concluded that only the physical interpretation of the problem can give a clue which approach should be taken.

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Key Words and Phrases: left and right Riemann-Liouville fractional derivatives, fractional differential equation, Euler-Lagrange equation, variational principle

1. Introduction

This paper concerns the problem of "fractionalization" of differential equations of Mechanics. Differential equations of fractional orders appear in many branches of Physics and Mechanics. There are numerous solutions of concrete problems, collected in the books [2], [1] and [3], for example. The monograph [11] contains, among other results, many references to fractional differential equations.

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In all mentioned works, differential equations of the form

$$D^{\alpha}y = f\left(y,t\right), \quad t \in \left(0,b\right), \tag{1}$$

where $\alpha \in \mathbb{R}^{+}$, are treated. Let m be an integer such that $m-1<\alpha<m$. The left Reimann-Liouville fractional derivative of order α, which appears in (1) is defined as

$$D^{\alpha}y(t) = \frac{d^{m}}{dt^{m}}\left[\frac{1}{\Gamma\left(m-\alpha\right)}\int_{0}^{t}\frac{y\left(\tau\right)}{\left(t-\tau\right)^{\alpha+1-m}}d\tau\right], \; m-1<\alpha<m, \tag{2}$$

where Γ is Euler's gamma function. Similarly, the right Riemann-Liouville derivative of order α is defined as

$$D_{\alpha}y(t) = (-1)^{m}\frac{d^{m}}{dt^{m}}\left[\frac{1}{\Gamma\left(m-\alpha\right)}\int_{t}^{b}\frac{y\left(\tau\right)}{\left(\tau-t\right)^{\alpha+1-m}}d\tau\right], \; m-1<\alpha<m. \tag{3}$$

There are two different approaches in formulating differential equations of fractional order in Mechanics and Physics. In the first "direct" approach, the ordinary (integer order) derivative in a differential equation is replaced by a fractional derivative. Such a procedure gives reasonable results in many areas, for example, in the viscoelasticity. In the second approach, one modifies Hamilton's principle (least action principle) by replacing the integer order derivative by a fractional one. Then, minimization of the action integral leads to the differential equation of the system. This second approach is considered to be, from the stand point of Physics, the more sound one (see for example, [7]).

Note that in the second approach the resulting fractional differential equation is not of the form (1) but in the form that we discuss next. Namely, if the modification is made on the level of the variational principle, we are faced with the following type of minimization problem with fractional derivatives (see [4], [5], [6], [7] and [12]): find a minimum of the functional

$$I\left[y\right] = \int_{0}^{b} F\left(t,y,D^{\alpha}y\right)dt, \tag{4}$$

where

$$y\left(0\right) = y_{0}, \quad y\left(b\right) = y_{1}. \tag{5}$$

In (4) and (5), $y(t)$ is a function having continuous left Riemann-Liouville derivative $D^{\alpha}y$ of order α, and $F\left(t,y,D^{\alpha}y\right)$ is a function with continuous first and second partial derivatives with respect to all its arguments. It is shown in [5], [6] and [7] that a necessary condition that $y(t)$ gives an extremum to (5) is that it satisfies the Euler-Lagrange equation

$$D_\alpha \left[\frac{\partial F}{\partial D^\alpha y} \right] + \frac{\partial F}{\partial y} = 0. \tag{6}$$

In the special case treated in [7] (motion of a particle in a fractal medium) the function F has the form

$$F = \frac{m}{2} \left[D^\alpha y \right]^2 - U\left(y, t\right), \tag{7}$$

where m and U are the "usual" mass, assumed to be constant, and the potential energy of the particle. With (7), equation (6) becomes

$$m \left(D_\alpha \circ D^\alpha y \right)(t) = -\frac{\partial U}{\partial y}.$$

In the special case when $U = \frac{\lambda}{2} y^2 + yg + h$, where $\lambda = const$ and g and h are given functions, we obtain

$$\left(D_\alpha \circ D^\alpha y \right)(t) = \lambda y\left(t\right) + g\left(t\right). \tag{8}$$

We shall analyze (8) for the case when $0 < \alpha < 1$. Our main result concerns the existence of a solution $y\left(t, \alpha\right)$ to (8) and its behavior in the limit when $\alpha \to 1^-$.

It is important to note that (4) is not the only type of functional used in fractional order physics. In [8] and [9] convolution type functionals are considered resulting in two Euler-Lagrange equations, called advanced and retarded equations. Another type of functionals are used in [10]. Namely, in [10] in the function $F\left(t, y, D^\alpha y\right)$ the derivative $D^\alpha y$ is replaced with the symmetric fractional differential operator $Dy = \frac{1}{2}[D^\alpha y + D_\alpha y]$. In this way, one is not, ab initio, favoring left or right fractional derivatives.

As far as we are aware, equations of type (8), are solved only in [6] and [7] in some very special cases. In the next section we construct a solution to (8), reducing it to a Fredholm integral equation of the second kind with singular kernel $K\left(t, u\right)$. If $1/2 \leq \alpha < 1$, then $K\left(t, u\right) \in \mathbf{L}^2\left(\left(0, b\right) \times \left(0, b\right)\right)$ and it can be easily proved that (8) has a solution in $\mathbf{L}^2\left(0, b\right)$, cf. [14] (for the theory of integral equations with singular kernels on Hölder spaces one can consult [11] and [16]). But we usually need a continuous solution to (8). That was the reason that we restricted ourselves in this paper to the problem of finding a solution in the space $\mathbf{C}\left(\left[0, b\right]\right)$ of continuous functions on $[0, b]$.

2. A solution to equation (8)

2.1. Preliminaries

First we list some properties of fractional integrals and derivatives. Let I^{α} and I_{α} denote

$$\left(I^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau,$$
$$\left(I_{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{b}\frac{f(\tau)}{(\tau-t)^{1-\alpha}}d\tau.$$

If $0<\alpha<1$ and $f\in\mathbf{L}^{1}(0,b)$, then $I^{\alpha}f$ and I_{α} exist almost everywhere in $(0,b)$. Also $D^{\alpha}I^{\alpha}f=f$ (cf. [11], Theorem 2.4).

It is also easily seen that $D_{\alpha}I_{\alpha}f=f$. Let Q_b be the operator $\left(Q_{b}f\right)(t)=f(b-t)$. For this operator we know that $Q_{b}\circ I^{\alpha}=I_{\alpha}\circ Q_{b}$ and $D_{\alpha}\circ Q_{b}=Q_{b}\circ D^{\alpha}$. Now it is easily seen that $D_{\alpha}\circ I_{\alpha}f=D_{\alpha}\circ Q_{b}\circ Q_{b}\circ I_{\alpha}f=Q_{b}\circ D^{\alpha}\circ I^{\alpha}\circ Q_{b}f=Q_{b}\circ Q_{b}f=f$.

The following lemma can be easily proved:

Lemma 1. *If $g\in\mathbf{C}([0,b])$, then $I^{\alpha}g$, $I_{\alpha}g$ and $I^{\alpha}\circ I_{\alpha}g$ belong to $\mathbf{C}([0,b])$, as well.*

With the change of the variable $\tau=t-u$

$$(I^{\alpha}g)(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{g(\tau)d\tau}{(t-\tau)^{1-\alpha}}=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{g(t-u)}{u^{1-\alpha}}du$$

and

$$\begin{aligned}(I^{\alpha}g)(t+h)-(I^{\alpha}g)(t) &= \frac{1}{\Gamma(\alpha}\int_{0}^{t}\frac{1}{u^{1-\alpha}}(g(t+h-u)-g(t-u))du\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t}^{t+h}\frac{g(t+h-u)}{u^{1-\alpha}}du\,,\ 0\le t\le b.\end{aligned}$$

Since $g\in\mathbf{C}([0,b])$, it is also uniformly continuous on $[0,b]$. For $\varepsilon>0$ there exist $\delta_1>0$ and $\delta_2>0$ such that $|g(t+h-u)-g(t-u)|<\varepsilon, |h|<\delta_1$ and $|(t+h)^{\alpha}-t^{\alpha}|<\varepsilon, |h|<\delta_2$.

Let $\delta=\min(\delta_1,\delta_2)$ and $M=\max|g(t)|,\ 0\le t\le b$. Then

$$|(I^{\alpha}g)(t+h)-(I^{\alpha}g)(t)|\le\frac{1}{\Gamma(\alpha+1)}(b^{\alpha}+M)\varepsilon,\ |h|<\delta\,.$$

This proves that $I^{\alpha}g\in\mathbf{C}([0,b])$.

To prove that $I_\alpha g \in \mathbf{C}([0,b])$ we use the operator Q_b, $(Q_b f)(t) = f(b-t)$. If g is continuous, then $Q_b g$ is also continuous on $[0,b]$. With the property of Q_b (cf. [11], p. 34),

$$I_\alpha g = I_\alpha \circ Q_b \circ Q_b g = Q_b \circ I^\alpha \circ Q_b g\,. \tag{9}$$

This proves that $I_\alpha g \in \mathbf{C}([0,b])$. It is now easily seen that $I^\alpha \circ I_\alpha g \in \mathbf{C}([0,b])$.

2.2. Integral equation which corresponds to equation (8)

LEMMA 2. *If $0 < \alpha < 1$ and $g \in \mathbf{C}([0,b])$, then every solution $f \in \mathbf{C}([0,b])$ to the integral equation*

$$f(t) - \lambda \int_0^b K_\alpha(t,u) f(u)du = G_\alpha(t),\ 0 \le t \le b, \tag{10}$$

is also a solution to equation

$$(D_\alpha \circ D^\alpha f)(t) = \lambda f(t) + g(t),\ 0 \le t \le b, \tag{11}$$

where

$$\begin{aligned} K_\alpha(t,u) &= \frac{1}{\Gamma^2(\alpha)} \int_0^t \frac{q(u,\tau)}{(t-\tau)^{1-\alpha}} d\tau\,; \\ q(u,\tau) &= \begin{cases} (u-\tau)^{\alpha-1}, & b \ge u > \tau \ge 0 \\ 0 & 0 \le u < \tau \le b; \end{cases} \end{aligned} \tag{12}$$

$$G_\alpha(t) = (I^\alpha \circ I_\alpha g)(t) \in \mathbf{C}([0,b])\,. \tag{13}$$

P r o o f. Let us give to the expression

$$\int_0^b K_\alpha(t,u) f(u) du$$

another form:

$$\begin{aligned} \int_0^b K_\alpha(t,u) f(u) du &= \frac{1}{\Gamma^2(\alpha)} \int_0^b f(u) \int_0^t \frac{q(u,\tau)}{(t-\tau)^{1-\alpha}} d\tau du \\ &= \frac{1}{\Gamma^2(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} \int_0^b f(u) q(u,\tau) du \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} \frac{1}{\Gamma(\alpha)} \int_\tau^b \frac{f(u)}{(u-\tau)^{1-\alpha}} du = (I^\alpha \circ I_\alpha f)(t)\,. \tag{14}$$

Consequently, equation (10) can be given in the form:

$$f(t) - \lambda(I^\alpha \circ I_\alpha f)(t) = (I^\alpha \circ I_\alpha g)(t)\,. \tag{15}$$

Suppose that we have a solution $f \in \mathbf{C}([0,b])$ to (10). This is also a solution to (15). But we will show that it is a solution to (11), as well. For that it is enough to prove the assertion that for $h \in \mathbf{C}([0,b])$ we have $D_\alpha \circ D^\alpha(I^\alpha \circ I_\alpha h) = h$.

We have seen that if $h \in \mathbf{L}^1(0,b)$, then $D^\alpha \circ I^\alpha h = h$ and $D_\alpha \circ I_\alpha h = h$. Thus

$$D_\alpha \circ D^\alpha \circ (I^\alpha \circ I_\alpha h) = D_\alpha \circ (D^\alpha \circ I^\alpha) \circ I_\alpha h = h\,.$$

By applying the operator $D_\alpha \circ D^\alpha$ to (15) we obtain (11), which proves Lemma 2. ■

2.3. Construction of a solution to (8)

THEOREM 3. *Let $0 < \alpha < 1$ and let $g \in \mathbf{C}([0,b])$ and $g \not\equiv 0$. If $|\lambda| < \frac{\Gamma^2(\alpha+1)}{b^{2\alpha}}$, then*

$$(D_\alpha \circ D^\alpha f)(t) = \lambda f(t) + g(t),\ 0 \le t \le b, \tag{16}$$

has a solution belonging to $\mathbf{C}([0,b])$. This solution f_α is given by the Neumann series:

$$f_\alpha(t) = G_\alpha(t) + \lambda K_\alpha G_\alpha(t) + \lambda^2 K_\alpha^2 G_\alpha(t) + \dots,\ 0 \le t \le b\,, \tag{17}$$

where

$$K_\alpha G_\alpha(t) = \int\limits_0^b K_\alpha(t,u) G_\alpha(u)du \ \text{ and } \ K_\alpha^n G_\alpha(t) = K_\alpha(K_\alpha^{n-1} G_\alpha)(t),\ n \ge 2;$$

the function $K_\alpha(t,u)$ is given by (12) and $G_\alpha(t)$ by (13). The series (17) converges in $\mathbf{C}([0,b])$.

P r o o f. Let us construct the Neumann series using the sequence:

$$f_1 = G_\alpha,\ \ f_n = K_\alpha f_{n-1},\ \ n \ge 2\,,$$

which gives

$$f_n = \sum_{j=0}^{n-1} \lambda^j K_\alpha^j G_\alpha$$

(cf. (17)). Since $g \in \mathbf{C}([0,b])$, by Lemma 1 the function G_α, given by (13) belongs to $\mathbf{C}([0,b])$, as well. Also by (15) $K_\alpha G_\alpha$ and $K_\alpha^n G_\alpha$, $n \ge 2$, belong to $\mathbf{C}([0,b])$. Consequently, the addends in the Neumann series (17) are continuous functions. Now to prove that the series converges in $\mathbf{C}([0,b])$, we need a majorant for $K_\alpha^n G_\alpha(t),\ n \geq 1$. Let us find it:

$$\begin{aligned}
\|K_\alpha G_\alpha(t)\| = \|(I^\alpha \circ I_\alpha G_\alpha)(t)\| &= \frac{1}{\Gamma^2(\alpha)} \Big\| \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \Big(\int_\tau^b \frac{G_\alpha(u)}{(u-\tau)^{1-\alpha}} du \Big) d\tau \Big\| \\
&\leq \Big\| \frac{1}{\Gamma^2(\alpha)} \int_0^t \frac{1}{(t-\tau)^{1-\alpha}} \int_\tau^b \frac{1}{(u-\tau)^{1-\alpha}} du\, d\tau \Big\| \|G_\alpha\| \\
&\leq \Big\| \frac{1}{\alpha} \frac{1}{\Gamma^2(\alpha)} \int_0^t (b-\tau)^\alpha (t-\tau)^{\alpha-1} d\tau \Big\| \|G_\alpha\| \\
&\leq \frac{1}{\alpha} \frac{1}{\Gamma^2(\alpha)} \Big\| b^\alpha \int_0^t (t-\tau)^{\alpha-1} d\tau \Big\| \|G_\alpha\| \leq \frac{1}{\Gamma^2(\alpha+1)} b^{2\alpha} \|G_\alpha\|,
\end{aligned}$$

where $\|f\| = \max\limits_{0 \leq t \leq b} |f(t)|$ is the norm in $\mathbf{C}([0,b])$. It follows that $\|K_\alpha G_\alpha(t)\| \leq \dfrac{b^{2\alpha}}{\Gamma^2(\alpha+1)} \|G_\alpha\|$ and $\|K_\alpha^n G_\alpha(t)\| \leq \left(\frac{b^{2\alpha}}{\Gamma^2(\alpha+1)}\right)^n \|G_\alpha\|$. Since $|\lambda| \frac{b^{2\alpha}}{\Gamma^2(\alpha+1)} < 1$, the series in (17) converges in $\mathbf{C}([0,b])$. If we apply the operator K_α to this series, then K_α can be applied on every addend of the series:

$$K_\alpha \sum_{j=0}^{\infty} \lambda^j K_\alpha^j G_\alpha = \sum_{j=0}^{\infty} \lambda^{j+1} K_\alpha^{j+1} G_\alpha = f - G_\alpha,$$

or

$$K_\alpha \lim_{n\to\infty} \sum_{j=0}^{n} \lambda^j K_\alpha^j G_\alpha = \lim_{n\to\infty} \sum_{j=1}^{n+1} \lambda^j K^j G = \lim_{n\to\infty} f_n - G_\alpha,$$

in $\mathbf{C}([0,b])$. Consequently the function given by the Neumann series (17) is a solution to (16). ■

Remarks:

1. In case $\lambda = 0,\ g \in \mathbf{L}^1(0,b)$ and $|g(b-t)| \leq K t^{\varepsilon-\alpha},\ \varepsilon > 0, 0 < t < b$, all the solutions to equation

$$(D_\alpha \circ D^\alpha f)(t) = g(t), \ \ 0 < t < b,$$

in $\mathbf{L}^1(0,b)$ are of the form:

$$f_\alpha(t) = (I^\alpha \circ I_\alpha g)(t) + C_1 (I^\alpha (b-\tau)^{\alpha-1})(t) + C_2 t^{\alpha-1},\ 0 < t < b. \qquad (18)$$

This follows from the fact that $D^\alpha f = 0$ if and only if $f(t) = C t^{\alpha-1}$. Also, $D_\alpha f = 0$ if and only if $f(t) = C(b-t)^{\alpha-1}$ (cf. [15]).

2. If in equation (16) $g(t)$ is of the form

$$g(t) = C_1(I^\alpha(b-\tau)^{\alpha-1})(t) + C_2 t^{\alpha-1},$$

then a solution to this equation is

$$f_\alpha(t) = -\frac{1}{\lambda}C_1(I^\alpha(b-\tau)^{\alpha-1})(t) - \frac{1}{\lambda}C_2 t^{\alpha-1}. \tag{19}$$

3. In both cases (cf. (18) and (19)) the solution to (16) contains constants C_1 and C_2 which can be determined so that the solution f_α satisfies additional conditions of the form $f_\alpha(0) = f_0, f_\alpha(b) = f_1$. However, in the general case, to satisfy these conditions we have to call for an appropriate additional condition for g. We find such condition on $g(t)$ by analyzing boundary conditions $f_\alpha(0) = f_\alpha(b) = 0$. Let $g(t)$ be given in the form $g(t) = h(t) + cp(t)$, where $h(t) \in \mathbf{C}([0,b])$ and $p(t) = \left(I^\alpha(b-\tau)^{\alpha-1}\right)(t)$, with $c = const.$ and $1/2 < \alpha < 1$. By the results of Theorem 3, equation (16), with $g = h$, has a solution $f_\alpha(t)$ given by (17). This solution satisfies initial condition $f_\alpha(0) = 0$. If it also satisfies $f_\alpha(b) = 0$, then $g(t)$ can be just $h(t)$. Otherwise, for $g(t) = h(t) - cp(t), c = \frac{\lambda h(b)}{p(b)}$, equation (16) has a solution which satisfies the prescribed boundary conditions (cf. Remarks, 1.).

2.4. Example

We treat an example that will show relation between two approaches of fractional generalizations of equations of physics (see [7]). Thus, we consider the problem of minimizing the following functional

$$I = \int_0^1 \left\{\frac{1}{2}[D^\alpha f]^2 + Af\right\} dt \tag{20}$$

with $1/2 < \alpha < 1$ and $A \in \mathbf{L}^1(0,1), |A(1-t)| \le at^{\varepsilon-\alpha}, \varepsilon > 0, a$ being a given constant. Also we assume that

$$f(0) = f(1) = 0. \tag{21}$$

The Euler–Lagrange equation for the functional (20) reads

$$(D_\alpha \circ D^\alpha f)(t) + A(t) = 0. \tag{22}$$

The boundary conditions corresponding to (22) are (21). Applying the result in Remark 1. we have a family of solutions to (22)

$$f_\alpha(t) = -(I^\alpha \circ I_\alpha A)(t) + C_1\left(I^\alpha(1-t)^{\alpha-1}\right)(t) + C_2 t^{\alpha-1}, \quad 0 < t < 1. \tag{23}$$

To satisfy the boundary condition $(21)_1$ we have to take $C_2 = 0$. The second constant C_1 can be found from the condition $f_\alpha(1) = 0$, i.e.,

$$\lim_{t\to 1^-} \frac{1}{\Gamma^2(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} \int_\tau^1 \frac{A(u)}{(u-\tau)^{1-\alpha}} du$$
$$+ \lim_{t\to 1^-} C_1 \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}(1-\tau)^{1-\alpha}}.$$

Both limits exist because of assumptions on $A(t)$ and α.

3. The limit of solution (17) when $\alpha \to 1^-$

THEOREM 4. *Let* $|\lambda| < \max\limits_{\alpha\in[\varepsilon,1-\varepsilon]} \frac{\Gamma^2(\alpha+1)}{b^{2\alpha}}, \varepsilon > 0$. *Then:*

1) The series (17) which defines a solution to (16) converges uniformly on $[0,b]\times[\varepsilon,1-\varepsilon]$ *to a function* $f(t,\alpha) \in \mathbf{C}([0,b]\times[\varepsilon,1-\varepsilon])$. *Let* $f(t)$ *denote the function* $f(t) = \lim\limits_{\alpha\to 1^-} f(t,\alpha)$. *Then the function* $f \in \mathbf{C}([0,b])$.

2) If in addition $f \in \mathbf{C}^2([0,b]), f^{(3)} \in \mathbf{L}^1(0,b)$ *and* $f^{(2)}(t,\alpha) \to f^{(2)}(t)$, $\alpha \to 1^-$, *uniformly in* $t \in [\eta, 1-\eta]$ *for every* $\eta > 0$, $f^{(3)}(t,\alpha) \to f^{(3)}(t), \alpha \to 1^-, t \in (0,b)$, *then* f *satisfies:*

$$-\frac{d^2 f}{dt^2} = \lambda f(t) + g(t), \quad 0 < t < b.$$

P r o o f. 1) In the proof of Theorem 3 we have only to change the condition $|\lambda| < \frac{\Gamma^2(\alpha+1)}{b^{2\alpha}}$, by $|\lambda| < \max\limits_{\alpha\in[\varepsilon,1-\varepsilon]} \frac{\Gamma^2(\alpha+1)}{b^{2\alpha}}$. Consequently, $f \in \mathbf{C}([0,b])$.

2) If $t \in (0,b)$, then there is an $\eta > 0$ such that $t \in [\eta, b-\eta] \equiv J_\eta$. Consider $D^\alpha h$ for an $h \in \mathbf{C}^2([0,b])$ and $h^{(3)} \in \mathbf{L}^1(0,b)$ using twice the partial integration we have:

$$(D^\alpha h)(t) = \frac{t^{-\alpha} h(0)}{\Gamma(1-\alpha)} + \frac{t^{1-\alpha} h^{(1)}(0)}{\Gamma(2-\alpha)} + \int_0^t \frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} h^{(2)}(\tau)\, d\tau.$$

Whence, for $t \in [\eta, b-\eta]$:

$$\lim_{\alpha\to 1^-} (D^\alpha h)(t) = h^{(1)}(0) + \int_0^t h^{(2)}(\tau)\, d\tau = h^{(1)}(t).$$

In the same way we have

$$\left(D_\alpha h^{(1)}\right)(t) = \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)} h^{(1)}(b) - \frac{(b-t)^{1-\alpha} h^{(2)}(b)}{\Gamma(2-\alpha)} + \int_t^b \frac{(\tau-t)^{1-\alpha}}{\Gamma(2-\alpha)} h^{(3)}(\tau)\, d\tau$$

and for $t \in [\eta, b-\eta]$:

$$\lim_{\alpha \to 1^-} \left(D_\alpha h^{(1)} \right)(t) = -h^{(2)}(b) + \int_t^b h^{(3)}(\tau)\, d\tau = -h^{(2)}(t).$$

The last two limits are valid, uniformly in $t, t \in [\eta, b-\eta]$. By construction

$$\left(D_\alpha \circ D^\alpha f(t,\alpha) \right)(t) = \lambda f(t,\alpha) + g(t), \quad 0 < t < b.$$

For a $t \in (0,b)$ we can take

$$\lim_{\alpha \to 1^-} \left(D_\alpha \circ D^\alpha f(t,\alpha) \right)(t) = \lim_{\alpha \to 1^-} f(t,\alpha) + g(t).$$

With the above results, the last limit gives

$$-\frac{d^2}{dt^2} f(t) = \lambda f(t) + g(t). \tag{24}$$

This proves Theorem 4. ■

4. Conclusion

We analyzed the differential equation (8)

$$\left(D_\alpha \circ D^\alpha y \right)(t) = \lambda y(t) + g(t) \tag{25}$$

which follows from the minimization of (4) with F given by (7) and $U = \frac{\lambda}{2} y^2 + yg + h$. This equation may be considered as fractional generalization of (24) for $0 < \alpha \leq 1$. The direct way to write fractional generalization of (24) is to consider

$$-D^\beta y(t) = \lambda y(t) + g(t), \tag{26}$$

with $1 \leq \beta \leq 2$. For (26) the solution is known and it reads (see [3], p. 140)

$$\begin{aligned} y_\beta(t) = & \sum_{k=1}^{2} C_k t^{\beta-k} E_{\beta,\beta-k+1}\left(-\lambda t^\beta\right) \\ & + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}\left(-\lambda (t-\tau)^\beta\right) g(\tau)\, d\tau, \end{aligned} \tag{27}$$

where $\beta > 1, C_k, k = 1,2$ are constants and $E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0$, is a two-parameter Mittag-Leffler function. It is interesting to compare (27) for $1 < \beta < 2$ and (17), i.e.,

$$y_\alpha(t) = \sum_{j=0}^{\infty} \lambda^j K_\alpha^j G_\alpha(t), \tag{28}$$

for $\alpha = \beta/2$. We do this in the special case of (25) when $\lambda = 0, g(t) = C = const., b = 1$ and $y(0) = 0, y(1) = 0$. From (23) (with $A(t) = -C$) we obtain

$$y_\alpha(t) = \frac{C}{\Gamma(\alpha)\Gamma(1+\alpha)} \int_0^t (1-\tau)^\alpha (t-\tau)^{\alpha-1} d\tau \\ + C_1 \int_0^t (1-\tau)^{\alpha-1} (t-\tau)^{\alpha-1} d\tau, \tag{29}$$

where

$$C_1 = -\frac{C\int_0^1 (1-\tau)^\alpha (1-\tau)^{\alpha-1} d\tau}{\Gamma(\alpha)\Gamma(1+\alpha)\int_0^1 (1-\tau)^{\alpha-1} (1-\tau)^{\alpha-1} d\tau} = -\frac{C(2\alpha-1)}{2\Gamma^2(\alpha+1)}.$$

The direct approach leads to the solution of $D^\beta y_\beta = C, y_\beta(0) = 0, y_\beta(1) = 0$ that reads

$$y_\beta(t) = -\frac{Ct^{\beta-1}}{\Gamma(1+\beta)}(1-t), \quad 1 < \beta < 2. \tag{30}$$

We believe that only physics of the problem can give a clue which approach should be taken.

References

[1] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations.* John Wiley& Sons, New York (1993).

[2] K.B. Oldham and J. Spanier, *The Fractional Calculus.* Academic Press, New York (1974).

[3] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[4] F. Riewe, Mechanics with fractional derivatives. *Phys. Rev.* **E 55** (1997), 3582-3592.

[5] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev.* **E 53** (1996), 1890-1899.

[6] O.P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **272** (2002), 368-379.

[7] S.Sh. Rekhviashvili, The Lagrange formalism with fractional derivatives in problems of mechanics. *Technical Physics Letters* **30** (2004), 33-37.

[8] D.W. Dreisigmeyer and P.M. Young, Nonconservative Lagrangian mechanics: A generalized function approach. *J. Phys. A: Math. Gen.* **36** (2003), 8297-8310.

[9] D.W. Dreisigmeyer and P.M. Young, Extending Bauer's corollary to fractional derivatives. *J. Phys. A: Math. Gen.* **37** (2004), L117-L121.

[10] M. Klimek, Fractional sequential mechanics - models with symmetric fractional derivative. *Czechoslovak Journal of Physics* **51**, No 1 (2001), 348-354.

[11] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives.* Gordon and Breach, Amsterdam (1993).

[12] S.I. Muslih and D. Baleanu, Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives. *J. Math. Anal. Appl.* **304** (2005), 599-606.

[13] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi (Ed-s), *Higher Transcedental Functions.* Mc Graw-Hill Book Company, New York (1953).

[14] J. Favard, *Cours d'analyse de l' École Polytechnique, T. III, Fasc. II.* Gauthier - Villars, Paris (1963).

[15] R. Gorenflo, F. Mainardi, *Fractional Calculus, Integral and Differential Equations of Fractional Order*, Chapter published in: A. Carpinteri and F. Mainardi (Ed-s), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien - New York (1997).

[16] N.I. Mushelishvili, *Singular Integral Equations.* Akademie-Verlag, Berlin (1965).

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