

INVERSION FORMULAS FOR THE  
 $q$ -RIEMANN-LIOUVILLE AND  $q$ -WEYL TRANSFORMS  
USING WAVELETS

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**Abstract**

This paper aims to study the  $q$ -wavelets and the continuous  $q$ -wavelet transforms, associated with the  $q$ -Bessel operator for a fixed  $q \in ]0, 1[$ . Using the  $q$ -Riemann-Liouville and the  $q$ -Weyl transforms, we give some relations between the continuous  $q$ -wavelet transform, studied in [3], and the continuous  $q$ -wavelet transform associated with the  $q$ -Bessel operator, and we deduce formulas which give the inverse operators of the  $q$ -Riemann-Liouville and the  $q$ -Weyl transforms.

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**1. Introduction**

In [7], A. Fitouhi and K. Trimèche generalized the theory of continuous wavelet transforms as presented by T. H. Koornwinder [11] and studied the generalized wavelets and the generalized continuous wavelet transforms associated with a class of singular differential operators. This class contains, in particular, the so called Bessel operator, which was studied extensively by K. Trimèche in [13].

In [1], F. Bouzeffour studied fractional transforms associated with the  $q$ -Bessel operator and as an application, he gave inversion formulas for the  $q$ -Riemann-Liouville and  $q$ -Weyl transforms, introduced in [6].

In this paper, we try to generalize our results from [3] by studying wavelets and continuous wavelet transforms associated with the  $q$ -Bessel operator, studied in [6]. The basic tool in this work are some elements of  $q$ -harmonic analysis related to the just mentioned operator. Next, using the  $q$ -Riemann-Liouville and the  $q$ -Weyl operators, we will give some relations between the continuous  $q$ -wavelet transform, studied in [3], and the continuous  $q$ -wavelet transform associated with the  $q$ -Bessel operator, and we deduce other formulas which give the inverse operators of the  $q$ -Riemann-Liouville and the  $q$ -Weyl transforms. These formulas are better than those given in [6] and [1] because they are simple and we have a large choice of  $q$ -wavelets associated with the  $q$ -Bessel operator, that can be used in these formulas.

We are not in a position to claim that all our results here are new, but the methods used are direct and constructive, and have a good resemblance with the classical ones. Our approach in this paper is very similar to the classical picture developed in [7] and [13].

The paper is organized as follows: in Section 2, we present some  $q$ -harmonic results associated with the  $q$ -Bessel operator. In Section 3, we define the  $q$ -wavelets and the continuous  $q$ -wavelet transforms associated with the  $q$ -Bessel operator. In Section 4, we give a characterization of the image set of the  $q$ -wavelet transform associated with the  $q$ -Bessel operator. Section 5 is devoted to give some inversion formulas of the  $q$ -Riemann-Liouville and the  $q$ -Weyl transforms. Finally, in Section 6, we give the inversion formulas for the  $q$ -Riemann-Liouville and the  $q$ -Weyl transforms using wavelets.

## 2. Preliminaries on $q$ -harmonic analysis related to the $q$ -Bessel operator

Throughout this paper, we fix  $q \in ]0, 1[$  such that  $\frac{\text{Log}(1-q)}{\text{Log}q} \in \mathbb{Z}$  and  $\alpha > -\frac{1}{2}$ . We refer to [8] and [9] for the definitions, notations and properties of the  $q$ -shifted factorials, the  $q$ -hypergeometric functions, the Jackson's  $q$ -derivative and the Jackson's  $q$ -integrals. For the definitions and proprieties of the special functions used here, we refer to the papers ([12], [6], [4], [2], [1]). The main sets and functional spaces used are:

- $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}$ ,  $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$  and  $\tilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}$ .
- $\mathcal{D}_{*q}(\mathbb{R}_q)$  the space of restrictions on  $\mathbb{R}_q$  of even infinitely  $q$ -differentiable functions on  $\mathbb{R}$  with compact supports.

- $\mathcal{C}_{*q,0}(\mathbb{R}_q)$  the space of restrictions on  $\mathbb{R}_q$  of even smooth functions, continued in 0 and vanishing at  $\infty$ .
- $\mathcal{S}_{*q}(\mathbb{R}_q)$  the space of restrictions on  $\mathbb{R}_q$  of infinitely  $q$ -differentiable and even functions satisfying:  $\sup_{x \in \mathbb{R}_q; 0 \leq k \leq n} |(1+x^2)^m D_q^k f(x)| < +\infty, n, m \in \mathbb{N}$ .
- For  $p > 0, L_{q,\alpha}^p(\mathbb{R}_{q,+}) = \left\{ f : \|f\|_{p,\alpha,q} = \left( \int_0^\infty |f(x)|^p x^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\}$ .

The  $q$ -Bessel operator is defined and studied in [6] by

$$\Delta_{\alpha,q} f(z) = \left( \frac{1}{x^{2\alpha+1}} D_q [x^{2\alpha+1} D_q f] \right) (q^{-1}z).$$

We recall (see [6]) that for  $\lambda \in \mathbb{C}$ , the problem

$$\begin{cases} \Delta_{\alpha,q} u(x) = -\lambda^2 u(x), \\ u(0) = 1, u'(0) = 0 \end{cases} \tag{1}$$

has as unique solution the function  $z \mapsto j_\alpha^{(3)}(\lambda z; q^2)$ , where  $j_\alpha^{(3)}(\cdot; q^2)$  is the normalized  $q$ -Bessel function.

The generalized  $q$ -Bessel translation operator  $T_{q,x}^\alpha, x \in \mathbb{R}_{q,+}$  is defined (see [6]) on  $\mathcal{D}_{*q}(\mathbb{R}_q)$  by

$$T_{q,x}^\alpha(f)(y) = \sum_{n=0}^\infty \frac{q^{n^2}}{(q^2, q^{2\alpha+2}; q^2)_n} \left(\frac{x}{y}\right)^{2n} \sum_{k=-n}^n (-1)^{n-k} U_k(n) f(q^k y),$$

$y \in \mathbb{R}_{q,+}$  and  $T_{q,0}^\alpha(f) = f$ , where

$$U_k(n) = q^{k(k-1)+2n(k+\alpha)} \sum_{p=0}^k \begin{bmatrix} n \\ p \end{bmatrix}_{q^2} \begin{bmatrix} n \\ n+k-p \end{bmatrix}_{q^2} q^{-2p(n+k+\alpha-p)}$$

is the  $q$ -Bessel-Binomial coefficient associated with the  $q$ -Bessel operator [6]. It verifies, in particular

$$\int_0^\infty T_{q,x}^\alpha(f)(y) g(y) y^{2\alpha+1} d_q y = \int_0^\infty f(y) T_{q,x}^\alpha(g)(y) y^{2\alpha+1} d_q y, \quad x \in \widetilde{\mathbb{R}}_{q,+}, \tag{2}$$

and

$$T_{q,x}^\alpha j_\alpha^{(3)}(ty; q^2) = j_\alpha^{(3)}(tx; q^2) j_\alpha^{(3)}(ty; q^2), \quad x, y, t \in \widetilde{\mathbb{R}}_{q,+}. \tag{3}$$

The  $q$ -Bessel Fourier transform and the  $q$ -convolution product are defined (see [6]) for  $f, g \in \mathcal{D}_{*q}(\mathbb{R}_q)$ , by

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_0^\infty f(x) j_\alpha^{(3)}(\lambda x; q^2) x^{2\alpha+1} d_q x, \tag{4}$$

$$f *_B g(x) = c_{\alpha,q} \int_0^\infty T_{q,x}^\alpha f(y)g(y)y^{2\alpha+1}d_qy, \tag{5}$$

where  $c_{\alpha,q} = \frac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}$ .

Using the properties of the generalized  $q$ -Bessel translation and the normalized  $q$ -Bessel function, one can prove easily the following results.

**THEOREM 1.** For  $f, g \in \mathcal{D}_{*q}(\mathbb{R}_q)$ , we have

$$\mathcal{F}_{\alpha,q}(f *_B g) = \mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g), \tag{6}$$

$$\mathcal{F}_{\alpha,q}(T_{q,x}^\alpha f)(\lambda) = j_\alpha^{(3)}(\lambda x; q^2)\mathcal{F}_{\alpha,q}(f)(\lambda), \quad x \in \tilde{\mathbb{R}}_{q,+}, \lambda \in \mathbb{R}_{q,+} \tag{7}$$

and

$$\mathcal{F}_{\alpha,q}(\Delta_{\alpha,q}f)(\lambda) = -\frac{\lambda^2}{q^{2\alpha+1}}\mathcal{F}_{\alpha,q}(f)(\lambda), \quad \lambda \in \mathbb{C}. \tag{8}$$

**THEOREM 2.** For  $f \in L_{q,\alpha}^1(\mathbb{R}_{q,+})$ , we have

$$\mathcal{F}_{\alpha,q}(f) \in \mathcal{C}_{*q,0}(\mathbb{R}_q) \quad \text{and} \quad \|\mathcal{F}_{\alpha,q}(f)\|_{\mathcal{C}_{*q,0}(\mathbb{R}_q)} \leq \frac{c_{\alpha,q}}{(q; q^2)_\infty^2} \|f\|_{1,\alpha,q}. \tag{9}$$

**THEOREM 3.**  $\mathcal{F}_{\alpha,q}$  is an isomorphism of  $S_{*q}(\mathbb{R}_q)$  onto itself and can be extended continuously to an isomorphism of  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  onto itself,  $\mathcal{F}_{\alpha,q}^{-1} = q^{4\alpha+2}\mathcal{F}_{\alpha,q}$  and

$$\forall f \in L_{q,\alpha}^2(\mathbb{R}_{q,+}), \quad \|\mathcal{F}_{\alpha,q}(f)\|_{2,\alpha,q} = q^{2\alpha+1} \|f\|_{2,\alpha,q}. \tag{10}$$

**PROPOSITION 1.** Let  $f$  and  $g$  be in  $L_q^2(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ , then:

- 1)  $f *_B g \in L_{q,\alpha}^2(\mathbb{R}_{q,+})$  iff  $\mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g) \in L_{q,\alpha}^2(\mathbb{R}_{q,+})$ .
- 2)

$$q^{4\alpha+2} \int_0^\infty |f *_B g(x)|^2 x^{2\alpha+1}d_qx = \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 |\mathcal{F}_{\alpha,q}(g)(x)|^2 x^{2\alpha+1}d_qx, \tag{11}$$

where both sides are finite or infinite.

**REMARK 1.** Using Theorem 3, the relation (7) and the fact that  $\sup_{x \in \mathbb{R}_q} |j_\alpha^{(3)}(x; q^2)| \leq \frac{1}{(q; q^2)_\infty^2}$ , one can see that, for  $f \in L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ), we have for  $x \in \mathbb{R}_{q,+}$ ,  $T_{q,x}^\alpha f$  is in  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ) and

$$\|T_{q,x}^\alpha f\|_{2,\alpha,q} \leq \frac{1}{(q; q^2)_\infty^2} \|f\|_{2,\alpha,q}. \tag{12}$$

**3.  $q$ -Wavelet transforms associated with the  $q$ -Bessel operator**

DEFINITION 1. A  $q$ -wavelet associated with the  $q$ -Bessel operator is an even function  $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$  satisfying the following admissibility condition:

$$0 < C_{\alpha,g} = \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a)|^2 \frac{d_q a}{a} < \infty. \tag{13}$$

REMARKS:

1) For all  $\lambda \in \mathbb{R}_{q,+}$ , we have  $C_{\alpha,g} = \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a\lambda)|^2 \frac{d_q a}{a}$ .

2) Let  $f$  be a nonzero function in  $\mathcal{S}_{*q}(\mathbb{R}_q)$  (resp.  $\mathcal{D}_{*q}(\mathbb{R}_q)$ ). Then  $g = \Delta_{\alpha,q} f$  is a  $q$ -wavelet associated with the  $q$ -Bessel operator, in  $\mathcal{S}_{*q}(\mathbb{R}_q)$  (resp.  $\mathcal{D}_{*q}(\mathbb{R}_q)$ ) and we have  $C_{\alpha,g} = \frac{1}{q^{4\alpha+2}} \int_0^\infty a^3 |\mathcal{F}_{\alpha,q}(f)(a)|^2 d_q a$ .

EXAMPLE: Consider the functions  $G(x; q^2) = A_\alpha e_{q^2}^{-\frac{q^{-(2\alpha+1)}}{(1+q)^2} x^2}$  and  $g = \Delta_{\alpha,q} G(\cdot; q^2)$ , where  $A_\alpha = c_{\alpha,q} \int_0^\infty x^{2\alpha+1} e_{q^2}^{-x^2} d_q x$  and  $e_{q^2}^x = \frac{1}{(x; q^2)_\infty}$ . We have  $G(\cdot; q^2) \in \mathcal{S}_{*q}(\mathbb{R}_q)$  and with the use of the relation [6],  $\mathcal{F}_{\alpha,q}(G(\cdot; q^2))(x) = q^{4\alpha+2} e_{q^2}^{-x^2}$ , we get  $0 < \int_0^\infty |\mathcal{F}_{\alpha,q}(g)|^2(a) \frac{d_q a}{a} = \frac{q^{4\alpha}}{(1+q)}$ . So,  $g$  is a  $q$ -wavelet associated with the  $q$ -Bessel operator.

PROPOSITION 2. Let  $g \neq 0$  be a function in  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$  satisfying:

- 1)  $\mathcal{F}_{\alpha,q}(g)$  is continuous at 0.
- 2)  $\exists \beta > 0$  such that  $\mathcal{F}_{\alpha,q}(g)(x) - \mathcal{F}_{\alpha,q}(g)(0) = O(x^\beta)$ , as  $x \rightarrow 0$ .

Then, (13) is equivalent to

$$\mathcal{F}_{\alpha,q}(g)(0) = 0. \tag{14}$$

P r o o f. • If  $\mathcal{F}_{\alpha,q}(g)(0) \neq 0$ , then from the condition 1) there exist  $p_0 \in \mathbb{N}$  and  $M > 0$ , such that for all  $n \geq p_0$ ,  $|\mathcal{F}_{\alpha,q}(g)(q^n)| \geq M$ . Then, the  $q$ -integral in (13) would be equal to  $\infty$ .

• Conversely, we suppose that  $\mathcal{F}_{\alpha,q}(g)(0) = 0$ .

As  $g \neq 0$ , we deduce from Theorem 3, that the first inequality in (13) is satisfied.

On the other hand, from the condition 2), there exist  $n_0 \in \mathbb{N}$  and  $\epsilon > 0$ , such that for all  $n \geq n_0$ ,  $|\mathcal{F}_{\alpha,q}(g)(q^n)| \leq \epsilon q^{n\beta}$ . Then using the definition of the  $q$ -integral and Theorem 3, we obtain

$$\int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a)|^2 \frac{d_q a}{a} = (1-q) \sum_{n=-\infty}^\infty |\mathcal{F}_{\alpha,q}(g)(q^n)|^2$$

$$\leq \frac{\|\mathcal{F}_{\alpha,q}(g)\|_{2,\alpha,q}^2}{q^{(2\alpha+2)n_0}} + \frac{1-q}{1-q^{2\beta}}\epsilon^2.$$

This proves the second inequality of (13). ■

REMARK 2. For  $g \in L_{q,\alpha}^1(\mathbb{R}_{q,+})$ , the continuity assumption in the previous proposition holds. Then (14) can be written as  $\int_0^\infty g(x)x^{2\alpha+1}d_qx = 0$ , which proves that  $g$  changes the sign on  $\mathbb{R}_{q,+}$  and tends to 0 at  $\infty$ .

THEOREM 4. For  $a \in \mathbb{R}_{q,+}$  and  $g \in L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ), the function  $g_a : x \mapsto \frac{1}{a^{2\alpha+2}}g\left(\frac{x}{a}\right)$  belongs to  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ) and we have

$$\|g_a\|_{2,\alpha,q} = \frac{1}{a^{\alpha+1}}\|g\|_{2,\alpha,q}, \tag{15}$$

$$\mathcal{F}_{\alpha,q}(g_a)(\lambda) = \mathcal{F}_{\alpha,q}(g)(a\lambda), \quad \lambda \in \mathbb{R}_{q,+}. \tag{16}$$

P r o o f. The change of variable  $u = \frac{x}{a}$  gives the result. ■

THEOREM 5. Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ). Then for all  $a \in \mathbb{R}_{q,+}$  and  $b \in \widehat{\mathbb{R}}_{q,+}$ , the function

$$g_{(a,b),\alpha}(x) = \sqrt{a} T_{q,b}^\alpha(g_a), \tag{17}$$

is a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ) and we have

$$C_{\alpha,g_{(a,b),\alpha}} = a \int_0^\infty \left( j_\alpha^{(3)}\left(\frac{xb}{a}; q^2\right) \right)^2 |\mathcal{F}_{\alpha,q}(g)(x)|^2 \frac{d_qx}{x}. \tag{18}$$

P r o o f. As  $g_a$  is in  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ), Remark 1 shows that the relation (17) defines an element of  $L_{q,\alpha}^2(\mathbb{R}_{q,+})$  (resp.  $\mathcal{S}_{*q}(\mathbb{R}_q)$ ). Furthermore, from the relations (16) and (7), we have for all  $\lambda \in \mathbb{R}_{q,+}$ ,

$$\mathcal{F}_{\alpha,q}(g_{(a,b),\alpha})(\lambda) = \sqrt{a} j_\alpha^{(3)}(b\lambda; q^2)\mathcal{F}_{\alpha,q}(g)(a\lambda).$$

This relation implies (18).

On the other hand, as  $g \neq 0$ , we deduce from (18) and Theorem 3 that  $C_{\alpha,g_{(a,b),\alpha}} \neq 0$ . Moreover, from the relation (13) and the fact that  $\sup_{x \in \mathbb{R}_q} |j_\alpha^{(3)}(x; q^2)| \leq \frac{1}{(q; q^2)_\infty^2}$ , we deduce that  $C_{\alpha,g_{(a,b),\alpha}} \leq \frac{a}{(q; q^2)_\infty^4} C_{\alpha,g}$ , which gives the result. ■

**PROPOSITION 3.** *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ . Then the mapping  $F : (a, b) \mapsto g_{(a,b),\alpha}$  is continuous from  $\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$  into  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ .*

**P r o o f.** It is clear that  $F$  is a mapping from  $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$  into  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$  and it is continuous at all  $(a, b) \in \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ . The properties of the generalized  $q$ -Bessel translation, Theorem 3 and the Lebesgue theorem prove that  $F$  is continuous at all points  $(a, 0)$ ,  $a \in \mathbb{R}_{q,+}$ . ■

**DEFINITION 2.** Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ . We define the continuous  $q$ -wavelet transform associated with the  $q$ -Bessel operator, for  $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , by

$$\Psi_{q,g}^\alpha(f)(a, b) = c_{\alpha,q} \int_0^\infty f(x) \overline{g_{(a,b),\alpha}(x)} x^{2\alpha+1} d_q x, \quad a \in \mathbb{R}_{q,+}, \quad b \in \widetilde{\mathbb{R}}_{q,+}. \quad (19)$$

**REMARK 3.** The relation (19) can also be written in the form

$$\Psi_{q,g}^\alpha(f)(a, b) = \sqrt{a} f *_B \overline{g_a}(b) = \sqrt{a} q^{-4\alpha-2} \mathcal{F}_{\alpha,q} [\mathcal{F}_{\alpha,q}(f) \cdot \mathcal{F}_{\alpha,q}(\overline{g_a})] (b).$$

We give some properties of  $\Psi_{q,g}^\alpha$  in the following proposition.

**PROPOSITION 4.** *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$  and  $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , then, for all  $a \in \mathbb{R}_{q,+}$ , the function  $b \mapsto \Psi_{q,g}^\alpha(f)(a, b)$  is continuous on  $\widetilde{\mathbb{R}}_{q,+}$ ,  $\lim_{b \rightarrow \infty} \Psi_{q,g}^\alpha(f)(a, b) = 0$  and*

$$\forall b \in \widetilde{\mathbb{R}}_{q,+}, \quad |\Psi_{q,g}^\alpha(f)(a, b)| \leq \frac{c_{\alpha,q}}{(q; q^2)_\infty^2 a^{\alpha+1/2}} \|f\|_{2,\alpha,q} \|g\|_{2,\alpha,q}. \quad (20)$$

*Additionally, if  $f$  and  $g$  are in  $\mathcal{S}_{*q}(\mathbb{R}_q)$ , then for all  $a \in \mathbb{R}_{q,+}$ , the function  $b \mapsto \Psi_{q,g}^\alpha(f)(a, b)$  is in  $\mathcal{S}_{*q}(\mathbb{R}_q)$ .*

**P r o o f.** The result follows from the properties of the generalized  $q$ -Bessel translation, the properties of the normalized  $q$ -Bessel function, the properties of the  $q$ -Bessel convolution product and Theorem 3. ■

**THEOREM 6.** *Let  $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$  be a  $q$ -wavelet associated with the operator  $\Delta_{\alpha,q}$ .*

i) **Plancherel formula for  $\Psi_{q,g}^\alpha$ :** *For  $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , we have*

$$\frac{1}{C_{\alpha,g}} \int_0^\infty \int_0^\infty |\Psi_{q,g}^\alpha(f)(a, b)|^2 b^{2\alpha+1} \frac{d_q b d_q a}{a^2} = \|f\|_{2,\alpha,q}^2. \quad (21)$$

ii) **Parseval formula for  $\Psi_{q,g}^\alpha$ :** *For  $f_1, f_2 \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , we have*

$$\int_0^\infty f_1(x)\bar{f}_2(x)x^{2\alpha+1}d_qx = \frac{1}{C_{\alpha,g}} \int_0^\infty \int_0^\infty \Psi_{q,g}^\alpha(f_1)(a,b)\overline{\Psi_{q,g}^\alpha(f_2)}(a,b)b^{2\alpha+1} \frac{d_qad_qb}{a^2}. \tag{22}$$

P r o o f. i) By using Fubini’s theorem, Theorem 3, Remark 3 and the relations (16) and (11), we get the relation (21).

ii) The result is easily deduced from (21). ■

**THEOREM 7.** *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , then for all  $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , we have*

$$f(x) = \frac{c_{\alpha,q}}{C_{\alpha,g}} \int_0^\infty \int_0^\infty \Psi_{q,g}^\alpha(f)(a,b)g_{(a,b),\alpha}(x)b^{2\alpha+1} \frac{d_qad_qb}{a^2}, \quad x \in \mathbb{R}_{q,+}. \tag{23}$$

P r o o f. For  $f \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$  and  $x \in \mathbb{R}_{q,+}$ , take in (22)  $f_1 = f$  and  $f_2 = \delta_x$ . The result follows then from the definitions of  $\Psi_{q,g}^\alpha$  and the  $q$ -Jackson’s integral. ■

### 4. Coherent states

Theorem 6 shows that the continuous  $q$ -wavelet transform associated with the  $q$ -Bessel operator  $\Psi_{q,g}^\alpha$  is isometry from the Hilbert space  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$  into the Hilbert space  $L^2_q(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; b^{2\alpha+1} \frac{d_qad_qb}{a^2 C_{\alpha,g}})$  (the space of square integrable functions on  $\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$  with respect to the measure  $b^{2\alpha+1} \frac{d_qad_qb}{a^2 C_{\alpha,g}}$ ). For the characterization of the image of  $\Psi_{q,g}^\alpha$ , we consider the vectors  $g_{(a,b),\alpha}$ ,  $(a,b) \in \mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$ , as a set of coherent states in the Hilbert space  $L^2_{q,\alpha}(\mathbb{R}_{q,+})$  (see [11]).

**DEFINITION 3.** A set of coherent states in a Hilbert space  $\mathcal{H}$  is a subset  $\{g_l\}_{l \in \mathcal{L}}$  of  $\mathcal{H}$  such that:

i)  $\mathcal{L}$  is a locally compact topological space and the mapping  $l \mapsto g_l$  is continuous from  $\mathcal{L}$  into  $\mathcal{H}$ .

ii) There is a positive Borel measure  $dl$  on  $\mathcal{L}$  such that, for  $f \in \mathcal{H}$ ,

$$\| f \|^2 = \int_{\mathcal{L}} | (f, g_l) |^2 dl,$$

where  $(.,.)$  and  $\| . \|$  are respectively the scalar product and the norm of  $\mathcal{H}$ .

Let now  $\mathcal{H} = L^2_{q,\alpha}(\mathbb{R}_{q,+})$ ,  $\mathcal{L} = \mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}$  equipped with the induced topology of  $\mathbb{R}^2$ . Choose a nonzero function  $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$  and let



$g_l = g_{(a,b),\alpha}$ ,  $l = (a, b) \in \mathcal{L}$  be given by the relation (17). Then we have a set of coherent states. Indeed, i) of Definition 3 is satisfied, because of Proposition 3, and ii) of Definition 3 is satisfied for the measure  $b^{2\alpha+1} \frac{d_q a d_q b}{a^2 C_{\alpha,g}}$  (see Theorem 6). By adaptation of the approach introduced by T.H. Koornwinder in [11], we obtain the following result:

**THEOREM 8.** *Let  $F$  be in  $L^2_q(\mathbb{R}_{q,+} \times \widetilde{\mathbb{R}}_{q,+}; b^{2\alpha+1} \frac{d_q a d_q b}{a^2 C_{\alpha,g}})$ . Then  $F$  belongs to  $Im \Psi_{q,g}^\alpha$  if and only if*

$$F(a, b) = \frac{1}{C_{\alpha,g}} \int_0^\infty \int_0^\infty F(a', b') \left( \int_0^\infty g_{(a',b'),\alpha}(x) \overline{g_{(a,b),\alpha}(x)} x^{2\alpha+1} d_q x \right) (b')^{2\alpha+1} \frac{d_q a' d_q b'}{(a')^2}.$$

**5. Inversion formulas for the  $q$ -Riemann-Liouville and the  $q$ -Weyl operators**

In the sequel, we will use the following spaces:

- $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q) = \left\{ f \in \mathcal{S}_{*q}(\mathbb{R}_q) : \int_0^\infty f(x) x^{2k+2\alpha+1} d_q x = 0, k = 0, 1, \dots \right\}$ .
- $\mathcal{S}_{*q}^0(\mathbb{R}_q) = \left\{ f \in \mathcal{S}_{*q}(\mathbb{R}_q) : D_q^{2k} f(0) = 0, k = 0, 1, \dots \right\}$ .

The  $q$ -Riemann-Liouville transform  $R_{\alpha,q}$  is defined on  $\mathcal{D}_{*q}(\mathbb{R}_q)$  by (see [6])

$$R_{\alpha,q}(f)(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_0^1 \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{2\alpha+1}; q^2)_\infty} f(xt) d_q t. \tag{24}$$

The  $q$ -Weyl transform is defined on  $\mathcal{D}_{*q}(\mathbb{R}_q)$  by (see [6])

$$W_{\alpha,q}(f)(x) = \frac{q(1+q^{-1})^{-\alpha+\frac{1}{2}} \Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{qx}^\infty \frac{(x^2/t^2 q^2; q^2)_\infty}{(q^{2\alpha+1} x^2/t^2; q^2)_\infty} f(t) t^{2\alpha} d_q t. \tag{25}$$

On  $\mathcal{D}_{*q}(\mathbb{R}_q)$  we have the relations (see [6])

$$\Delta_{\alpha,q} \circ R_{\alpha,q} = R_{\alpha,q} \circ \Delta_q \quad \text{and} \quad R_{\alpha,q}(f *_q g) = R_{\alpha,q}(f) *_B R_{\alpha,q}(g),$$

where " $*_q$ " is the  $q$ -even convolution product associated with the operator  $\Delta_q : f \mapsto D_q^2(f)(q^{-1} \cdot)$  studied in [5].

The  $q$ -Fourier-cosine transform  $\mathcal{F}_q$  (studied in [5]) and the  $q$ -Bessel transform are linked by the following relation (see [6]):

PROPOSITION 5. For  $f \in \mathcal{S}_{*q}(\mathbb{R}_q)$ , we have

$$\mathcal{F}_{\alpha,q}(f) = \mathcal{F}_q \circ W_{\alpha,q}(f). \tag{26}$$

We state the following results, useful in the sequel.

THEOREM 9. The  $q$ -Fourier-cosine transform  $\mathcal{F}_q$  is a topological isomorphism from  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  into  $\mathcal{S}_{*q}^0(\mathbb{R}_q)$ .

P r o o f. The result follows from the Plancherel formula of  $\mathcal{F}_q$  (see [2]) and the fact that  $D_q^2 \cos(x; q^2) = -\cos(qx; q^2)$ . ■

Similarly, we have the following result.

THEOREM 10. The  $q$ -Fourier-Bessel transform  $\mathcal{F}_{\alpha,q}$  is a topological isomorphism from  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$  into  $\mathcal{S}_{*q}^0(\mathbb{R}_q)$ .

COROLLARY 1. The  $q$ -Weyl transform  $W_{\alpha,q}$  is a topological isomorphism from  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$  into  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ .

P r o o f. We deduce the result from the relation  $\mathcal{F}_{\alpha,q} = \mathcal{F}_q \circ W_{\alpha,q}$  and Theorems 9 and 10. ■

PROPOSITION 6. For  $f$  in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  (resp.  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ ) and  $g$  in  $\mathcal{S}_{*q}(\mathbb{R}_q)$  the function  $f *_q g$  (resp.  $f *_B g$ ) belongs to  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  (resp.  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ ).

P r o o f. The result follows from Theorem 9 (resp. Theorem 10) and the fact that  $f *_q g = \mathcal{F}_q(\mathcal{F}_q(f) \cdot \mathcal{F}_q(g))$  (resp.  $f *_B g = q^{-4\alpha-2} \mathcal{F}_{\alpha,q}(\mathcal{F}_{\alpha,q}(f) \cdot \mathcal{F}_{\alpha,q}(g))$ ). ■

PROPOSITION 7. The operator  $K_{\alpha,q,1}$  defined by

$$K_{\alpha,q,1}(f) = \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} \mathcal{F}_q^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_q(f))$$

is a topological isomorphism from  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  onto itself.

P r o o f. The multiplication operator

$$f \mapsto \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} |\lambda|^{2\alpha+1} f$$

is a topological isomorphism from  $\mathcal{S}_{*q}^0(\mathbb{R}_q)$  into itself. The inverse is given by  $f \mapsto \frac{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)|\lambda|^{2\alpha+1}} f$ . The result follows then from Theorem 9. ■

PROPOSITION 8. *The operator  $K_{\alpha,q,2}$  defined by*

$$K_{\alpha,q,2}(f)(x) = \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)} \mathcal{F}_{\alpha,q}^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_{\alpha,q}(f))(x)$$

*is a topological isomorphism from  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$  onto itself.*

P r o o f. From the relation  $\mathcal{F}_{\alpha,q} = \mathcal{F}_q \circ W_{\alpha,q}$  and the definition of  $K_{\alpha,q,1}$ , we have for all  $f \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$

$$K_{\alpha,q,2} = W_{\alpha,q}^{-1} \circ K_{\alpha,q,1} \circ W_{\alpha,q}. \tag{27}$$

We deduce the result from Proposition 7 and Corollary 1. ■

PROPOSITION 9. *i) For all  $f \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  and  $g \in \mathcal{S}_{*q}(\mathbb{R}_q)$ , we have*

$$K_{\alpha,q,1}(f *_q g) = K_{\alpha,q,1}(f) *_q g.$$

*ii) For all  $f \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$  and  $g \in \mathcal{S}_{*q}(\mathbb{R}_q)$ , we have*

$$K_{\alpha,q,2}(f *_B g) = K_{\alpha,q,2}(f) *_B g.$$

P r o o f. The result follows from the properties of the  $q$ -convolution products and the definitions of  $K_{\alpha,q,1}$  and  $K_{\alpha,q,2}$ . ■

THEOREM 11. *For all  $f \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ , we have the following inversion formulas for the operator  $R_{\alpha,q}$*

$$f = R_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}(f), \tag{28}$$

$$f = R_{\alpha,q} \circ W_{\alpha,q} \circ K_{\alpha,q,2}(f). \tag{29}$$

P r o o f. Using the properties of the operator  $R_{\alpha,q}$ , studied in [6], Theorem 3 and Proposition 5, we obtain for  $x \in \tilde{\mathbb{R}}_{q,+}$ ,

$$\begin{aligned} q^{4\alpha+2} f(x) &= c_{\alpha,q} \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda) j_\alpha^{(3)}(\lambda x; q^2) \lambda^{2\alpha+1} d_q \lambda \\ &= R_{\alpha,q} \left[ c_{\alpha,q} \int_0^\infty \mathcal{F}_{\alpha,q}(f)(\lambda) \cos(\lambda \cdot; q^2) \lambda^{2\alpha+1} d_q \lambda \right] (x) \\ &= R_{\alpha,q} \left\{ \frac{c_{\alpha,q}}{c_{-1/2,q}} \mathcal{F}_q^{-1} [\lambda^{2\alpha+1} \mathcal{F}_q \circ W_{\alpha,q}(f)] \right\} (x) \\ &= q^{4\alpha+2} R_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}(f)(x). \end{aligned}$$

We deduce the second from the first relation and the the relation (27). ■

COROLLARY 2. *The operator  $R_{\alpha,q}$  is a topological isomorphism from  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  into  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ .*

P r o o f. We deduce the result from Proposition 7, Corollary 1 and relation (28). ■

Similarly, we have the following result.

THEOREM 12. *For all  $f \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ , we have the following inversion formulas for the operator  $W_{\alpha,q}$*

$$f = W_{\alpha,q} \circ R_{\alpha,q} \circ K_{\alpha,q,1}(f), \tag{30}$$

$$f = W_{\alpha,q} \circ K_{\alpha,q,2} \circ R_{\alpha,q}(f). \tag{31}$$

P r o o f. For  $f \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ , Corollary 1 (resp. Corollary 2) implies that  $W_{\alpha,q}^{-1}(f)$  (resp.  $R_{\alpha,q}(f)$ ) belongs to  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ . Then by writing the relation (28) (resp. (29)) for  $W_{\alpha,q}^{-1}(f)$  (resp.  $R_{\alpha,q}(f)$ ), we obtain the result. ■

COROLLARY 3. i) *For all  $f, g \in \mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ , we have*

$$W_{\alpha,q}(f *_B g) = W_{\alpha,q}(f) *_q W_{\alpha,q}(g). \tag{32}$$

ii) *For all  $f, g \in \mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  we have*

$$R_{\alpha,q}(f *_q g) = R_{\alpha,q}(f) *_B W_{\alpha,q}^{-1}(g). \tag{33}$$

### 6. Inversion formulas for the $q$ -Riemann-Liouville and the $q$ -Weyl operators using wavelets

In this section, we assume that the reader is familiar with the notions and notations presented in [3]. In particular, we recall the following two notations:

$$H_a(f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right) \quad \text{and} \quad C_g = \int_0^\infty |\mathcal{F}_q(g)|^2(a) \frac{d_q a}{a}.$$

We begin by the next useful and easily verified result.

PROPOSITION 10. *For all  $a \in \mathbb{R}_{q,+}$  and  $g \in L^2_{q,\alpha}(\mathbb{R}_{q,+})$ , we have*

$$g_a = \frac{1}{a^{2\alpha+3/2}} H_a(g) = \frac{q^{-4\alpha-2}}{\sqrt{a}} \mathcal{F}_{\alpha,q} \circ H_{a^{-1}} \circ \mathcal{F}_{\alpha,q}(g) = \frac{1}{\sqrt{a}} W_{\alpha,q}^{-1} \circ H_a \circ W_{\alpha,q}(g). \tag{34}$$

PROPOSITION 11. *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ . Then for all  $f$  in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ , we have the following relation*

$$\Psi_{q,g}^\alpha(f)(a, \cdot) = W_{\alpha,q}^{-1} [\Phi_{q,W_{\alpha,q}(g)}(W_{\alpha,q}(f))(a, \cdot)], \quad a \in \mathbb{R}_{q,+}. \quad (35)$$

P r o o f. Let  $a \in \mathbb{R}_{q,+}$ , from the properties of the continuous  $q$ -wavelet transform (see [3]), and relations (32) and (34), we have

$$\begin{aligned} \Psi_{q,g}^\alpha(f)(a, \cdot) &= \sqrt{a}f *_B \bar{g}_a = \sqrt{a}W_{\alpha,q}^{-1} [W_{\alpha,q}(f) *_q W_{\alpha,q}(\bar{g}_a)] \\ &= W_{\alpha,q}^{-1} [W_{\alpha,q}(f) *_q \overline{H_a \circ W_{\alpha,q}(g)}] \\ &= W_{\alpha,q}^{-1} [\Phi_{q,W_{\alpha,q}(g)}(W_{\alpha,q}(f))(a, \cdot)]. \end{aligned}$$

■

THEOREM 13. *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ . Then:*

1) *For all  $f$  in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ , we have for  $a \in \mathbb{R}_{q,+}$ ,  $b \in \tilde{\mathbb{R}}_{q,+}$ ,*

$$\Psi_{q,g}^\alpha(f)(a, b) = R_{\alpha,q} [\Phi_{q,W_{\alpha,q}(g)}(R_{\alpha,q}^{-1}(f))(a, \cdot)](b). \quad (36)$$

2) *For all  $f$  in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ , we have for  $a \in \mathbb{R}_{q,+}$ ,  $b \in \tilde{\mathbb{R}}_{q,+}$ ,*

$$\Phi_{q,W_{\alpha,q}(g)}(f)(a, b) = W_{\alpha,q} [\Psi_{q,g}^\alpha(W_{\alpha,q}^{-1}(f))(a, \cdot)](b). \quad (37)$$

P r o o f. We deduce the result from Corollary 3, the properties of the continuous  $q$ -wavelet transform (see [3]) and the relation (34). ■

PROPOSITION 12. 1) *If  $g$  is a  $q$ -wavelet in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ , then  $K_{\alpha,q,1}(g)$  is a  $q$ -wavelet in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$  and we have*

$$K_{\alpha,q,1} \circ H_a(g) = \frac{1}{a^{2\alpha+1}} H_a \circ K_{\alpha,q,1}(g), \quad a \in \mathbb{R}_{q,+}. \quad (38)$$

2) *If  $g$  is a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ , then  $K_{\alpha,q,2}(g)$  is a  $q$ -wavelet in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$  and we have*

$$K_{\alpha,q,2}(g_a) = \frac{1}{a^{2\alpha+1}} (K_{\alpha,q,2}(g))_a, \quad a \in \mathbb{R}_{q,+}. \quad (39)$$

P r o o f. 1) Let  $g$  be a  $q$ -wavelet in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ . From the definition of  $K_{\alpha,q,1}$ , we have for  $\lambda \in \mathbb{R}_{q,+}$ ,

$$\mathcal{F}_q(K_{\alpha,q,1}(g))(\lambda) = \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}\lambda^{2\alpha+1}\mathcal{F}_q(g)(\lambda).$$

Proposition 4 of [3], implies that  $K_{\alpha,q,1}(g)$  is a  $q$ -wavelet. On the other hand, using the fact  $\mathcal{F}_q \circ H_a = H_{a^{-1}} \circ \mathcal{F}_q$ ,  $a \in \mathbb{R}_{q,+}$  and the above equality, we obtain

$$\begin{aligned} &\mathcal{F}_q(H_a \circ K_{\alpha,q,1}(g))(\lambda) \\ &= a^{2\alpha+1} \frac{\Gamma_{q^2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_{q^2}(\alpha+1)}\lambda^{2\alpha+1}\mathcal{F}_q(H_a(g))(\lambda), \end{aligned}$$

which gives the result.

2) The same way of 1) leads to the result. ■

**THEOREM 14.** *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ . Then for  $a \in \mathbb{R}_{q,+}$  and  $b \in \tilde{\mathbb{R}}_{q,+}$ , we have:*

1) For all  $f$  in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ ,

$$\Psi_{q,g}^\alpha(f)(a,b) = \frac{1}{a^{2\alpha+1}}R_{\alpha,q}[\Phi_{q,K_{\alpha,q,1} \circ W_{\alpha,q}(g)}(W_{\alpha,q}(f))(a,\cdot)](b). \tag{40}$$

2) For all  $f$  in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ ,

$$\Phi_{q,W_{\alpha,q}(g)}(f)(a,b) = \frac{1}{a^{2\alpha+1}}W_{\alpha,q}[\Psi_{q,K_{\alpha,q,2}(g)}^\alpha(R_{\alpha,q}(f))(a,\cdot)](b). \tag{41}$$

**P r o o f.** 1) Let  $f$  be in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ ,  $a \in \mathbb{R}_{q,+}$  and  $b \in \tilde{\mathbb{R}}_{q,+}$ . Using Corollary 3, we obtain

$$\Psi_{q,g}^\alpha(f)(a,b) = \sqrt{a}f *_B \bar{g}_a(b) = \sqrt{a}R_{\alpha,q}[W_{\alpha,q}(f) *_q R_{\alpha,q}^{-1}(\bar{g}_a)](b).$$

So, Theorem 11, Proposition 12 and the relation (34), achieve the proof.

2) Follows from Corollary 3, Theorem 12 and Propositions 9 and 12. ■

**THEOREM 15.** *Let  $g$  be a  $q$ -wavelet associated with the  $q$ -Bessel operator in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ . Then for all  $x \in \mathbb{R}_{q,+}$ :*

1) For all  $f$  in  $\mathcal{S}_{*q,-1/2}(\mathbb{R}_q)$ , we have

$$W_{\alpha,q}^{-1}(f)(x) = \frac{c_{\alpha,q}}{C_{\alpha,g}} \int_0^\infty \left( \int_0^\infty R_{\alpha,q}[\Phi_{q,K_{\alpha,q,1}} \circ W_{\alpha,q}(g)(f)(a, \cdot)](b) \times g_{(a,b),\alpha}(x) \frac{b^{2\alpha+1}}{a^{2\alpha+3}} d_q b \right) d_q a.$$

2) For all  $f$  in  $\mathcal{S}_{*q,\alpha}(\mathbb{R}_q)$ , we have

$$R_{\alpha,q}^{-1}(f)(x) = \frac{c_{-\frac{1}{2},q}}{C_g} \int_0^\infty \left( \int_0^\infty W_{\alpha,q} \left[ \Psi_{q,K_{\alpha,q,2}}^\alpha(f)(a, \cdot) \right] (b) g_{a,b}(x) \frac{d_q b}{a^{2\alpha+3}} \right) d_q a.$$

**P r o o f.** The result follows from the previous theorem, Theorem 7 and ([3], Theorem 7). ■

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