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ON A CLASS OF FRACTIONAL TYPE INTEGRAL EQUATIONS IN VARIABLE EXPONENT SPACES *

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Abstract

We obtain a criterion of Fredholmness and formula for the Fredholm index of a certain class of one-dimensional integral operators M with a weak singularity in the kernel, from the variable exponent Lebesgue space $L^{p(\cdot)}([a,b],\varrho)$ to the Sobolev type space $L^{\alpha,p(\cdot)}([a,b],\varrho)$ of fractional smoothness. We also give formulas of closed form solutions $\varphi \in L^{p(\cdot)}$ of the 1st kind integral equation $M_0\varphi = f$, known as the generalized Abel equation, with $f \in L^{\alpha,p(\cdot)}$, in dependence on the values of the variable exponent $p(x)$ at the endpoints $x = a$ and $x = b$.

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1. Introduction

We study the solvability of the following integral equation of the first kind

$$(M\varphi)(x) := \int_a^b \frac{c(x,t)}{|x-t|^{1-\alpha}}\varphi(t)\,dt = f(x), \quad x \in (a,b), \tag{1.1}$$

where $0 < \alpha < 1$ and the function $c(x,t)$ may have a jump at the diagonal:

$$c(x,t) = \begin{cases} u(x,t), & \text{if } t < x \\ v(x,t), & \text{if } t > x. \end{cases}$$

The interval (a,b) may be finite or infinite. For definiteness we give the final results for the case of a finite interval, $-\infty < a < b < \infty$, the case of a half axis or axis requiring some technical modifications. However, occasionally we formulate some auxiliary results for the case $b = \infty$, when this does not require special modifications.

Equations of type (1.1), including especially the case of jump of $c(x,t)$ at the diagonal $t = x$, have various applications. They were widely studied in the setting when the solution φ was looked in the Lebesgue space $L^p(a,b)$ or $L^p([a,b],\varrho)$, see §30 in [21] and references therein.

In applications it may happen that a solution of the equation is looked for in a more general setting: the solution may have L^p-behavior at one end point of the interval and be for instance bounded at another end point. Or, more generally, it may belong to L^{p_1} near one end point and to L^{p_2} near another end point, with different p_1 and p_2. This generalization, in its turn is a particular case of a more general setting related to the so-called variable exponent Lebesgue spaces, when the order of integrability p may be a function of x. Last years there was observed an enormous rise of interest to the study of the so-called variable exponent Lebesgue spaces and operator theory in such spaces, influenced both by theoretical interest and possible applications revealed in [18], we refer in particular to the existing surveys [3, 9, 20], in this topic, and papers [10, 12] mostly related to the content of this paper.

This paper deals with such a setting when the solution $\varphi(x)$ is integrable with variable exponent $p(x)$. The right-hand side $f(x)$ of the equation in this case runs the fractional Sobolev space $L^{\alpha,p(\cdot)}$ with variable $p(x)$. One of the main points in the study of equations of type (1.1) is to reveal the influence of the endpoints of the interval onto the picture of solvability. Making use of the recent progress in the variable exponent analysis, we show that it is possible to "localize" the values

of $p(x)$ in this influence, namely we show, under the natural assumptions on $p(x)$, that only the values $p(a)$ and $p(b)$ are important in the study of the solvability of the equation, the Fredholmness criterion and formula for the index depending on the values $p(a)$ and $p(b)$, but not depending on the values of $p(x)$ in the inner points of the interval.

The paper is organized as follows.

An essential part of the paper contains necessary preliminaries, given in Section 2, where we have to present necessary tools related to variable exponent spaces (Subsection 2.1, on maximal, singular, convolution and potential operators in such spaces (Subsections 2.2-2.3), Marchaud fractional derivative in connection with variable exponent (Subsection 2.4), fractional Sobolev spaces with variable exponent (Subsection 2.5) and Fredholmness of singular integral operators in variable exponent Lebesgue spaces (Subsection 2.6). Making use of these tools, we investigate the Fredholm nature of the operator M in Section 3. The main result on Fredholmness is given in Theorem 3.23 and closed form solution formulas for the generalized Abel equation are considered in Subsection 3.4.

NOTATION:

$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$;
ϱ is a weight, *i.e.*, an a.e. finite and a.e. positive function;
$\mathcal{P}(\Omega)$ and $\mathcal{P}_1(\Omega)$, see (2.2)-(2.3);
w-$Lip\,(\Omega)$, see (2.4);
w-$Lip_{x_0}\,(\Omega)$, see (2.13);
$\mathcal{M}$ is the maximal operator, see (2.7);
$\mathbb{P}_{\varrho}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ such that $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega, \varrho)$.

2. Preliminaries

2.1. On spaces $L^{p(\cdot)}$ with variable exponents

Although our main results concern the one-dimensional case $n = 1$, some auxiliary statements below are given for the multidimensional case. We refer to [14, 19] for details on variable Lebesgue spaces over domains in $\mathbb{R}^n$, but give some necessary definitions. For a measurable function $p : \Omega \to [1, \infty)$, where $\Omega \subset \mathbb{R}^n$ is an open set, we put

$$p^+ = p^+(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad p^- = p^-(\Omega) := \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

In the sequel we use the notation

$$\mathcal{P}(\Omega) := \{p \in L^\infty(\Omega) : 1 < p^- \leqq p(x) \leqq p^+ < \infty\} \tag{2.2}$$

and

$$\mathcal{P}_1(\Omega) := \{p \in L^\infty(\Omega) : 1 \leqq p^- \leqq p(x) \leqq p^+ < \infty\}. \tag{2.3}$$

The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent is introduced as the set of functions φ on Ω for which

$$\mathfrak{I}_p(\varphi) := \int_\Omega |\varphi(x)|^{p(x)} dx < \infty.$$

By $w\text{-}Lip\,(\Omega)$ we denote the class of exponents $p \in L^\infty(\Omega)$ satisfying the log-condition

$$|p(x) - p(y)| \leqq \frac{C}{-\ln|x-y|}, \quad |x-y| \leqq \frac{1}{2},\ x, y \in \Omega. \tag{2.4}$$

In case of unbounded domains we also refer to the decay condition

$$|p(x) - p(\infty)| \leqq \frac{C}{\ln(1+|x|)}, \quad x \in \Omega. \tag{2.5}$$

By $p'(x)$ we denote the conjugate exponent: $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$.

The weighted Lebesgue space $L^{p(\cdot)}(\Omega, \varrho)$ is defined as the set of all measurable on Ω functions φ for which

$$\|\varphi\|_{L^{p(\cdot)}(\Omega,\varrho)} = \|\varrho\varphi\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \mathfrak{I}_p\left(\frac{\varrho\varphi}{\lambda}\right) \leqq 1\right\} < \infty.$$

The notation $\|\varphi\|_{L^{p(\cdot)}(\Omega)}$ and $\|\varphi\|_{p(\cdot)}$ will be used interchangeably, when no ambiguity arise.

In [12] the following theorem was proved.

THEOREM 2.1. *The class $C_0^\infty(\mathbb{R}^n)$ is dense in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ if*

$$p \in \mathcal{P}_1(\mathbb{R}^n) \quad \textit{and} \quad [\varrho(x)]^{p(x)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{2.6}$$

LEMMA 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $p \in \mathcal{P}(\Omega) \bigcap w\text{-}Lip\,(\Omega)$. There exists an extension $\widetilde{p}(x)$ of $p(x)$ to the whole space $\mathbb{R}^n$ such that $\widetilde{p}(x) \equiv p(x)$ for $x \in \Omega$, $\widetilde{p} \in \mathcal{P}(\mathbb{R}^n) \bigcap w\text{-}Lip\,(\mathbb{R}^n)$ and $\widetilde{p}(x)$ is constant outside some large fixed ball.*

2.2. On maximal and convolution operators in $L^{p(\cdot)}$

Let

$$(\mathcal{M}\varphi)(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{B(x,r)\bigcap\Omega} |\varphi(y)|dy \tag{2.7}$$

be the Hardy-Littlewood maximal operator. For dilations

$$(\mathbb{K}_\varepsilon f)(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) f(y)dy$$

there is known the following statement, see [2] for the non-weighted case, which was extended to the weighted case in [16] as stated below.

THEOREM 2.3. *Let ϱ be a weight, $\varrho^{-1} \in L^{p'(\cdot)}_{\rm loc}$, $p \in \mathcal{P}_1(\mathbb{R}^n)$ and $k(x)$ be an integrable function on $\mathbb{R}^n$ with $A := \int_{\mathbb{R}^n} \sup_{|y|\geqq|x|} |k(y)|dx < \infty$. Then:*

i) $$\left|\sup_{\varepsilon>0} (\mathbb{K}_\varepsilon f)(x)\right| \leqq A(\mathcal{M}f)(x) \qquad \textit{for all} \qquad f \in L^{p(\cdot)}(\mathbb{R}^n, \varrho),$$

so that

ii) $$\left\|\sup_{\varepsilon>0} (\mathbb{K}_\varepsilon f)(x)\right\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leq C_1\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)}$$

in the case $p(\cdot) \in \mathbb{P}_\varrho(\mathbb{R}^n)$. If in addition $\int_{\mathbb{R}^n} k(y)dy = 1$, and $\varrho(x)$ satisfies condition (2.6)*, then also*

iii) $$(\mathbb{K}_\varepsilon f)(x) \to f$$

as $\varepsilon \to 0$ in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ and almost everywhere.

By Theorem 2.3, the boundedness in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ of the maximal operator guarantees that of convolution operators

$$(Af)(x) = \int\limits_{\mathbb{R}^n} k(y)f(x-y)dy$$

whose kernel $k(x)$ has a decreasing integrable dominant. However, the boundedness of the maximal operator requires in general the local log-condition (2.4). Meanwhile, for rather *nice* kernels $k(x)$ this condition may be avoided, see [4, 5, Corollary 4.7].

2.3. Boundedness of potential and singular operators in weighted $L^{p(\cdot)}$ - spaces

We consider power type weights of the form

$$\varrho(x)=\prod_{k=1}^{m}|x-x_k|^{\beta_k},\qquad x_k\in\Omega,\ \ k=1,2,\dots,m, \tag{2.8}$$

where

$$-\frac{n}{p(x_k)}<\beta_k<\frac{n}{p'(x_k)},\quad k=1,2,...,m. \tag{2.9}$$

The following result was proved in [13] for a single power weight, but its validity is extended to the case of a finite product of power weights as in (2.8), by standard means using unity partition, see for instance, [15, Remark 2.7] or [16, Section 5].

THEOREM 2.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, let $\alpha(x)\in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_{x\in\Omega}\alpha(x)>0$, let $p\in\mathcal{P}(\Omega)\bigcap w\text{-}Lip\,(\Omega)$ and let $\varrho(x)$ be weight of form* (2.8) *with $x_k\in\overline{\Omega}$. Under condition* (2.9) *the operator*

$$\left(I^{\alpha(\cdot)}_{\varrho}f\right)(x)=\varrho(x)\int_\Omega\frac{f(y)\,dy}{\varrho(y)|x-y|^{n-\alpha(x)}}$$

is bounded in the space $L^{p(\cdot)}(\Omega)$.

We will consider the "unilateral" one-dimensional potential operators

$$\left(I^{\alpha}_{a+}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt,\qquad x>a; \tag{2.10}$$

$$\left(I^{\alpha}_{b-}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{\varphi(t)}{(t-x)^{1-\alpha}}dt,\qquad x<b, \tag{2.11}$$

as well, where $\alpha>0$, known also as Riemann-Liouville fractional integrals, left-sided and right-sided, respectively [21].

The following theorem on the boundedness of the singular operator

$$(S\varphi)(x)=\frac{1}{\pi}\int\limits_a^b\frac{\varphi(t)\,dt}{t-x},\qquad x\in(a,b)$$

was proved in [12].

THEOREM 2.5. *Let $-\infty < a < b < \infty$ and let $p \in \mathcal{P}(a,b) \bigcap w\text{-}Lip\,(a,b)$. The operator S is bounded in the space $L^{p(\cdot)}[(a,b),\varrho]$, where ϱ is weight of form* (2.8) *with $x_k \in [a,b]$, $k = 1,2,\dots,m$, if and only if*

$$-\frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}, \qquad k = 1,2,\dots,m.$$

2.4. Marchaud fractional derivative

Let now $n = 1$ and $\Omega = [a,b]$, where $-\infty < a < b \leqq \infty$, and consider the space $L^{p(\cdot)}[(a,b),\varrho]$ with the weight

$$\varrho(x) = \begin{cases} |x-a|^{\mu(x)}|b-x|^{\nu(x)} & \text{when } b < \infty \\ |x-a|^{\mu(x)}(1+|x|)^{\nu(x)} & \text{when } b = \infty \end{cases}, \tag{2.12}$$

where the exponents $\mu(x), \nu(x)$ are bounded functions which have finite limits $\mu(a) = \lim\limits_{x\to a}\mu(x), \nu(b) = \lim\limits_{x\to b}\nu(x)$. We need the following notation for the class of exponents of the weight admissible in the sequel.

DEFINITION 2.6. Let $\Omega = (a,b)$, where $-\infty \leqq a < b \leqq \infty$ and let $x_0 \in [a,b]$. By $w\text{-}Lip_{x_0}(\Omega)$ we denote the class

$$w\text{-}Lip_{x_0}(\Omega) = \left\{ \mu \in L^\infty(\Omega) : |\mu(x) - \mu(x_0)| \leqq \frac{A}{\ln\frac{1}{|x-x_0|}},\ |x-x_0| \leqq \frac{1}{2} \right\}, \tag{2.13}$$

in case $x_0 \neq \infty$, and

$$w\text{-}Lip_{\infty}(\Omega) = \left\{ \mu \in L^\infty(\Omega) : |\mu(x) - \mu(\infty)| \leqq \frac{A}{\ln(2+|x|)} \right\}. \tag{2.14}$$

For $\mu \in w\text{-}Lip_a(a,b) \bigcap w\text{-}Lip_b(a,b)$ with $-\infty < a < b < \infty$ one has

$$|x-a|^{\mu(x)}|b-x|^{\nu(x)} \approx |x-a|^{\mu(a)}|b-x|^{\nu(b)}. \tag{2.15}$$

Similarly, for $\mu \in w\text{-}Lip_a(\mathbb{R}^1) \bigcap w\text{-}Lip_b(\mathbb{R}^1) \bigcap w\text{-}Lip_\infty(\mathbb{R}^1)$

$$|x-a|^{\mu(x)}|b-x|^{\nu(x)} \approx |x-a|^{\mu(a)}|b-x|^{\nu(b)}(1+|x|)^{\mu(\infty)+\nu(\infty)-\mu(a)-\nu(b)}. \tag{2.16}$$

REMARK 2.7. From Theorem 2.1 it is easy to derive that the class $C_0^\infty\big((a,b)\big)$ of infinitely differentiable functions with support in (a,b), $-\infty < a < b < \infty$ is dense in the space $L^{p(\cdot)}[(a,b),\varrho]$ with the weight of form (2.12), if $p \in \mathcal{P}_1(a,b)$ and $\mu(a)p(a) > -1, \nu(b)p(b) > -1$.

Everywhere in the sequel we assume that

$$p(x)\equiv p_\infty=\text{const}\quad\text{for large}\quad |x|\geq R\qquad\text{in the case}\quad b=\infty. \tag{2.17}$$

For the weight $|x-a|^{\mu(x)}|b-x|^{\nu(x)}$, in the sequel we will refer to the following conditions

$$-\frac{1}{p(a)}<\mu(a)<\frac{1}{p'(a)},\quad -\frac{1}{p(b)}<\nu(b)<\frac{1}{p'(b)}; \tag{2.18}$$

$$-\frac{1}{p_\infty}<\nu(\infty)+\mu(\infty)<\frac{1}{p'_\infty}. \tag{2.19}$$

The Marchaud fractional derivative [21, p. 200]

$$(\mathbb{D}^\alpha_{a+}f)(x)=\frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}\int_a^x\frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt, \tag{2.20}$$

of order $0<\alpha<1$, for "not so nice" functions $f(x)$ is understood as

$$\lim_{\varepsilon\to0}\left(\mathbb{D}^\alpha_{a+,\varepsilon}f\right)(x)=\frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha}+ \\ +\frac{\alpha}{\Gamma(1-\alpha)}\lim_{\varepsilon\to0}\int_a^{x-\varepsilon}\frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt,\quad \varepsilon>0,$$

where $f(x)$ is assumed to be continued as zero beyond the interval $[a,b]$. It is known [21, p. 200] that

$$\left(\mathbb{D}^\alpha_{a+,\varepsilon}f\right)(x)=\frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}(\mathbb{A}_\varepsilon f)(x),$$

where

$$(\mathbb{A}_\varepsilon f)(x)=\int_a^{x-\varepsilon}\frac{f(x)-f(t)}{(x-t)^{1+\alpha}}dt\quad\text{for}\quad a+\varepsilon\leqq x\leqq b, \tag{2.21}$$

$$(\mathbb{A}_\varepsilon f)(x)=\frac{f(x)}{\alpha}\left[\frac{1}{\varepsilon^\alpha}-\frac{1}{(x-a)^\alpha}\right]\quad\text{for}\quad a\leqq x\leqq a+\varepsilon. \tag{2.22}$$

The following statement was proved in [16].

THEOREM 2.8. *Let* $-\infty<a<b<\infty$, $0<\alpha<1$ *and*

$$f=I^\alpha_{a+}\varphi,\ \ \varphi\in L^{p(\cdot)}[(a,b),\varrho],$$

where $p \in \mathcal{P}(a,b)\bigcap w\text{-}Lip\,(a,b)$ *and* $\varrho = (x-a)^{\mu(x)}(b-x)^{\nu(x)}$ *with* $\mu \in w\text{-}Lip_a(a,b)$, $\nu \in w\text{-}Lip_b(a,b)$. *Then, under conditions* (2.18),

$$\mathbb{D}^{\alpha}_{a+}f = \varphi,$$

where $\mathbb{D}^{\alpha}_{a+}f = \lim\limits_{\varepsilon\to 0}\mathbb{D}^{\alpha}_{a+,\varepsilon}f$ *with the limit in the norm of the space* $L^{p(\cdot)}[(a,b),\varrho]$.

This is also valid in the case $b=\infty$ *for the weight* $(x-a)^{\mu(x)}(1+|x|)^{\nu(x)}$, *if additionally* $\mu,\nu \in w\text{-}Lip_{\infty}(a,b)$ *and* (2.17) *and* (2.19) *hold.*

2.5. Fractional Sobolev space $L^{\alpha,p(\cdot)}\,[(a,b),\varrho]$

In the following definition we introduce a Sobolev space of fractional smoothness of functions on $[a,b]$ as the restriction of Bessel potentials onto $[a,b]$ under the corresponding extensions of the variable exponent $p(x)$ and the exponents $\mu(x),\nu(x)$ initially defined on $[a,b]$.

DEFINITION 2.9. Let $-\infty < a < b < \infty$. We define the fractional Sobolev type space $L^{\alpha,p(\cdot)}[(a,b),\varrho]$ with weight $\varrho = (x-a)^{\mu(x)}(b-x)^{\nu(x)}$ as

$$L^{\alpha,p(\cdot)}[(a,b),\varrho] = \mathcal{B}^{\alpha}[L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})]\Big|_{(a,b)} \tag{2.23}$$

where $\widetilde{p}$ is an arbitrary extension of p to $\mathbb{R}^1$ with preservation of the log-continuity and with the decay condition at infinity; we may take for instance, the extension $\widetilde{p}$ as described in Lemma 2.2. The extension $\widetilde{\varrho}$ of the weight ϱ is taken in the form $\widetilde{\varrho}(x) = |x-a|^{\tilde{\mu}(x)}\cdot|b-x|^{\tilde{\nu}(x)}$ with $\widetilde{\mu} \in w\text{-}Lip_a(\mathbb{R}^1)\bigcap w\text{-}Lip_{\infty}(\mathbb{R}^1), \widetilde{\nu} \in w\text{-}Lip_b(\mathbb{R}^1)\bigcap w\text{-}Lip_{\infty}(\mathbb{R}^1)$. We define the norm of $f = \mathcal{B}\varphi\Big|_{(a,b)}$ by

$$\|f\|_{L^{\alpha,p(\cdot)}[(a,b),\varrho]} = \inf\ \|\varphi\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R}^1,\widetilde{\varrho})}$$

where the infimum is taken with respect to all possible φ in the representation $f = \mathcal{B}\varphi\Big|_{(a,b)}$ and all the extensions $\widetilde{p},\widetilde{\mu}$ and $\widetilde{\nu}$.

The following proposition, proved in [16] and important for our further applications, states that the space $L^{\alpha,p(\cdot)}\,[(a,b),\varrho]$ coincides with the range of the operators of fractional integration.

THEOREM 2.10. *Let* $-\infty < a < b < \infty$, $p(\cdot) \in \mathcal{P}(a,b)\bigcap w\text{-}Lip\,(a,b)$ *and* $\varrho(x)$ *be weight of form* (3.43) *with the exponents* $\mu(x)$ *and* $\nu(x)$ *satisfying assumption* (2.13). *Under the conditions*

$$\alpha - \frac{1}{p(a)} < \mu(a) < \frac{1}{p'(a)}, \quad \alpha - \frac{1}{p(b)} < \nu(b) < \frac{1}{p'(b)}\,, \tag{2.24}$$

the following coincidence of the spaces

$$I^{\alpha}_{a+}\left[L^{p(\cdot)}[(a,b),\varrho]\right]=I^{\alpha}_{b-}\left[L^{p(\cdot)}[(a,b),\varrho]\right]=L^{\alpha,p(\cdot)}[(a,b),\varrho] \tag{2.25}$$

holds, and

$$C_1\|\mathbb{D}^{\alpha}_{a+}f\|_{L^{p(\cdot)}[(a,b),\varrho]}\leq\|f\|_{L^{\alpha,p(\cdot)}[(a,b),\varrho]}\leq C_2\|\mathbb{D}^{\alpha}_{a+}f\|_{L^{p(\cdot)}[(a,b),\varrho]}. \tag{2.26}$$

2.6. Fredholmness of singular integral operators in the weighted space $L^{p(\cdot)}(\Gamma,\varrho)$

We recall the notion of Fredholmness of a linear operator in a Banach space. Let X and Y be Banach spaces and $[X\to Y]$ the algebra of all the linear operators bounded from X into Y. By $Z_X(A)=\{\varphi: A\varphi=0,\varphi\in X\}$ we denote the kernel of an operator $A\in[X\to Y]$, and by $Z_Y(A)=\{\psi: A^*\psi=0,\psi\in Y^*\}$ its co-kernel. We also denote $n_A=\dim Z_X(A)$ and $m_A=\dim Z_{Y^*}(A^*)$ and define the *index* of a Fredholm operator as

$$\varkappa=\varkappa_{X\to Y}=n_A-m_A.$$

We will need the result on Fredholmness, in weighted variable exponent spaces $L^{p(\cdot)}(\Gamma,\varrho)$, of singular integral operators

$$\mathbb{A}=\mathcal{A}P_++\mathcal{B}P_-, \tag{2.27}$$

where $P_\pm=\frac{1}{2}(I\pm S)$ and $\mathcal{A},\mathcal{B}\in PC(\Gamma)$. The Fredholm theory of such operators for constant p is well known and extensively developed, see the books [1], [6], [7]. For the case of the spaces $L^{p(\cdot)}(\Gamma,\varrho)$ with variable $p(\cdot)$, we will derive the required statement on Fredholmness from a general such result within the framework of abstract Banach function spaces proved in [11, p.73]. To formulate the result from [11], we need to introduce the following notions from [11].

DEFINITION 2.11. A Banach space $X=X(\Gamma)$ of functions on a closed simple Jordan rectifiable curve Γ is called *admissible*, if:

($\mathfrak{A}1$) $C(\Gamma)\subset X\subset L_1(\Gamma)$,

($\mathfrak{A}2$) any $a\in L_\infty(\Gamma)$ is a multiplier in X,

($\mathfrak{A}3$) the operator S is bounded in $X(\Gamma)$,

($\mathfrak{A}4$) $C^\infty(\Gamma)$ is dense in X.

In the sequel we assume that the space X satisfies the following two axioms.

AXIOM 1. *For the space X there exist two functions $\alpha(t)$ and $\beta(t)$, $0 < \alpha(t) < 1,\ 0 < \beta(t) < 1,$ such that the operator $|t-t_0|^{\gamma(t_0)} S |t-t_0|^{-\gamma(t_0)} I$ with an arbitrary $t_0 \in \Gamma$ is bounded in the space $X(\Gamma)$ for all $\gamma(t_0)$ such that $-\alpha(t_0) < \gamma(t_0) < 1-\beta(t_0)$ and is unbounded in X if $\gamma(t_0) \notin (-\alpha(t_0), 1-\beta(t_0))$.*

Let $X(\Gamma, |t-t_0|^{\gamma}) = \{f : |t-t_0|^{\gamma} f(t) \in X(\Gamma)\}$.

AXIOM 2. *For any $\gamma < 1-\beta(t_0)$ the imbedding $X(\Gamma, |t-t_0|^{\gamma}) \subset L^1(\Gamma)$ is valid and $C^{\infty}(\Gamma)$ is dense in $X(\Gamma, |t-t_0|^{\gamma})$, whatsoever $t_0 \in \Gamma$ is.*

DEFINITION 2.12. Let X be a Banach function space satisfying Axiom 1. A function $a \in PC(\Gamma)$ is called X-nonsingular if $\inf\limits_{t\in\Gamma} |a(t)| > 0$ and $\frac{1}{2\pi} \arg \frac{a(t_k-0)}{a(t_k+0)} \notin [\alpha(t_k), \beta(t_k)] + \mathbb{Z}$ for every discontinuity point $t_k, k = 1, 2, \ldots, m$, of $a(t)$, where $[\cdots] + \mathbb{Z}$ stands for the set $\bigcup\limits_{\xi\in[\cdots]} \{\xi, \xi \pm 1, \xi \pm 2, ...\}$.

Let

$$\theta(t_k) = \frac{1}{2\pi} \int\limits_{t_k+0}^{t_{k+1}-0} d\, arg\, a(t).$$

DEFINITION 2.13. Let $X(\Gamma)$ satisfy Axiom 1 and $a \in PC(\Gamma)$ be X-nonsingular. The integer

$$\operatorname*{Ind}_X a := \sum_{k=1}^{m} [\theta(t_k) - \Re\Gamma(t_k)], \tag{2.28}$$

where $\Re\Gamma(t_k)$ are chosen in the interval $\beta(t_k) - 1 < \Re\Gamma(t_k) < \alpha(t_k)$, is called X-index of the function a.

In [11] the following statement was proved.

THEOREM 2.14. *Let X be any space admissible in the sense of Definition 2.11, satisfying Axioms 1-2. The operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ is Fredholm in the space X if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \quad \inf_{t\in\Gamma} |b(t)| \neq 0 \tag{2.29}$$

and the function $\frac{a(t)}{b(t)}$ is X-nonsingular. In this case

$$\operatorname*{Ind}_X A = -\operatorname*{Ind}_X \frac{a}{b}. \tag{2.30}$$

Condition (2.29) *is also necessary for the operator A to be Fredholm in X. If the functions $\alpha(t)$ and $\beta(t)$ of the space X coincide at the points t_k of discontinuity of the coefficients $a(t), b(t)$: $\alpha(t_k) = \beta(t_k), k = 1, 2, \dots, m$, then the condition of X-nonsingularity of $\frac{a(t)}{b(t)}$ is necessary as well.*

The space $X(\Gamma) = L^{p(\cdot)}(\Gamma, \varrho) = \{f : \varrho(t)f(t) \in L^{p(\cdot)}(\Gamma)\}$, where Γ is a Carleson curve and

$$\varrho(t) = \prod_{k=1}^{m} |t - t_k|^{\mu_k}, \quad -\frac{1}{p(t_k)} < \mu_k < \frac{1}{p'(t_k)}, \quad t_k \in \Gamma,\ k = 1, 2, \dots, m, \tag{2.31}$$

is admissible in the sense of Definition 2.11 and Axioms 1 and 2 are fulfilled with

$$\alpha(t) = \beta(t) = \begin{cases} \frac{1}{p(t)}, & t \neq t_k \\ \frac{1}{p(t)} + \mu_k, & t = t_k,\ k = 1, 2, \dots, m \end{cases} \tag{2.32}$$

which follows from Theorem 2.5.

As a result, we arrive at the following corollary of Theorem 2.14.

COROLLARY 2.15. *Let Γ be a closed Carleson curve, let $\varrho(t) = \prod\limits_{k=1}^{m} |t - t_k|^{\mu_k}$ and let $p(t), t \in \Gamma$ and also $p(\cdot) \in \mathcal{P}(\Gamma) \bigcap w\text{-}Lip\,(\Gamma)$. The operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ is Fredholm in the space $L^{p(\cdot)}(\Gamma, \varrho)$, if and only if condition* (2.29) *is satisfied and*

$$\frac{1}{2\pi} \arg \frac{g(t_k - 0)}{g(t_k + 0)} \neq \frac{1}{p(t_k)} + \mu_k \pmod 1, \tag{2.33}$$

where $g(t) = \frac{a(t)}{b(t)}$. Under these conditions

$$\operatorname*{Ind}_{L_{p(\cdot)}(\Gamma,\varrho)} \mathbb{A} = -\sum_{k=1}^{m} \left[\theta(t_k) - \frac{1}{2\pi} \arg \frac{g(t_k - 0)}{g(t_k + 0)}\right], \tag{2.34}$$

where $\theta(t_k) = \frac{1}{2\pi} \int\limits_{t_k+0}^{t_{k+1}-0} d\, arg\, g(t)$ and the values of $\frac{1}{2\pi} \arg \frac{g(t_k-0)}{g(t_k+0)}$ are chosen in the interval

$$\mu_k - \frac{1}{p'(t_k)} < \frac{1}{2\pi} \arg \frac{g(t_k - 0)}{g(t_k + 0)} < \mu_k + \frac{1}{p(t_k)}. \tag{2.35}$$

For the case where $\Gamma = [a,b]$ is an interval of the real line, we obtain the following statement, in which

$$\theta(x) := \arg \frac{\mathcal{B}(x)}{\mathcal{A}(x)} \quad \text{and} \quad 0 \leqq \theta(a) < 2\pi. \tag{2.36}$$

THEOREM 2.16. *The operator $\mathbb{A} = \mathcal{A}P_+ + \mathcal{B}P_-$ with $\mathcal{A}, \mathcal{B} \in C([a,b])$ is Fredholm in the space $L^{p(\cdot)}[(a,b), \varrho]$, where $\varrho(x) = (x-a)^\mu (b-x)^\nu$, if and only if*

$$\min_{x\in[a,b]} |\mathcal{A}(x)| \neq 0, \qquad \min_{x\in[a,b]} |\mathcal{B}(x)| \neq 0 \tag{2.37}$$

and

$$\frac{\theta(a)}{2\pi} \neq \frac{1}{p(a)} + \mu, \quad \frac{\theta(b)}{2\pi} \neq \frac{1}{p'(b)} - \nu \ (\mathrm{mod}\, 1). \tag{2.38}$$

Under these conditions

$$\operatorname*{Ind}_{L^{p(\cdot)}([a,b],\varrho)} \mathbb{A} = \left[\frac{1}{p(a)} + \mu - \frac{\theta(a)}{2\pi}\right] + \left[\frac{1}{p(b)} + \nu + \frac{\theta(b)}{2\pi}\right]. \tag{2.39}$$

P r o o f. To obtain Theorem 2.16 from Corollary 2.15, we may take any closed smooth curve Γ which contains the interval $[a,b]$ and continue the coefficients $c(t), d(t)$ to $\Gamma\backslash[a,b]$ as $c(t) \equiv 1$ and $d(t) \equiv 0$. It is known that the exponent $p(t)$ may be also continued with the preservation of the log-condition (see [22], Chapter 6, Section 2, where an extension of functions with preservation of the continuity modulus is described in case of domains in $\mathbb{R}^n$; this gives a similar result for smooth curves). Formula (2.39) for the index may be obtained from the general formula (2.34) by direct recalculation (see Remark 2.15 and formula (2.33) in [8] for similar formulas in the non-weighted case and for constant p). ∎

3. On Fredholmness of the operator M

In the sequel $-\infty < a < b < \infty$. We consider the equation

$$\begin{aligned}(M\varphi)(x) &\equiv \int_a^x \frac{u(x,t)\varphi(t)dt}{(x-t)^{1-\alpha}} + \int_x^b \frac{v(x,t)\varphi(t)dt}{(t-x)^{1-\alpha}} \\ &=: (M_1\varphi)(x) + (M_2\varphi)(x) = f(x),\end{aligned} \tag{3.40}$$

where $0 < \alpha < 1$, and the functions $u(x,t)$ and $v(x,t)$ are supposed to satisfy the following assumptions:

1°. $u(x,t)$ and $v(x,t)$ are Hölderian in x uniformly in t:

$$|u(x_1,t)-u(x_2,t)| \leqq A|x_1-x_2|^\lambda, \quad x_1 \geqq t, x_2 \geqq t, \tag{3.41}$$

$$|v(x_1,t)-v(x_2,t)| \leqq B|x_1-x_2|^\lambda, \quad x_1 \leqq t, x_2 \leqq t, \tag{3.42}$$

where $\alpha < \lambda \leq 1$ and $A > 0$ and $B > 0$ do not depend on x_1, x_2 and t;

2°. $u(x,x) := u(x, x-0) \in C([a,b])$, $v(x,x) := v(x, x+0) \in C([a,b])$.

We consider the operator M in the space $L^{p(\cdot)}[(a,b),\varrho]$ with the weight

$$\varrho(x) = (x-a)^{\mu(x)}(b-x)^{\nu(x)} \tag{3.43}$$

where the exponents $\mu(x), \nu(x) \in L^\infty(a,b)$ have finite limits $\mu(a) = \lim\limits_{x\to a}\mu(x)$, $\nu(b) = \lim\limits_{x\to b}\nu(x)$ and satisfy the log-conditions at the end-points:

$$|\mu(x)-\mu(a)| \leqq \frac{A}{\ln\frac{D}{x-a}}, \ \ |\nu(x)-\nu(b)| \leqq \frac{A}{\ln\frac{D}{b-x}}, \quad D = 2(b-a). \tag{3.44}$$

Under conditions (3.44) we have

$$(x-a)^{\mu(x)}(b-x)^{\nu(x)} \approx (x-a)^{\mu(a)}(b-x)^{\nu(b)}. \tag{3.45}$$

We represent the operator M as

$$M = M_0 + T_1 + T_2, \tag{3.46}$$

where

$$(M_0\varphi)(x) \equiv \int_a^x \frac{u(t,t)\varphi(t)dt}{(x-t)^{1-\alpha}} + \int_x^b \frac{v(t,t)\varphi(t)dt}{(t-x)^{1-\alpha}}, \tag{3.47}$$

and

$$\begin{aligned}(T_1\varphi)(x) &= \int_a^x \frac{u(x,y)-u(y,y)}{(x-y)^{1-\alpha}}\varphi(y)dy,\\ (T_2\varphi)(x) &= \int_x^b \frac{v(x,y)-v(y,y)}{(y-x)^{1-\alpha}}\varphi(y)dy.\end{aligned} \tag{3.48}$$

3.1. A representation for the operator M_0

The boundedness of the operator M_0 in the space $L^{p(\cdot)}[(a,b),\varrho]$, under suitable conditions on ϱ, follows from Theorem 2.4 and the fact that the functions $u(t,t)$ and $v(t,t)$ are bounded. However, we need a stronger statement, namely its boundedness from $L^{p(\cdot)}[(a,b),\varrho]$ into $L^{\alpha,p(\cdot)}[(a,b),\varrho]$. That statement, given in Corollary 3.18 will follow from Theorem 2.10 and the next theorem.

THEOREM 3.17. *Let $p(x) \in \mathcal{P}(a,b)\bigcap w\text{-}Lip\,(a,b)$ and $\varrho(x)$ be weight of form* (3.43) *with the exponents $\mu(x)$ and $\nu(x)$ satisfying assumption* (3.44) *and the conditions*

$$\alpha - \frac{1}{p(a)} < \mu(a) < \frac{1}{p'(a)}, \quad -\frac{1}{p(b)} < \nu(b) < \frac{1}{p'(b)}. \tag{3.49}$$

Then on functions $\varphi \in L^{p(\cdot)}[(a,b),\varrho]$ the operator M_0 can be represented in the form

$$M_0\varphi = \Gamma(\alpha) I_{a+}^{\alpha} N_{\alpha}\varphi, \tag{3.50}$$

where

$$N_{\alpha}\varphi = a_1(x)\varphi(x) + \frac{1}{\pi}\int_a^b \left(\frac{y-a}{x-a}\right)^{\alpha} \frac{a_2(y)}{y-x}\varphi(y)dy \tag{3.51}$$

and $a_1(x) = u(x,x) + v(x,x)\cos\alpha\pi$, $a_2(x) = v(x,x)\sin\alpha\pi$.

P r o o f. The representation (3.50)-(3.51) is known within the frameworks of constant exponents p, see [21, p. 653], so it is valid for C_0^{∞}-functions.

Using Theorem 2.1 and taking into account (3.44) and (2.4), we see that C_0^{∞} is dense in the $L^{p(\cdot)}[(a,b),\varrho]$. Therefore, the validity of representation (3.50) should follow from the boundedness in $L^{p(\cdot)}[(a,b),\varrho]$ of all the operators involved in (3.50). The boundedness of the operator M_0 was already observed above. The operator I_{a+}^{α} is bounded according to Theorem 2.4 (take $\varrho(x) \equiv 1$ in Theorem 2.4). To state that the operator N_{α} is bounded in $L^{p(\cdot)}[(a,b),\varrho]$, it suffices to observe that $a_1(x)$ in the first term on the right-hand side of (3.51) is a bounded function, while the boundedness in $L^{p(\cdot)}[(a,b),\varrho]$ of the second term is equivalent to the boundedness in $L^{p(\cdot)}[(a,b),\varrho]$ of the singular operator S in the space $L^{p(\cdot)}[(a,b),\varrho_1]$ with $\varrho_1(x) = (x-a)^{\mu(x)-\alpha}(b-x)^{\nu(x)}$. The latter is covered by Theorem 2.5. ∎

COROLLARY 3.18. *Under the assumptions of Theorem 3.17, the operator M_0 is bounded from the weighted Lebesgue space $L^{p(\cdot)}[(a,b),\varrho]$ to the weighted fractional Sobolev space $L^{\alpha,p(\cdot)}[(a,b),\varrho]$.*

3.2. On compactness of the operators T_1 and T_2

We have to prove the compactness of the operators T_1 and T_2 from $L^{p(\cdot)}[(a,b), \varrho]$ to $L^{\alpha,p(\cdot)}[(a,b), \varrho]$. It suffices to consider only the operator T_1, the proof for T_2 being symmetrical (one can also reduce the case of T_2 to the case of T_1 by the reflection change of variables $x = a + b - x_1$, $y = a + b - y_1$ and working with the reflected variable exponent $p_1(x_1) = p(a + b - x_1)$.

We will base ourselves on the following Krasnoselski type dominated compactness theorem for integral operators

$$\mathbb{K}f(x) = \int_\Omega \mathcal{K}(x,y)f(y)d\mu(y), \quad \Omega \subset \mathbb{R}^n, \tag{3.52}$$

proved for variable exponents in [17]. We recall that the $\mathbb{K}$ is called a regular operator from a Banach space X into a Banach space Y, if a similar operator with the kernel $|\mathcal{K}(x,y)|$ is bounded from X to Y. By $W_{p(\cdot)}(\Omega)$ we denote the class of weights ϱ of the form

$$\varrho(x) = \prod_{k=1}^{N} |x - x_k|^{\beta_k}, \quad x_k \in \Omega, \tag{3.53}$$

where $-\frac{n}{p(x_k)} < \beta_k < \frac{n}{q(x_k)}, \quad k = 1, 2, ..., N.$

PROPOSITION 3.19. *Let $\mathbb{K}$ and $\mathbb{K}_0$ be two regular linear integral operators of form* (3.52) *with the kernels $\mathcal{K}(x,y)$ and $\mathcal{K}_0(x,y)$, acting from the space $L^{p_1(\cdot)}(\Omega, \varrho_1)$ into $L^{p_2(\cdot)}(\Omega, \varrho_2)$, where $|\Omega| < \infty$, let $p_1(\cdot)$ and $p_2(\cdot)$ satisfy conditions* (2.2)-(2.4) *and $\varrho_j \in W_{p_j(\cdot)}(\Omega), j = 1, 2$. If*

$$|\mathcal{K}(x,t)| \le \mathcal{K}_0(x,t),$$

and the majorizing operator $\mathbb{K}_0$ is compact from $L^{p_1(\cdot)}(\Omega, \varrho_1)$ to $L^{p_2(\cdot)}(\Omega, \varrho_2)$, then the operator $\mathbb{K}$ is also compact.

By means of Proposition 3.19, in [17] there was also proved the following weighted compactness theorem, which we formulate here for the one-dimensional case.

PROPOSITION 3.20. *Let $\Omega = (a,b), -\infty < a, b < \infty$. Under the conditions $A(x,y) \in L^\infty(\Omega \times \Omega)$ and $\operatorname{ess\,inf}_{y\in\Omega} \alpha(y) > 0$, the operator*

$$\int_\Omega \frac{A(x,y)}{|x-y|^{1-\alpha(y)}}\, \varphi(y)dy$$

is compact in the space $L^{p(\cdot)}(\Omega, \varrho)$, if $p(\cdot) \in \mathcal{P}(\Omega) \bigcap w\text{-}Lip(\Omega)$ and ϱ is a weight of form (3.43) *with the functions μ, ν satisfying conditions* (2.18), (3.44).

THEOREM 3.21. *Let $u(x,t)$ satisfy assumption 1°. The operator T_1 is compact from $L^{p(\cdot)}[(a,b),\varrho]$ into $L^{\alpha,p(\cdot)}[(a,b),\varrho]$, $-\infty < a < b < \infty$, if $p(\cdot) \in \mathcal{P}(\Omega)\bigcap w\text{-}Lip[(a,b)]$ and ϱ is a weight of form* (3.43) *with the functions μ, ν satisfying conditions* (2.18), (3.44).

P r o o f. We will use the fact that the operator T_1 is known, see [21, p. 654], to be representable as a composition of the fractional operator I^{α}_{a+} with an operator having a weak singularity:

$$T_1\varphi = I^{\alpha}_{a+}V_1\varphi, \tag{3.54}$$

where

$$V_1\varphi = \frac{\alpha}{\Gamma(1-\alpha)}\int_a^x \varphi(s)K(x,s)ds, \quad K(x,s) = \int_s^x \frac{u(x,s)-u(t,s)}{(t-s)^{1-\alpha}(x-t)^{1+\alpha}}ds. \tag{3.55}$$

Then by Theorems 2.10 and 2.8, we have only to prove that the operator V_1 is compact in the space $L^{p(\cdot)}[(a,b),\varrho]$. From (3.41) it easily follows that

$$|K(x,s)| \le c(x-s)^{\lambda-1}.$$

Then the compactness of the operator V_1 follows from Proposition 3.19 and the compactness of operators with weak singularity in variable exponent spaces. The latter was proved in [13] in non-weighted case and in [17] in the weighted case, as presented in Proposition 3.20. ∎

COROLLARY 3.22. *Let $u(x,t)$ and $v(x,t)$ satisfy assumption 1°-2°. The operators M_1 and M_2 are bounded from $L^{p(\cdot)}[(a,b),\varrho]$ into $L^{\alpha,p(\cdot)}[(a,b),\varrho]$, $-\infty < a < b < \infty$, if $p(\cdot) \in \mathcal{P}(\Omega)\bigcap w\text{-}Lip[(a,b)]$ and ϱ is a weight of the form* (3.43) *with the functions μ, ν satisfying conditions* (2.18), (3.44).

P r o o f. We have

$$M_1\varphi = \int_a^x \frac{u(t,t)\varphi(t)}{(x-t)^{1-\alpha}}dt + T_1\varphi,$$

where T_1 is the operator defined in (3.48). By representation (3.54), we obtain

$$M_1\varphi = I^{\alpha}_{a+}\left(\mathfrak{u}\,\varphi + V_1\varphi\right), \tag{3.56}$$

where $\mathfrak{u}(x) = u(x,x)$. The boundedness of M_1 in $L^{\alpha,p(\cdot)}[(a,b),\varrho]$ follows from the boundedness in $L^{p(\cdot)}[(a,b),\varrho]$ of the operator $\mathfrak{u}I + V_1$. Similarly the operator M_2 is considered. ∎

3.3. Fredholmness statement for the operator $M:\ L^{p(\cdot)}[(a,b),\varrho]\to L^{\alpha,p(\cdot)}[(a,b),\varrho]$

Observe that the function $\theta(x)$, introduced in (2.36), for the operator (3.51) takes the form

$$\theta(x)=\arg\frac{u(x,x)+v(x,x)e^{-\alpha\pi i}}{u(x,x)+v(x,x)e^{\alpha\pi i}} \tag{3.57}$$

and as usual we choose the initial value of $\arg$ by the condition $0\le\theta(a)<2\pi$.

THEOREM 3.23. *Let the functions $u(x,t)$ and $v(x,t)$ satisfy assumptions 1° and 2°. Then the operator M is Fredholm from $L^{p(\cdot)}[(a,b),\varrho]$ into $L^{\alpha,p(\cdot)}[(a,b),\varrho]$ with $p(x)\in\mathcal{P}(\Omega)\bigcap w\text{-}Lip(\Omega)$ and the weight ϱ of form* (3.43) *satisfying conditions* (3.44) *and* (3.49)*, if and only if:*

$$\text{i) } u^2(x,x)+v^2(x,x)\neq 0,\ x\in[a,b]; \tag{3.58}$$

$$\text{ii) } \frac{\theta(a)}{2\pi}\neq\frac{1}{p(a)}+\mu(a)-\alpha\pmod 1,\quad \frac{\theta(b)}{2\pi}\neq\frac{1}{p'(b)}-\nu(b)\pmod 1. \tag{3.59}$$

These conditions being satisfied, the index of the operator M is given by

$$\operatorname{Ind}M=\left[\frac{1}{p(a)}+\mu(a)-\alpha-\frac{\theta(a)}{2\pi}\right]+\left[\frac{1}{p(b)}+\nu(b)+\frac{\theta(b)}{2\pi}\right]. \tag{3.60}$$

P r o o f. By representation (3.46), compactness of T_1, T_2 and by (3.50), the investigation of the Fredholmness of the operator M from $L^{p(\cdot)}[(a,b),\varrho]$ into $L^{\alpha,p(\cdot)}[(a,b),\varrho]$ is equivalent to that of the operator N_α in $L^{p(\cdot)}[(a,b),\varrho]$, the latter being equivalent to the study of the Fredholmness of the singular integral operator

$$a_1(x)\varphi(x)+\frac{1}{\pi}\int_a^b\frac{a_2(y)}{y-x}\varphi(y)dy$$

in the space $L^{p(\cdot)}[(a,b)\varrho^\star]$ where $\varrho^\star(x)=(x-a)^{\mu(x)-\alpha}(b-x)^{\nu(x)}$. The result on its Fredholmness follows from Theorem 2.16 with $\mathcal{A}(x)=a_1(x)+a_2(x)$ and $\mathcal{B}(x)=a_1(x)-a_2(x)$, if we take (3.45) into account. ∎

3.4. Closed form formulas for solutions $\varphi\in L^{p(\cdot)}[(a,b),\varrho]$ of the equation $M_0\varphi=f$ with $f\in L^{\alpha,p(\cdot)}[(a,b),\varrho]$

We consider the equation $M_0\varphi=f$ in the form

$$\int_a^x\frac{u(t)\varphi(t)dt}{(x-t)^{1-\alpha}}+\int_x^b\frac{v(t)\varphi(t)dt}{(t-x)^{1-\alpha}}=f(x), \tag{3.61}$$

known as the generalized Abel equation, see [21, Ch. 6]. The functions u and v are assumed to be continuous on $[a,b]$. Since the equation $M_0\varphi = f$ was reduced to the singular integral equation, we may use the well known fact from the theory of the latter to obtain the closed form expression for the general solution of the equation $M_0\varphi = f$, assuming that the Fredholmness conditions are satisfied. The Fredholmness of the equation $M_0\varphi = f$ has already been characterized in Theorem 3.23. In view of conditions (3.59) and (3.49), we have the only two possibilities for $\theta(a)$:

$$0 \le \frac{\theta(a)}{2\pi} < \frac{1}{p(a)} - \alpha + \mu(a) \quad \text{or} \quad \frac{1}{p(a)} - \alpha + \mu(a) < \frac{\theta(a)}{2\pi} < 1$$

and similarly two possibilities for $T = \frac{\theta(b)}{2\pi} - \left[\frac{\theta(b)}{2\pi}\right]$:

$$0 \le T < \frac{1}{p'(b)} - \nu(b) \quad \text{or} \quad \frac{1}{p'(b)} - \nu(b) < T < 1.$$

To write down the general solution of the equation $M_0\varphi = f$ in the weighted space $L^{p(\cdot)}[(a,b),\varrho]$ where $f \in L^{\alpha,p(\cdot)}[(a,b),\varrho]$, we need the following numbers:

$$\lambda_a(p) = \begin{cases} -\frac{\theta(a)}{2\pi}, & \text{if } 0 \le \frac{\theta(a)}{2\pi} < \frac{1}{p(a)} - \alpha + \mu(a); \\ 1 - \frac{\theta(a)}{2\pi}, & \text{if } \frac{1}{p(a)} - \alpha + \mu(a) < \frac{\theta(a)}{2\pi} < 1, \end{cases} \tag{3.62}$$

and

$$\lambda_b(p) = \begin{cases} T, & \text{if } 0 \le T < \frac{1}{p'(b)} - \nu(b); \\ T - 1, & \text{if } \frac{1}{p'(b)} - \nu(b) < T < 1. \end{cases} \tag{3.63}$$

For brevity we define $\varkappa = \operatorname{Ind} M$, where $\operatorname{Ind} M$ is given by (3.60).

THEOREM 3.24. *Let conditions* (3.58)-(3.59) *hold, the functions* $u(x,t)$ *and* $v(x,t)$ *satisfy assumptions 1° and 2°,* $p(x) \in \mathcal{P}(\Omega) \bigcap w\text{-}Lip\,(\Omega)$ *and* ϱ *be of the form* (3.43).

If $\varkappa > 0$, *then for every* $f \in L^{\alpha,p(\cdot)}[(a,b),\varrho]$ *the equation* $M_0\varphi = f$ *is unconditionally solved in the space* $L^{p(\cdot)}[(a,b),\varrho]$ *and all its solutions in this space are given by the following formula*

$$\varphi(x) = \frac{v(x)}{u^2(x)+v^2(x)}(x-a)^{\lambda_a(p)}(b-x)^{\lambda_b(p)}\mathcal{Z}(x)P_{\varkappa-1}(x)$$
$$+\frac{u(x)f(x)}{u^2(x)+v^2(x)} - \frac{v(x)}{u^2(x)+v^2(x)}\frac{(x-a)^{\lambda_a(p)}(b-x)^{\lambda_b(p)}\mathcal{Z}(x)}{\pi}$$
$$\times \int_a^b \frac{(\mathbb{D}^{\alpha}_{a+}f)(t)\,dt}{(t-a)^{\lambda_a(p)}(b-t)^{\lambda_b(p)}\mathcal{Z}(t)(t-x)},$$

where

$$P_{\varkappa-1}(x)=\sum_{k=0}^{\varkappa-1}c_kx^k$$

is a polynomial of degree $\varkappa-1$ *with arbitrary coefficients and* $\mathcal{Z}(x)$ *is a non-vanishing continuous function defined by*

$$\mathcal{Z}(x)=\exp\Psi(x),\quad \Psi(x)=\frac{1}{2\pi}\int_a^b\frac{\theta(t)}{t-x}dt+\frac{\theta(a)}{2\pi}\ln(x-a)-\frac{\theta(b)}{2\pi}\ln(b-x).$$

If $\varkappa<0$*, the equation is solvable in the space* $L^{p(\cdot)}[(a,b),\varrho]$*, if and only if* f *satisfies the conditions*

$$\int_a^b\frac{f(t)t^kdt}{(t-a)^{\mu_a(p)}(b-t)^{\mu_b(p)}\mathcal{Z}(t)}=0,\quad k=1,2,\dots,|\varkappa|.$$

P r o o f. By the standard arguments well known in the theory of singular integral equations, it may be checked that the above formula gives the general solution in the space $L^{p(\cdot)}[(a,b),\varrho]$, if the singular operator involved in that formula, is bounded in this space. By Theorem 2.5 this is the case under the choice (3.62)-(3.63) of $\lambda_a(p)$ and $\lambda_b(p)$. ∎

We dwell on a special case when $u(t)=u=const$ and $v(t)=v=const$:

$$\int_a^b\frac{C_1+C_2\text{sign}\,(x-t)}{|x-t|^{1-\alpha}}\varphi(t)\,dt=f(x),\tag{3.64}$$

where $C_1+C_2=u, C_1-C_2=v$. This equation attracted attention of various authors in view of applications of such equation, see [21, Ch. 6] and references therein. We have denoted

$$\theta=\arg\frac{u+ve^{-i\alpha\pi}}{u+ve^{i\alpha\pi}}\in(0,2\pi),$$

and also,

$$A=\frac{1}{p(a)}+\mu(a)-\alpha,\quad B=\frac{1}{p'(b)}+\nu(b).$$

Note that within the frameworks of the well-posedness of the equation, that is, in the case where the operator generated by the left-hand side of the equation is bounded from $L^{p(\cdot)}[(a,b),\varrho]$ to $L^{\alpha,p(\cdot)}[(a,b),\varrho]$, we have

$$0<A<1-\alpha,\quad 0<B<1,$$

according to conditions (3.49). Recall also that the Fredholmness conditions imply the following restrictions on the value of θ:

$$\frac{\theta}{2\pi} \neq A \quad \text{and} \quad \frac{\theta}{2\pi} \neq B.$$

A direct calculation by formula (3.60) shows that the index $\varkappa$ may have only values $-1, 0$ and 1:

$$\varkappa = \begin{cases} 0, & \text{if } \frac{\theta}{2\pi} < \min(A,B) \text{ or } \frac{\theta}{2\pi} > \max(A,B) \\ -1, & \text{if } A < B \text{ and } A < \frac{\theta}{2\pi} < B, \\ +1, & \text{if } A > B \text{ and } B < \frac{\theta}{2\pi} < A. \end{cases}$$

Thus, we arrive at the following statement.

THEOREM 3.25. *Under the conditions of Theorem 3.24 on the exponent $p(x)$ and the weight $\varrho(x)$, equation* (3.64) *with $f \in L^{\alpha,p(\cdot)}[(a,b),\varrho]$ is uniquely and unconditionally solvable in the space $L^{p(\cdot)}[(a,b),\varrho]$ if and only if*

$$\frac{\theta}{2\pi} < \min\left(\frac{1}{p(a)} + \mu(a) - \alpha, \frac{1}{p'(b)} + \nu(b)\right),$$

or

$$\frac{\theta}{2\pi} > \max\left(\frac{1}{p(a)} + \mu(a) - \alpha, \frac{1}{p'(b)} + \nu(b)\right).$$

The solution of the equation is derived from the general formula of Theorem 3.24 with $P_{\varkappa-1} \equiv 0$ and $Z(x) \equiv 1$.

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