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DESIGN OF FRACTIONAL ORDER DIGITAL DIFFERENTIATORS AND INTEGRATORS USING INDIRECT DISCRETIZATION

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Abstract

In this paper, design of fractional order digital differentiators and integrators using indirect discretization is presented. The proposed approach is based on using continued fraction expansion to find the rational approximation of the fractional order operator, s^{α}. The rational approximation thus obtained is discretized by using s to z transforms. The proposed approach is tested for differentiators and integrators of orders $\frac{1}{4}$ and $\frac{1}{2}$. The results obtained compare favorably with the ideal characteristics

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1. Introduction

Fractional calculus is an old topic dealing with the generalization of the integration and differentiation to an arbitrary order, see e.g. [1]. Nowadays, the fractional calculus theory is applied in almost all the areas of science and engineering. There are different commonly used definitions for fractional order integration and differentiation operators, one of them is the Grünwald-Letnikov definition

$$D^{\alpha} f(t) = \lim_{\tau \to 0} \frac{1}{\tau^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t - k\tau), \tag{1.1}$$

where $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$. The fractional order systems involve fractional order differential equations, examples of such are: the fractance device, semi-infinite lossy transmission line, diffusion of heat into the semi-infinite solid, fractional order differentiators and integrators, $PI^{\lambda}D^{\mu}$ – controllers, etc (see for example [10], [11]). Fractional order differentiators and integrators are used to determine the fractional order time derivatives and integrals of the applied input signal, see [4]. They find applications also in control systems, signal processing, bio-medical engineering, radars, sonars etc, see [4]-[6]. A fractional order differentiator or integrator is defined by

$$H(s) = s^{\alpha}\,, \tag{1.2}$$

where α could be either positive or negative, for differentiator or integrator respectively. The design of digital differentiators and integrators involves the discretization of the fractional-order operators s^{α}, see [5]. The direct and the indirect discretizations are commonly used discretization techniques. The direct discretization involves direct power series expansion or continued fraction expansion of the s to z transform. Chen, Petras and Vinagre [5]-[6] have used direct discretization techniques in designing fractional order differentiators and integrators. In the indirect discretization, first, rational approximation for $H(s)$ in s-domain is obtained by limiting its order and then, it is digitized.

In this paper, the indirect discretization technique is followed. The paper is organized as follows. In Section 2, we present indirect discretization technique. Section 3 deals with the magnitude and phase responses. Finally, conclusions are drawn in Section 4.

2. Indirect discretization

A.N. Khovanskii [9] has obtained the continued fraction expansion for $(1+x)^{\alpha}$ as

$$(1+x)^{\alpha} = 1 + \frac{\alpha x}{1+}\,\frac{(1-\alpha)x}{2+}\,\frac{(1+\alpha)x}{3+}\,\frac{(2-\alpha)x}{2+\,\cdots}\,. \tag{2.1}$$

The above continued fraction expansion is shown to be convergent, see [9]. Replacing x by $s-1$, the numerator and denominator polynomials in the rational approximation limited to the fifth order:

$$s^{\alpha} = \frac{P_0 s^5 + P_1 s^4 + P_2 s^3 + P_3 s^2 + P_4 s + P_5}{Q_0 s^5 + Q_1 s^4 + Q_2 s^3 + Q_3 s^2 + Q_4 s + Q_5} \tag{2.2}$$

are obtained using the first ten terms of expansion (2.1) and are as below:

$$\begin{aligned}
P_0 = Q_5 &= -\alpha^5 - 15\alpha^4 - 85\alpha^3 - 225\alpha^2 - 274\alpha - 120\\
P_1 = Q_4 &= 5\alpha^5 + 45\alpha^4 + 5\alpha^3 - 1005\alpha^2 - 3250\alpha - 3000\\
P_2 = Q_3 &= -10\alpha^5 - 30\alpha^4 + 410\alpha^3 + 1230\alpha^2 - 4000\alpha - 12000\\
P_3 = Q_2 &= 10\alpha^5 - 30\alpha^4 - 410\alpha^3 + 1230\alpha^2 + 4000\alpha - 12000\\
P_4 = Q_1 &= -5\alpha^5 + 45\alpha^4 - 5\alpha^3 - 1005\alpha^2 + 3250\alpha - 3000\\
P_5 = Q_0 &= \alpha^5 - 15\alpha^4 + 85\alpha^3 - 225\alpha^2 + 274\alpha - 120\,.
\end{aligned} \tag{2.3}$$

The rational approximation thus obtained is to be discretized using s to z transforms. The most efficient s to z transforms are the Al-Alaoui and the bilinear transforms as defined in [2],[3],[7],[8]:

$$s = \frac{8(z-1)}{7T(z+1/7)}\,, \tag{2.4}$$

$$s = \frac{2(z-1)}{T(z+1)}\,. \tag{2.5}$$

The discretized transfer function in the z-domain is obtained by employing (2.4) and (2.5) in (2.2), and is given by $H(z)$ as follows:

$$H(z) = \frac{A_0z^5 + A_1z^4 + A_2z^3 + A_3z^2 + A_4z + A_5}{B_0z^5 + B_1z^4 + B_2z^3 + B_3z^2 + B_4z + B_5}\,, \tag{2.6}$$

where the digital filter coefficients $A_0, A_1, \ldots, A_5$ and $B_0, B_1, \ldots, B_5$ are given in **Table. 1**.

3. Simulation results

To verify the effectiveness of the indirect discretization technique, α is chosen as $\frac{1}{4}$ and $\frac{1}{2}$. The magnitude and phase responses,pole-zero diagrams of the integrators and differentiators evaluated at T=1 sec are shown in **Figs. 1-4**. **Figures 1 and 2** depict the magnitude and phase responses of differentiators and integrators of order $\frac{1}{4}$ using the bilinear and the Al-Alaoui transforms. **Figures 3 and 4** depict the magnitude and phase responses of differentiators and integrators of order $\frac{1}{2}$ for the same transforms. It can be inferred that the Al-Alaoui transform improves the high frequency magnitude response compared to the bilinear transform, whereas the bilinear transform provides better phase response compared to the Al-Alaoui transform. **Figures 5 and 6** are the pole-zero diagrams of differentiators and integrators of order $\frac{1}{2}$ and $\frac{1}{4}$. One can observe from these figures that the poles and zeros are lying inside of the unit circle. Further, the poles and zeros are interlacing on the segment of the real axis. So the proposed differentiators and integrators are stable and are of minimum phase.

Bilinear Transform	Al-Alaoui Transform
$A_0 = 32P_0 + 16P_1T + 8P_2T^2 + 4P_3T^3 + 2P_4T^4 + P_5T^5$	$A_0 = 32768P_0 + 28672P_1T + 25088P_2T^2 + 21952P_3T^3 + 19208P_4T^4 + 16807P_5T^5$
$A_1 = -160P_0 - 48P_1T - 8P_2T^2 + 4P_3T^3 + 6P_4T^4 + 5P_5T^5$	$A_1 = -163840P_0 - 110592P_1T - 68096P_2T^2 - 34496P_3T^3 - 8232P_4T^4 + 12005P_5T^5$
$A_2 = -320P_0 - 32P_1T + 16P_2T^2 + 8P_3T^3 - 4P_4T^4 - 10P_5T^5$	$A_2 = 327680P_0 + 155648P_1T + 54272P_2T^2 + 4480P_3T^3 - 8624P_4T^4 + 3430P_5T^5$
$A_3 = 320P_0 + 32P_1T - 16P_2T^2 - 8P_3T^3 + 4P_4T^4 + 10P_5T^5$	$A_3 = -327680P_0 - 90112P_1T - 5120P_2T^2 + 6784P_3T^3 - 2128P_4T^4 + 490P_5T^5$
$A_4 = 160P_0 - 48P_1T + 8P_2T^2 + 4P_3T^3 - 6P_4T^4 + 5P_5T^5$	$A_4 = 163840P_0 + 12288P_1T - 5632P_2T^2 + 1216P_3T^3 - 216P_4T^4 + 35P_5T^5$
$A_5 = -32P_0 - 32P_1T - 8P_2T^2 + 4P_3T^3 - 2P_4T^4 + P_5T^5$	$A_5 = -32768P_0 + 4096P_1T - 512P_2T^2 + 64P_3T^3 - 8P_4T^4 + P_5T^5$
$B_0 = 32Q_0 + 16Q_1T + 8Q_2T^2 + 4Q_3T^3 + 2Q_4T^4 + Q_5T^5$	$B_0 = 32768Q_0 + 28672Q_1T + 25088Q_2T^2 + 21952Q_3T^3 + 19208Q_4T^4 + 16807Q_5T^5$
$B_1 = -160Q_0 - 48Q_1T - 8Q_2T^2 + 4Q_3T^3 + 6Q_4T^4 + 5Q_5T^5$	$B_1 = -163840Q_0 - 110592Q_1T - 68096Q_2T^2 - 34496Q_3T^3 - 8232Q_4T^4 + 12005Q_5T^5$
$B_2 = 320Q_0 + 32Q_1T - 16Q_2T^2 - 8Q_3T^3 + 4Q_4T^4 + 10Q_5T^5$	$B_2 = 327680Q_0 + 155648Q_1T + 54272Q_2T^2 + 4480Q_3T^3 - 8624Q_4T^4 + 3430Q_5T^5$
$B_3 = -320Q_0 + 32Q_1T + 16Q_2T^2 - 8Q_3T^3 - 4Q_4T^4 + 10Q_5T^5$	$B_3 = -327680Q_0 - 90112Q_1T - 5120Q_2T^2 + 6784Q_3T^3 - 2128Q_4T^4 + 3430Q_5T^5$
$B_4 = 160Q_0 - 48Q_1T + 8Q_2T^2 + 4Q_3T^3 - 6Q_4T^4 + 5Q_5T^5$	$B_4 = 163840Q_0 + 12288Q_1T - 5632Q_2T^2 + 1216Q_3T^3 - 216Q_4T^4 + 35Q_5T^5$
$B_5 = -32Q_0 + 16Q_1T - 8Q_2T^2 + 4Q_3T^3 - 2Q_4T^4 + Q_5T^5$	$B_5 = -32768Q_0 + 4096Q_1T - 512Q_2T^2 + 646Q_3T^3 - 8Q_4T^4 + Q_5T^5$

Table 1: Expressions for the digital filter coefficients

4. Conclusions

In this paper a design of fractional order digital differentiators and integrators using indirect discretization technique has been presented. First the rational approximation for the fractional order operator is calculated and then it is digitized using s to z transforms. The magnitude response obtained by using the Al-Alaoui transform is more closer to the ideal one, compared to the bilinear transform. But the phase response is better when the bilinear transform is used. The differentiators and integrators obtained are stable and minimum phase. The thus proposed approach seems to be simple and accurate.

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For Figures 1 − 6, see next 3 pages

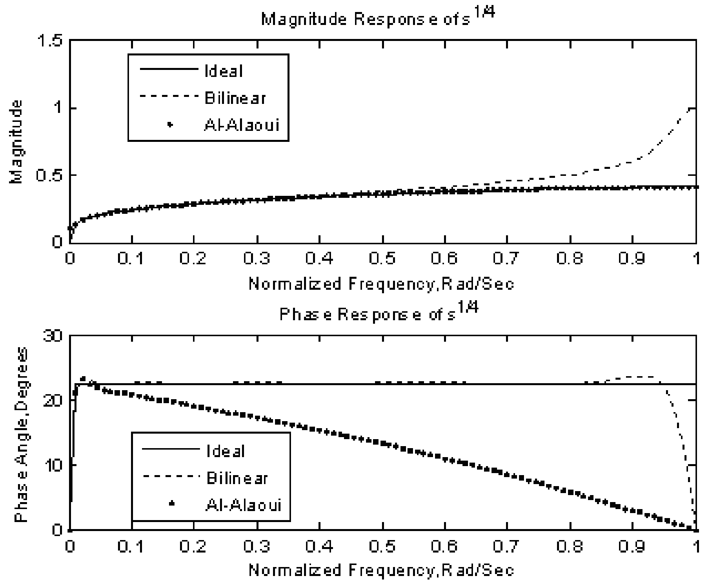

Figure 1: Magnitude and phase responses of digital differentiator of order $\frac{1}{4}$

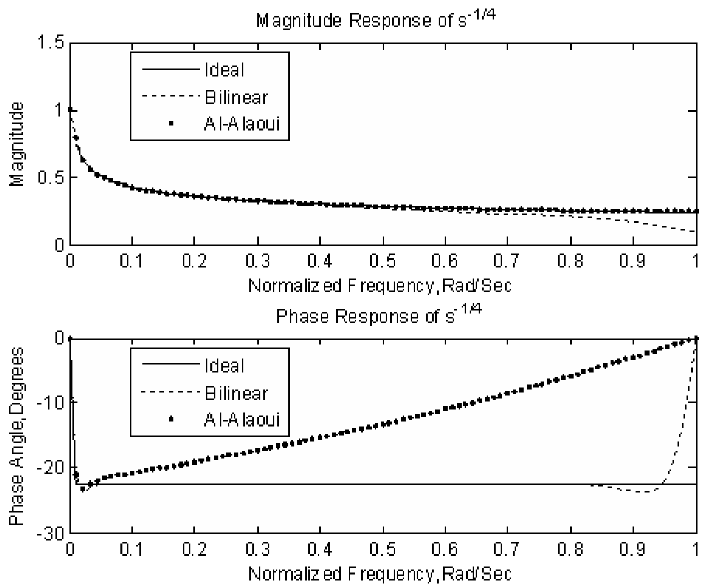

Figure 2: Magnitude and phase responses of digital integrator of order $\frac{1}{4}$

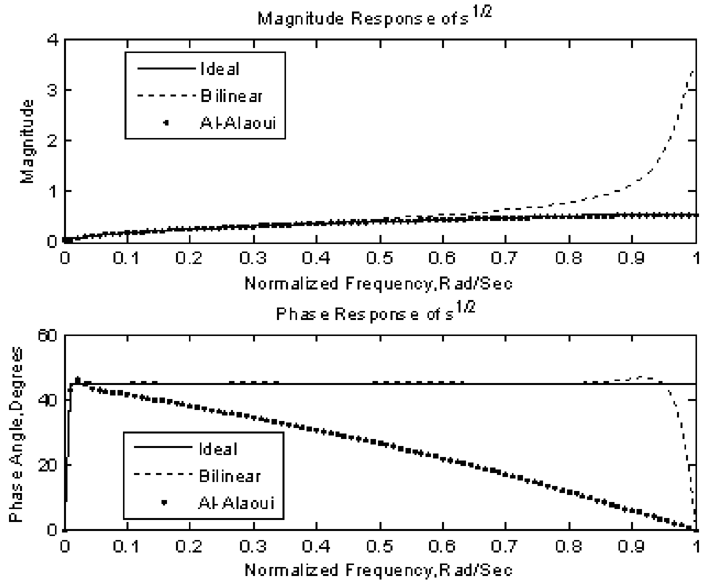

Figure 3: Magnitude and phase responses of digital differentiator of order $\frac{1}{2}$

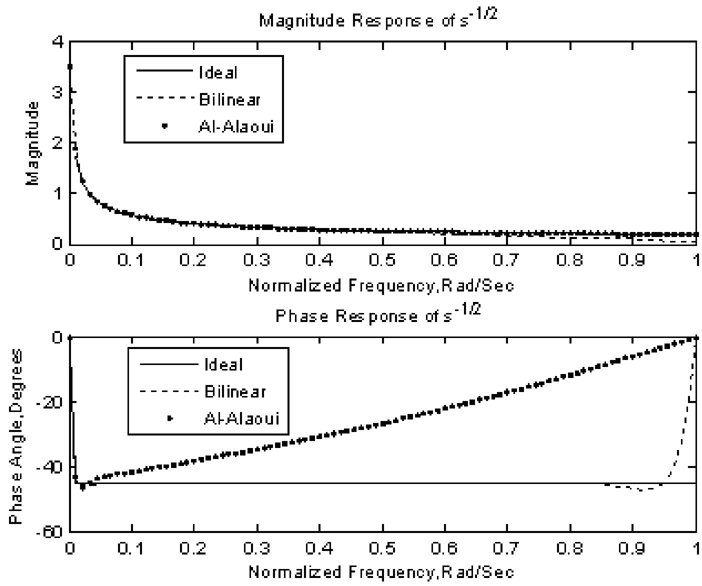

Figure 4: Magnitude and phase responses of digital integrator of order $\frac{1}{2}$

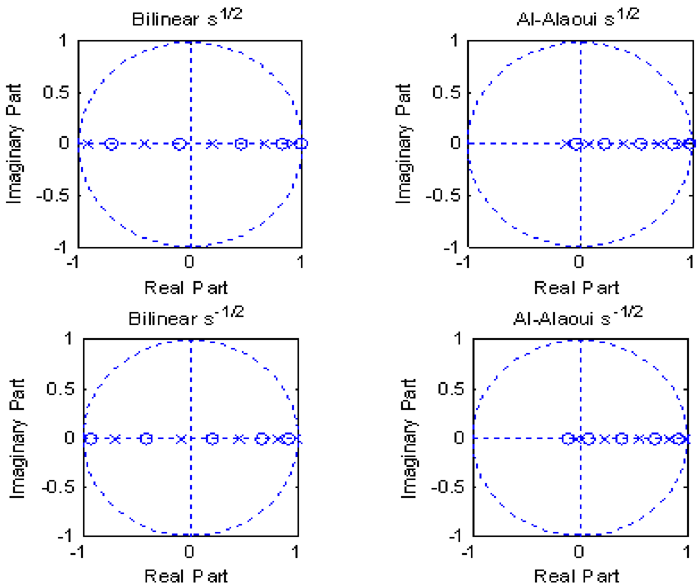

Figure 5: Pole-zero plot of differentiators and integrators of order $\frac{1}{2}$

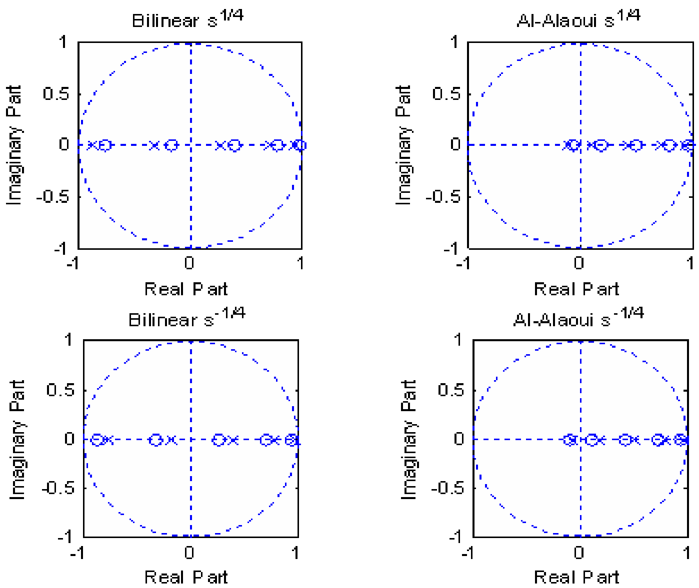

Figure 6: Pole-zero plot of differentiators and integrators of order $\frac{1}{4}$