

**FOPID CONTROLLER DESIGN FOR
ROBUST PERFORMANCE USING
PARTICLE SWARM OPTIMIZATION**

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Abstract

This paper proposes a novel method to design an H_∞ -optimal fractional order PID (FOPID) controller with ability to control the transient, steady-state response and stability margins characteristics. The method uses particle swarm optimization algorithm and operates based on minimizing a general cost function. Minimization of the cost function is carried out subject to the H_∞ -norm; this norm is also included in the cost function to achieve its lower value. The method is applied to a phase-locked-loop motor speed system and an electromagnetic suspension system as two examples to illustrate the design procedure and verify performance of the proposed controller. The results show that the proposed method is capable of improving system responses as compared to the conventional H_∞ -optimal controller while still maintains the H_∞ -optimality of the solutions.

Mathematics Subject Classification: 26A33; 93C15, 93C55, 93B36, 93B35, 93B51; 03B42; 70Q05; 49N05

Key Words and Phrases: FOPID controller, robust performance, particle swarm optimization

1. Introduction

As summarized in [1], many real-world physical systems are well characterized by fractional-order differential equations, i.e., equations involving noninteger-order derivatives. In particular, it has been shown that materials having memory and hereditary effects [2] and dynamical processes, such

as mass diffusion and heat conduction [3], in fractal porous media [4] can be more adequately modeled by fractional-order models than integer-order models. With the success in the synthesis of real noninteger differentiator and the emergence of new electrical circuit element called “fractance” [5], fractional-order controllers [6, 7, 8, 9], such as fractional-order *PID* (Proportional-Integral-Derivative) controllers [10, 11, 12, 13], have been designed and applied to control a variety of dynamical processes. The main advantage of using fractional-order controllers for a linear control system is that the time and frequency responses can be shaped using functions rather than of exponential type and, as a consequence, the performance of the feedback control loop can be improved over the use of integer-order controllers.

In recent years, mixed H_2/H_∞ -optimal control problems have received a great deal of attention [14, 15, 16] to improve H_∞ controller responses with respect to rational-type controllers such as PID. This paper is to improve those mixed cost functions by incorporating several different performance characteristics not necessarily representable as H_2 or H_∞ norms. This will partly degrade the H_∞ -norm of the controlled system. To overcome this drawback, the paper extends the controller structure from rational PID to the *fractional order PID (FOPID)*. As compared with PID, the FOPID offers more flexibility to obtain a lower H_∞ -norm which yields a more robust control.

Heuristic search algorithms such as genetic algorithm (GA) and simulated annealing (SA) have already been applied to the problem of mixed H_2/H_∞ -optimal control design [14, 15, 16]. The results confirm potential of these algorithms for H_2/H_∞ -optimal control design. GA is a population-based search algorithm which works with a population of strings that represent different potential solutions. Therefore, GA has implicit parallelism that enhances its search capability and the optima can be found more quickly when applied to complex optimization problems. SA is an efficient point-based optimization technique, which aims at escaping from local optima to find a globally optimal solution, and has been widely applied in various engineering problems. However, some deficiencies have been identified in GA and SA performance [17, 22].

Particle swarm optimization (PSO) is another evolutionary computation technique [19, 20]. This technique combines social psychology principles in socio-cognition human agents and evolutionary computations. PSO has been motivated by behaviors of organisms such as fish schooling and

bird flocking. Generally, PSO is characterized as a simple concept, easy to implement, and computationally efficient algorithm. PSO has a flexible and well-balanced mechanism to enhance global and local exploration abilities and it has more efficiency than GA and SA [21, 22]. We use PSO as the computational engine for the method presented in this paper and illustrate its advantages in designing the proposed H_∞ -optimal FOPID controller.

2. Fractional calculus

We extracted the following material of fractional calculus from [18].

2.1. Definitions

Fractional calculus is a generalization of the common sense calculus. The chief idea is to develop a functioning operator D , associated to an order v not limited to integer numbers, that generalizes the classical concepts of derivative (for a positive v) and integral (for a negative v).

Exactly like there are several optional definitions for the common sense integer-order integrals (according to Riemann, Lebesgue, Stieltjes, etc.), there are also different definitions for fractional derivatives. The most usual definition is due to Riemann and Liouville (see [23]) that generalizes the following definition corresponding to integer orders:

$${}_0D_x^{-n}f(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt, \quad n \in \mathbb{N}. \tag{1}$$

The generalized definition of D^v is

$${}_cD_x^v f(x) = \begin{cases} \int_c^x \frac{(x-t)^{-v-1}}{\Gamma(-v)} f(t)dt, & \text{if } v < 0; \\ f(x), & \text{if } v = 0; \\ D^n [{}_cD_x^{v-n} f(x)], & \text{if } v > 0; \\ n = \min\{K \in \mathbb{N} : k > v\}, & \end{cases} \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

The Laplace transform of D^v satisfies the following analogues of the classical rules:

$$L[{}_0D_x^v] = \begin{cases} s^v F(s), & \text{if } v \leq 0; \\ s^v F(s) - \sum_{k=0}^{n-1} s^k {}_0D_x^{v-k-1} f(0), & \text{if } n-1 < v < n \in \mathbb{N}. \end{cases} \tag{3}$$

This means that, if zero initial conditions are assumed, the systems with dynamic behavior described by differential equations including fractional derivatives give rise to transfer functions with fractional orders of s . More details are provided in [24, 25, 12].

2.2. Integer order approximation

The most common way of using, in both simulations and hardware implementations, of transfer functions including fractional orders of s is to approximate them with usual (integer order) transfer functions. To perfectly approximate a fractional transfer function, an integer transfer function would have to involve an infinite number of poles and zeroes. Nonetheless, it is possible to obtain logical approximations with a finite number of zeroes and poles.

One of the well-known approximations is due to Oustaloup, who uses the recursive distribution of poles and zeroes. The approaching transfer function is given by [1]:

$$s^v \approx k \prod_{n=1}^N \frac{1 + (s/\omega_{z,n})}{1 + (s/\omega_{p,n})}. \quad (4)$$

The approximation is legitimate in the frequency range $[\omega_l, \omega_h]$. Gain k is also regulated so that both sides of (4) shall have unit gain at 1 rad/s. The number of poles and zeros (N) is chosen in advance, and the desired performance of the approximation strongly depends on: low values cause simpler approximations, but may cause ripples in both gain and phase behaviors. Such ripples can be functionally removed by increasing N , but the approximation will become computationally heavier. Frequencies of poles and zeroes in (4) are given by:

$$\omega_{z,1} = \omega_l \sqrt{\eta}, \quad (5)$$

$$\omega_{p,n} = \omega_{z,n} \varepsilon, \quad ; n = 1, \dots, N, \quad (6)$$

$$\omega_{z,n+1} = \omega_{p,n} \eta, \quad ; n = 1, \dots, N - 1, \quad (7)$$

$$\varepsilon = (\omega_h/\omega_l)^{v/N}, \quad (8)$$

$$\eta = (\omega_h/\omega_l)^{(1-v)/N}. \quad (9)$$

The case $v < 0$ can be handled by inverting (4). For $|v| > 1$, the approximation becomes dissatisfactory. So it is common to separate the fractional orders of s as follows:

$$s^v = s^n s^\delta, \quad v = n + \delta, \quad n \in \mathbb{Z}, \quad \delta \in [0, 1] \quad (10)$$

and only the second term, i.e. s^δ , needs to be approximated.

The electric circuit shown in Fig. 1 can give an easy hardware implementation of the approximate fractional function of (4). Consider the circuit depicted in Fig. 1, such that:

$$I = \sum_{i=1}^n I_i, \quad R_{i+1} = \frac{R_i}{\varepsilon}, \quad C_{i+1} = \frac{C_i}{\eta}, \quad (11)$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the i -th branch of the circuit, respectively.

The admittance $Y(j\omega)$ is given by

$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \sum_{i=0}^{n-1} \frac{j\omega C \varepsilon^i}{j\omega CR + (\varepsilon\eta)^i}. \tag{12}$$

Figure 2 shows the asymptotic Bode diagram of magnitude and phase angle of $Y(j\omega)$. According to (5)-(9), the pole and zero frequencies ($\omega_i = \omega_{p,i}$ and $\omega'_i = \omega_{z,i}$) satisfy the following recursive relationships:

$$\frac{\omega_{i+1}}{\omega_i} = \frac{\omega'_{i+1}}{\omega'_i} = \varepsilon\eta, \quad \frac{\omega_i}{\omega'_i} = \varepsilon, \quad \frac{\omega'_{i+1}}{\omega_i} = \eta. \tag{13}$$

From the Bode diagram of amplitude or phase, the average slope \dot{m} can be calculated as

$$\dot{m} = \frac{\log \varepsilon}{\log \varepsilon + \log \eta}. \tag{14}$$

Thus, the circuit of Fig. 1 represents an approach to approximate the implementation of s^v , $0 < v < 1$, with $\dot{m} = v$, based on a recursive pole/zero placement in the frequency domain.

If a discrete transfer function approximation is sought, the above approximation in (4) may be discretized, see [26]. There are also methods that directly provide discrete approximations, as in [27]. Besides, electric circuits which can serve as exact fractional integrators and differentiators have also been reported in [23, 28].

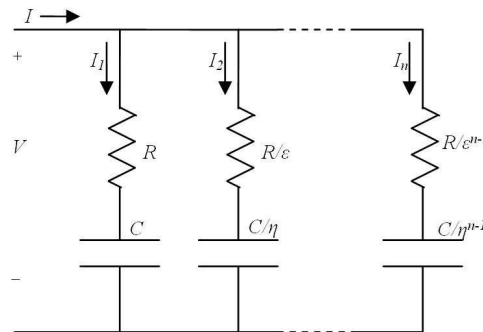
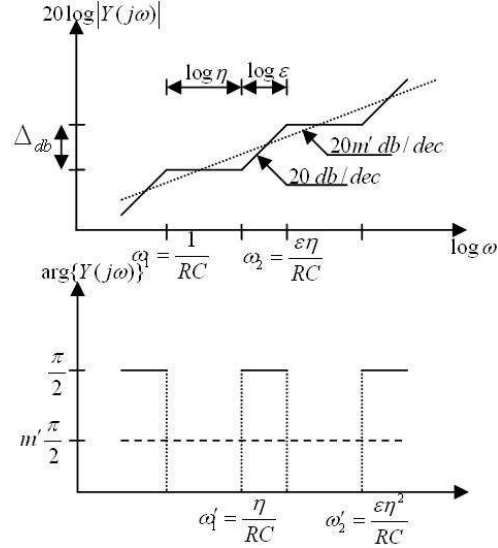


Figure 1: Electrical circuit with a recursive association of resistance and capacitance elements.

Figure 2: Bode diagrams of amplitude and phase of $Y(j\omega)$

2.3. FOPID controller

The differential equation of a fractional order $PI^\lambda D^\mu$ controller is described by (see Podlubny [10, 12]):

$$u(t) = k_P e(t) + k_I D_t^{-\lambda} e(t) + k_D D_t^\mu e(t). \quad (15)$$

The continuous transfer function of FOPID is obtained through Laplace transform and is given by:

$$G_c(s) = k_P + k_I s^{-\lambda} + k_D s^\mu. \quad (16)$$

Design of an FOPID controller involves design of three parameters k_P, k_I, k_D , and two orders λ, μ which are not necessarily integer. As shown in Fig. 3, the fractional order controller generalizes the conventional integer order PID controller from point to plane. This expansion can provide more flexibility in achieving control objectives.

3. Particle swarm optimization (PSO)

PSO is a population-based evolutionary algorithm that was developed from research on swarm such as fish schooling and bird flocking [19]. It has become one of the most powerful methods for solving optimization problems. The method is proved to be robust in solving problems featuring nonlinearity

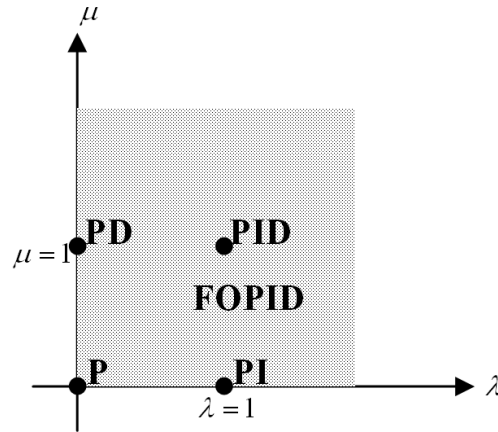


Figure 3: PID controller with fractional order PID

and nondifferentiability, multiple optima, and high dimensionality. The advantages of the PSO are its relative simplicity and stable convergence characteristic with good computational efficiency.

The PSO consists of a swarm of particles moving in a D dimensional search space where a certain quality measure and fitness are being optimized. Each particle has a position represented by a position vector $X_i = (x_{i1}, x_{i2}, \dots, x_{iD})$ and a velocity represented by a velocity vector $V_i = (v_{i1}, v_{i2}, \dots, v_{iD})$, which is clamped to a maximum velocity $V_{max} = (v_{max1}, v_{max2}, \dots, v_{maxD})$. Each particle remembers its own best position so far in a vector $P_i = (p_{i1}, p_{i2}, \dots, p_{iD})$, where i is the index of that particle. The best position vector among all the neighbors of a particle is then stored in the particle as a vector $P_g = (p_{g1}, p_{g2}, \dots, p_{gD})$. The modified velocity and position of each particle can be manipulated according to the following equations:

$$v_{id}^{(t+1)} = wv_{id}^{(t)} + c_1r_1(p_{id} - x_{id}^{(t)}) + c_2r_2(p_{gd} - x_{id}^{(t)}), \quad (17)$$

$$x_{id}^{(t+1)} = x_{id}^{(t)} + v_{id}^{(t+1)}, \quad d = 1, \dots, D, \quad (18)$$

where w can be expressed by the inertia weights approach [30] and often decreases linearly from w_{max} (of about 0.9) to w_{min} (of about 0.4) during a run. In general, the inertia weight w is set according to the following equation

$$w = w_{max} - \frac{w_{max} - w_{min}}{iter_{max}} \cdot iter, \quad (19)$$

where $iter_{max}$ represents the maximum number of iterations, and $iter$ is the number of current iteration or generation. Also c_1 and c_2 are the acceleration constants which influence the convergence speed of each particle and are often set to 2.0 according to the past experiences [31]. Moreover r_1 and r_2 are random numbers in the range of $[0,1]$, respectively. If V_{max} is too small, then the particles may not explore sufficiently beyond local solutions. In many experiences with PSO, V_{max} is often set to the maximum dynamic range of the variables on each dimension, $v_{dmax} = x_{dmax}$.

4. Design of H_∞ -optimal FOPID controller

4.1. Problem description

We consider application of PSO to effectively provide an accurate solution to the design problems of H_∞ -optimal FOPID controllers for systems with uncertainty and disturbance. The problem description is given as follows. Consider a control system as shown in Fig. 4, where $G(s)$ is the nominal and SISO plant, $\Delta G(s)$ is the plant perturbation, $K(s)$ is the FOPID controller, $r(t)$ is the reference input, $u(t)$ is the control input, $e(t)$ is the tracking error, $d(t)$ is the external disturbance and $y(t)$ is the output of the system [14]. Without loss of generality, the plant perturbation is assumed to be bounded by a known stable function

$$|\Delta G(j\omega)| \leq |\Delta W_1(j\omega)|, \quad \forall \omega \in [0, \infty). \quad (20)$$

A controller $K(s)$ is desired to achieve the following conditions: 1) the nominal closed loop system ($\Delta G(s) = 0$ and $d(t) = 0$) is asymptotically stable; and 2) the robust stability condition satisfies the following inequality:

$$\|W_1(s)T(s)\|_\infty < 1; \quad (21)$$

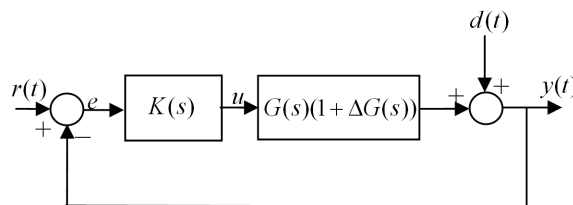


Figure 4: Control system with plant perturbation and external disturbance.

and 3) the disturbance attenuation performance satisfies the following inequality:

$$\|W_2(s)S(s)\|_\infty < 1, \quad (22)$$

where $\|\cdot\|_\infty$ denotes the H_∞ -norm, which is defined as

$$\|M(s)\|_\infty = \max_\omega |M(j\omega)|. \quad (23)$$

$W_2(s)$ is a stable weighting function specified by designer. $S(s)$ and $T(s) = 1 - S(s)$ are the sensitivity and complementary sensitivity functions of the system with the following representations:

$$S(s) = (1 + G(s)K(s))^{-1}, \quad (24)$$

$$T(s) = K(s)G(s)(1 + K(s)G(s))^{-1}. \quad (25)$$

A balanced performance criterion to minimize both (21) and (22) simultaneously is obtained as the following robust performance condition, [16]:

$$J_\infty = \max_\omega \sqrt{|W_1(s)T(s)|^2 + |W_2(s)S(s)|^2} < 1. \quad (26)$$

4.2. Enhancing the cost function

Robust stability and disturbance attenuation are often not enough in the control system design and desired transient, steady-state response characteristics and/or adequate stability margins must also be taken into account. In the proposed method, the handling of constraint (26) is to recast the constraint as an objective to be minimized and, consequently, a weighted-sum approach is conveniently used. We propose the following cost function to achieve the desired specifications:

$$J(k) = w_1 J_\infty + w_2 M_P + w_3 t_r + w_4 t_s + w_5 E_{SS} + w_6 \int_0^T e^2(t) dt + \frac{w_7}{PM} + \frac{w_8}{GM}. \quad (27)$$

Beside the constraint (26), the performance criterion (27) includes overshoot M_P , rise time t_r , settling time t_s , steady-state error E_{SS} , integral of squared-error (ISE), gain margin (GM) and phase margin (PM). The ISE is evaluated up to T which is chosen sufficiently large so that $e(t)$ is negligible for $t > T$. In (27), k is $[k_P, k_I, k_D, \lambda, \mu]$, as controller parameters. If we set all weights to zero except w_1 and w_6 , the cost is transformed to a mixed H_2/H_∞ -optimal controller which is studied in [14] and [16]. Consequently, our proposed cost generalizes the ones in [14] and [16].

The proposed performance criterion (27) comprises eight terms. The significance of each is determined by a weight factor w_i . It is up to the user to set the weight factors properly in order to attain the desired specification. An increase in w_i will result in some improvement in the corresponding feature at the expense of degrading other criteria. Because of particular significance in satisfying (26), we choose w_1 high. For the current study, selections are $w_1 = 10$, $w_2 = 1$, $w_3 = 1$, $w_4 = 1$, $w_5 = 1$, $w_6 = 1$, $w_7 = 200$ and $w_8 = 10$.

An approach using penalty function (see [32]) is employed to ensure the stability. Let the performance index (27) be $J(k)$. Then the value of the fitness of each particle of PSO $k_i (i = 1, \dots, n)$ is determined by the evaluation function, denoted by $F(k_i)$ as

$$F(k_i) = P(k_i) + J(k_i) \quad ; i = 1, \dots, n, \quad (28)$$

where n denotes the population size of PSO. The penalty for the individual k_i is calculated by means of the penalty function $P(k_i)$ given by

$$P(k_i) = \begin{cases} P_1 & ; \text{if } k_i \text{ is unstable,} \\ 0 & ; \text{else.} \end{cases} \quad (29)$$

If the individual k_i does not satisfy the stability then k_i is an unstable individual and it is penalized with a very large positive constant P_1 . Automatically, k_i does not survive the evolutionary process. Otherwise, the individual k_i is feasible and is not penalized.

4.3. Design of H_∞ -optimal FOPID controller using PSO

The H_∞ -optimal FOPID controller design problem is to find the optimal $k = [k_P, k_I, k_D, \lambda, \mu]$ from search space to minimize the objective function $F(k)$ in (28) subject to the inequality constraint (26). Chen and Cheng [16] used prior domain knowledge, i.e., the Routh-Hurwitz criterion, for decreasing the domain size of each design parameter k_i . In this study, we do not use any domain knowledge to confine the search space in order to demonstrate the strong search ability of PSO in efficiently obtaining a near-optimal solution to the investigated problem.

The proposed PSO-based method for finding a near-optimal solution to the H_∞ -optimal FOPID controller design problem is described as follows:

1. Randomly initialize the individuals of the population including searching points and velocities in the search space.
2. For each initial individual k_i of the population, calculate the values of the evaluation function in (28).

3. Compare each individual's evaluation value with its personal best p_{id} . The best evaluation value among the p_{ids} is denoted as p_{gd} .
4. Modify the member velocity of each individual k_i according to (17), where the value of w is set by (19).
5. Modify the member position of each individual k_i according to (18).
6. If the number of iterations reaches the maximum, then go to Step 7, otherwise, go to Step 2.
7. The latest P_g is the optimal controller parameter.

The flowchart of the algorithm is also shown in Fig. 5.

5. Illustrative examples

The following parameters are used for carrying out the FOPID design using PSO:

- The members of each individual are k_P , k_I , k_D , λ and μ .
- Population size = 30.
- Inertia weight factor w is set as (19), where $w_{\max} = 0.9$ and $w_{\min} = 0.4$.
- The limit of change in velocity is set to maximum dynamic range of the variables on each dimension.
- Acceleration constants $c_1 = 2$ and $c_2 = 2$.
- Maximum iteration is set to 1000.
- ω_l and ω_h in (5)-(9) are set to 10^{-5} and 10^5 rad/s respectively.
- The order of approximation in (4) is set to $N = 7$.
- T in (27) is set to 100s.
- P_1 in (29) is set to 10^6 .

5.1. Example 1

Consider a phase-locked-loop motor speed control system [16] as in Fig. 4, where

$$G(s) = \frac{68.76}{s(0.05s + 1)}.$$

Suppose the system suffers from the external disturbance $d(t) = 0.1e^{-0.1t} \sin(0.8\pi t)$ and the plant perturbation

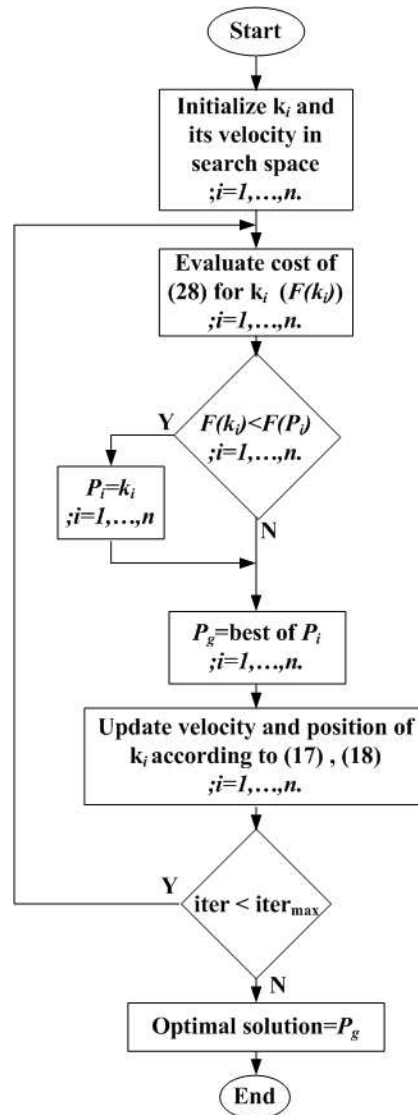


Figure 5: Flowchart of the proposed method

$$\Delta G(s) = \frac{0.5}{s^2 + 0.2s + 8}.$$

Note that the plant perturbation is unknown in fact but bounded by the following known stable function:

$$W_1(s) = \frac{0.6}{s^2 + 0.2s + 8}.$$

To treat the disturbance attenuation problem, the weighting function $W_2(s)$ is chosen as

$$W_2(s) = \frac{0.5s + 0.05}{s^2 + 0.2s + 6.3265}.$$

The plant perturbation and weights are taken from [16]. An FOPID controller is to be designed to minimize (28). The lower bounds of the five controller parameters are zero and their upper bounds are set to $k_{P_{\max}} = k_{I_{\max}} = k_{D_{\max}} = 100$ and $\lambda_{\max} = \mu_{\max} = 2$. We performed 10 trials for the proposed method. The best solution is summarized in Table 1 in comparison with conventional H_∞ controller, two mixed H_2/H_∞ controllers, and an optimal PID controller which is designed based on our proposed method and the same cost function (28). The step response of the system under the external disturbance is shown in Fig. 6 for different controllers. The conventional H_∞ controller which has an order of six has apparently the best H_∞ -norm among all controllers. However, its transient response characteristics are not desirable. The mixed H_2/H_∞ controllers include *ISE* and *ITSE* criterion in addition to the H_∞ -norm. They have been able to improve the transient response characteristics at the expense of some degradation in H_∞ -norm. We applied our proposed method to design an optimal PID controller. This controller further improves the transient response at the cost of further degradation in H_∞ -norm. The proposed FOPID upgrades the PID controller by reducing the H_∞ -norm from about 0.89 to 0.787. The proposed technique offers flexibility of control and compromise over different performance characteristics.

5.2. Example 2

Electromagnetic suspension systems can suspend objects without any contact. The increasing use of this technology in its various forms makes the research extremely active. The electromagnetic suspension technology has already applied to magnetically levitated vehicles, magnetic bearings, and so on. Recent advances on this field are shown in [33]. The structure of the electromagnetic suspension system is shown schematically in Fig. 7.

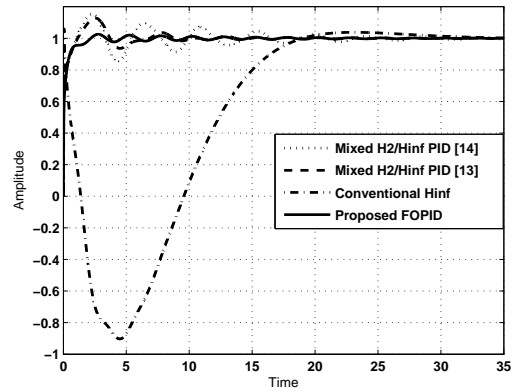


Figure 6: Step response of the system in the presence of external disturbance

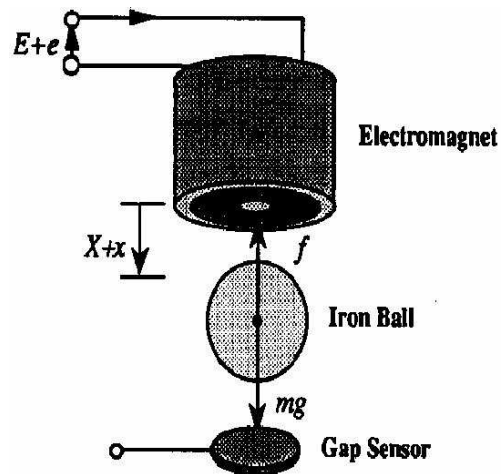


Figure 7: Schematic diagram of the electromagnetic suspension system

The plant considered is:

$$G(s) = \frac{-36.27}{(s + 66.94)(s - 66.94)(s + 45.69)}.$$

The plant perturbation is:

$$W_1(s) = \frac{1.4 \times 10^{-5}(1 + s/8)(1 + s/170)(1 + s/420)}{(1 + s/30)(1 + s/35)(1 + s/38)}.$$

The weighting function $W_2(s)$ is chosen as:

$$W_2(s) = \frac{200}{1 + 10s}.$$

The weights are taken from [33]. A PID and an FOPID controller are designed based on the cost (28). The lower bounds of the five controller parameters are zero and their upper bounds of them are set to, $k_{P\max} = k_{I\max} = k_{D\max} = 10^{10}$ and $\lambda_{\max} = \mu_{\max} = 2$. We performed 10 trials for the proposed method. The best solutions are summarized in Table 2 in comparison with conventional H_∞ controller which has an order of seven. Step response of the system with the proposed FOPID controller and conventional H_∞ controller is shown in Fig. 8 (note that the PID response has not been shown in this figure as it would mask the other two responses by its highly oscillatory nature and large overshoots).

The FOPID controller apparently provides a more desired transient response as compared with the conventional H_∞ controller and the PID controller. As compared to the H_∞ controller, the FOPID not only improves the transient response characteristics, it also improves the H_∞ -norm from 0.063 to 0.061. This is justifiable as we are implementing our FOPID controller using a rational transfer function of order 15 while the controller has an order of 7. The PID controller is marginally stable and, indeed, it cannot achieve our control objectives while the FOPID does it.

6. Conclusion

In this paper, an H_∞ -optimal FOPID controller is designed using the particle swarm optimization algorithm. To achieve desirable response characteristics as well as robustness features, we combined the H_∞ -norm with several response indices to generate an extended cost function. This cost function facilitates control over different features at the cost of some degradation in the H_∞ -norm. The fractional order PID offers further improvement in the H_∞ -norm and provides a more robust system as compared

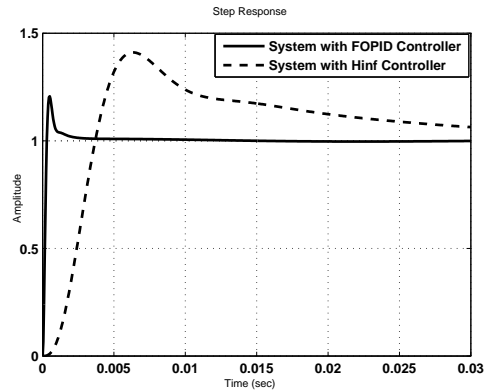


Figure 8: Step response of the system with both FOPID and H_∞ controller

to the rational PID. Simulated examples on a motor speed control and a suspension system confirm the desired features of our proposed FOPID controller.

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Table I
Comparison of the evaluation values between the FOPID, PID and conventional H_∞ controller.

Controller	k_P	k_I	k_D	λ	μ	$M_P(\%)$	t_r	t_s	E_{SS}	PM	GM	J_∞
FOPID	0	0.082	0.034	0	1.4	0	0.898	1.5	0	141.34	∞	0.787
PID	0.07711	0	0.01795	1	1	0	0.58	1.22	0	126.37	∞	0.8907
H_2 / H_∞ PID [15]	0.03763	0.08856	0.05537	1	1	13.6	0.6	8.55	0	108.07	∞	0.8129
H_2 / H_∞ PID [16]	0.01656	0.14189	0.05537	1	1	17	0.68	9.79	0	45.87	∞	0.8052
H_∞	-----	-----	-----	----	---	4	6.4	23.3	0	59.6	4.05	0.6683

Table II
Comparison of the evaluation values between the FOPID, PID and conventional H_∞ controller.

Controller	k_P	k_I	k_D	λ	μ	$M_P(\%)$	t_r	t_s	E_{SS}	PM	GM	J_∞
FOPID	-1.85×10^7	-7.965	-3130.6	0.4	1.653	20.7	0.0002	0.002	0	52.28	42.45	0.061
PID	-44639	-2.8×10^9	-1.7×10^5	1	1	98	0.0004	0.177	0	1.05	0.59	0.69
H_∞	-----	-----	-----	----	-----	41	0.0022	0.047	0	43.65	3.36	0.063

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