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SOLVING FRACTIONAL DIFFUSION-WAVE EQUATIONS USING A NEW ITERATIVE METHOD [1]

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Dedicated to Professor Shyam Kalla
on his 70th anniversary

Abstract

In the present paper a New Iterative Method [1] has been employed to find solutions of linear and non-linear fractional diffusion-wave equations. Illustrative examples are solved to demonstrate the efficiency of the method.

Mathematics Subject Classification: 26A33, 31B10
Key Words and Phrases: Caputo fractional derivative, fractional initial value problems, New Iterative Method, fractional diffusion-wave equation

1. Introduction

A space-time fractional diffusion-wave equation is obtained from classical diffusion equation by replacing the second order space derivative by fractional derivative of order $\beta\,(1 < \beta \leq 2)$, and first order time derivative by fractional derivative of order $\alpha\,(0 < \alpha \leq 1)$, [2]. Similar generalizations of classical wave equation have been discussed in the literature [3]-[10]. Diffusion-wave equations involving Riemann-Liouville derivative [3], [11], Caputo derivative [4], [5], [9], [14], [15] and Grünwald-Letnikov derivative [16] have been discussed by various researchers. Fujita [3] has presented the existence and uniqueness of the solution of the following equation

$$D_t^{\alpha} u = D_x^{\beta} u, \quad 0 < \alpha \leq 1,\ 0 < \beta \leq 2. \tag{1}$$

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Schneider and Wyss [4] have shown that the time-fractional diffusion equation ($\beta = 2$) represents sub-diffusion for $0 < \alpha < 1$. It has further been observed that $1 < \alpha < 2$ represents enhanced diffusion only in one-dimension. The solutions need not remain non-negative and can not represent physical diffusion of any kind [4], [5] in higher dimensions for $\alpha > 1$.

Fractional diffusion-wave equation has been used widely in many branches of Science and Engineering. These equations represent propagation of mechanical waves in visco-elastic media [6], [7], [8], a non-Markovian diffusion process with memory [9], charge transport in amorphous semiconductors [10] and many more. Mainardi et al. [6], [7], [8] studied the fractional wave equation governing the propagation of mechanical diffusive waves in viscoelastic media which exhibit a power-law creep. Nigmatullin [11] used fractional diffusion-wave equation to model electromagnetic acoustic, and mechanical responses. Roman and Alemany [12] investigated a continuous time random walks on fractals. Giona, Cerbelli and Roman [13] studied the relaxation phenomena in complex viscoelastic material using fractional diffusion equations.

Various methods such as Green's function method [4], Finite sine transform method [14], Method of images and Fourier transform [9], Separation of variables method [15], Finite difference method [16], Adomian decomposition method (ADM) [17], [18] have been used to solve these equations. Recently, Daftardar-Gejji and Jafari [1] have devised a New Iterative Method (NIM) to solve non-linear functional equations. This method is free from rounding off errors since it does not involve discretization and has fairly simple algorithm. Also it does not require prior knowledge of the concepts such as Lagrange multipliers or homotopy.

In this article we employ the NIM ([1]) to solve the following fractional diffusion-wave equation

$$D_t^{\alpha} u\left(\bar{x}, t\right) = \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u\left(\bar{x}, t\right) + A\left(u\left(\bar{x}, t\right)\right), \quad 1 < \beta_i \leq 2, \tag{2}$$

where $\bar{x} = (x_1, \cdots, x_n) \in \Re^n$, a_i are constants, $A(u)$ is non-linear function of u, D_t^{α} and $D_{x_i}^{\beta_i}$ denote Caputo partial fractional derivatives with respect to t and with respect to x_i, respectively.

2. Preliminaries

In this section we set up the notations and recall some basic definitions (see for example, [19], [20]).

Definition 1.1. A real function $f(x),\ x > 0$ is said to be in space C_α, $\alpha \in \Re$, if there exists a real number $p\,(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 1.2. A real function $f(x),\ x > 0$ is said to be in space C_α^m, $m \in I\!N \bigcup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 1.3. Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann-Liouville integral of order μ, $\mu > 0$ is given by

$$I_t^\mu f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(x,\tau)\, d\tau, \quad t > 0. \tag{3}$$

Definition 1.4. The (left sided) Caputo partial fractional derivative of f with respect to t, $f \in C_{-1}^m, m \in I\!N \bigcup \{0\}$, is defined as:

$$\begin{aligned} D_t^\mu f(x,t) \quad &= \frac{\partial^m}{\partial t^m} f(x,t), \quad \mu = m \\ &= I_t^{m-\mu} \frac{\partial^m f(x,t)}{\partial t^m}, \quad m-1 < \mu < m, \quad m \in I\!N. \end{aligned} \tag{4}$$

Note that

$$I_t^\mu D_t^\mu f(x,t) = f(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k f}{\partial t^k}(x,0) \frac{t^k}{k!}, \quad m-1 < \mu \leq m,\ m \in I\!N, \tag{5}$$

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \tag{6}$$

3. The New Iterative Method (NIM)

Daftardar-Gejji and Jafari [1] have introduced a new iterative method for solving the functional equation

$$u(\bar{x},t) = f(\bar{x},t) + L(u(\bar{x},t)) + N(u(\bar{x},t)), \tag{7}$$

where f is a given function, L and N are given linear and non-linear functions of u respectively, $\bar{x} = (x_1, x_2, \cdots, x_n)$. We are looking for a solution u of (7) having the series form:

$$u(\bar{x},t) = \sum_{i=0}^{\infty} u_i(\bar{x},t). \tag{8}$$

Since L is linear,

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i). \tag{9}$$

The nonlinear operator N is decomposed as (see [1])

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \qquad (10)$$

From equations (8)–(10), (7) is equivalent to

$$\sum_{i=1}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \qquad (11)$$

We define the recurrence relation:

$$\begin{aligned} u_0 &= f \\ u_1 &= L(u_0) + N(u_0) \\ u_{m+1} &= L(u_m) + N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}),\ m = 1, 2, \cdots. \end{aligned} \qquad (12)$$

Then,

$$(u_1 + \cdots + u_{m+1}) = L\left(u_0 + \cdots + u_m\right) + N\left(u_0 + \cdots + u_m\right),\ m = 1, 2, \cdots, \qquad (13)$$

and

$$\sum_{i=0}^{\infty} u_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \qquad (14)$$

The k-term approximate solution of (7)–(8) is given by $u = u_0 + u_1 + \cdots + u_{k-1}$. For the convergence of the method, we refer the reader to paper [1].

4. Fractional Initial Value Problem

We consider the following fractional initial value problem, for $\bar{x} \in \Re^n$:

$$D_t^{\alpha} u\left(\bar{x}, t\right) = \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u\left(\bar{x}, t\right) + A\left(u\left(\bar{x}, t\right)\right), \quad t > 0, \quad m - 1 < \alpha \leq m, \qquad (15)$$

$$\frac{\partial^j u}{\partial t^j}\left(\bar{x}, 0\right) = h_j(\bar{x}), \quad 0 \leq j \leq m - 1, \quad m = 1, 2, \quad 1 < \beta_i \leq 2, \qquad (16)$$

where a_i are constants, $A(u)$ is non-linear function of u and h_k are functions of $\bar{x}$. Applying I_t^{α} on both sides of (15) and using (16) we get

$$u\left(\bar{x}, t\right) = \sum_{j=0}^{m-1} h_j(\bar{x}) \frac{t^j}{j!} + I_t^{\alpha}\left(\sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u\left(\bar{x}, t\right)\right) + I_t^{\alpha} A(u). \qquad (17)$$

Equation (17) has the form (7) with $f = \sum\limits_{j=0}^{m-1} h_j \dfrac{t^j}{j!}$, $L(u) = I_t^\alpha \left(\sum\limits_{i=1}^{n} a_i D_{x_i}^{\beta_i} u \right)$ and $N(u) = I_t^\alpha A(u)$ and can be solved using NIM.

5. Illustrative Examples

Some illustrative examples are presented below.

EXAMPLE 1. Consider the time-fractional diffusion equation

$$D_t^\alpha u(x,t) = u_{xx}(x,t), \quad t > 0, \quad x \in \Re, \quad 0 < \alpha \le 1, \tag{18}$$

$$u(x,0) = \sin(x). \tag{19}$$

System (18)–(19) is equivalent to

$$u = \sin(x) + I_t^\alpha u_{xx}. \tag{20}$$

Using the algorithm (12) of NIM, we get the recurrence relation

$$u_0 = \sin(x), \; u_1 = -\sin(x) \frac{t^\alpha}{\Gamma(\alpha+1)}, ... \tag{21}$$

In general $u_j = (-1)^j \sin(x) \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}$, $j = 0, 1, 2, ...$ The solution of (18)–(19) is thus

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t) = \sin(x) \sum_{j=0}^{\infty} \frac{(-t^\alpha)^j}{\Gamma(j\alpha+1)}$$
$$= \sin(x) E_\alpha(-t^\alpha).$$

EXAMPLE 2. Consider the time-fractional wave equation

$$D_t^\alpha u(x,t) = k \cdot u_{xx}(x,t), \quad t > 0, \quad x \in \Re, \quad 1 < \alpha \le 2, \tag{22}$$

$$u(x,0) = x^2, \; u_t(x,0) = 0. \tag{23}$$

We get the equivalent integral equation of initial value problem (22)–(23) as

$$u = x^2 + k \cdot I_t^\alpha u_{xx}. \tag{24}$$

Applying the NIM, we get $u_0 = x^2$, $u_1 = 2k \cdot \frac{t^\alpha}{\Gamma(\alpha+1)}$, $u_2 = 0, \cdots$. The solution of (22)–(23) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i = x^2 + 2k \cdot \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{25}$$

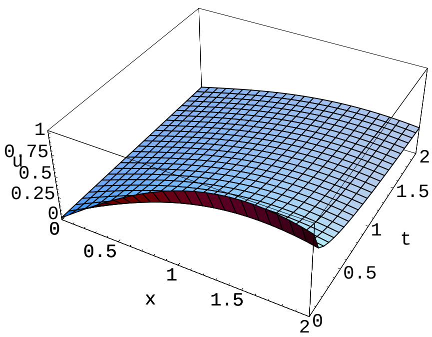

Fig. 1: (Ex. 1, $\alpha = 0.5$)

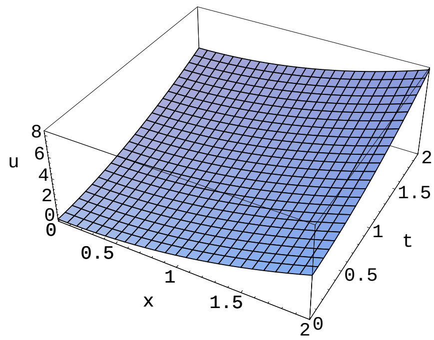

Fig. 2: (Ex. 2, $k = 1$, $\alpha = 1.7$)

Example 3. Consider the space-fractional diffusion equation

$$u_t(x,t) = k \cdot D_x^{\beta} u(x,t), \quad t > 0, \quad x \in \Re, \quad 1 < \beta \leq 2, \tag{26}$$

$$u(x,0) = \frac{2x^{\beta}}{\Gamma(1+\beta)}. \tag{27}$$

Integrating (26) and using (27) we get

$$u(x,t) = \frac{2x^{\beta}}{\Gamma(1+\beta)} + k \int_0^t \left(D_x^{\beta} u(x,t) \right) dt. \tag{28}$$

Applying the NIM, we get

$$u_0 = \frac{2x^{\beta}}{\Gamma(1+\beta)},\ u_1 = 2kt,\ u_2 = 0, \cdots. \tag{29}$$

The solution of (26)–(27) turns out to be

$$u(x,t) = \frac{2x^{\beta}}{\Gamma(1+\beta)} + 2kt. \tag{30}$$

Example 4. Now we consider the space and time fractional diffusion equation

$$D_t^{\alpha} u(x,t) = k \cdot D_x^{\beta} u(x,t), \quad t > 0, \quad x \in \Re, \tag{31}$$

$$u(x,0) = \frac{3x^{\beta}}{\Gamma(1+\beta)} \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2. \tag{32}$$

Applying I_t^{α} on both sides of (31) and using condition (32), we get

$$u(x,t) = \frac{3x^{\beta}}{\Gamma(1+\beta)} + I_t^{\alpha}(D_x^{\beta} u(x,t)). \tag{33}$$

Using the algorithm of NIM we get

$$u_0 = \frac{3x^{\beta}}{\Gamma(1+\beta)},\ u_1 = 3k \frac{t^{\alpha}}{\Gamma(\alpha+1)},\ u_2 = 0, \cdots. \tag{34}$$

Thus $u(x,t) = \frac{3x^{\beta}}{\Gamma(1+\beta)} + 3k\frac{t^{\alpha}}{\Gamma(\alpha+1)}$ is solution of (31)–(32).

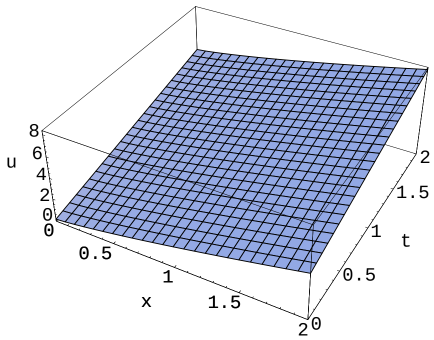

Fig. 3: (Ex. 3, $k = 1$, $\beta = 1.2$)

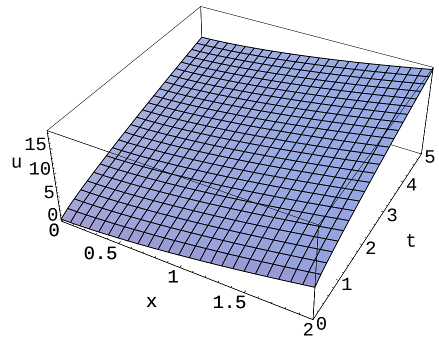

Fig. 4: (Ex. 4, $k = 1$, $\alpha = 0.8$)

Example 5. Consider the two-dimensional time fractional wave equation

$$D_t^{\alpha} u(\bar{x},t) = 2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u(\bar{x},t), \quad t > 0, \quad \bar{x} \in \Re^2, \tag{35}$$

$$u(\bar{x},0) = \sin(x_1)\cdot\sin(x_2),\ u_t(\bar{x},0) = 0, \quad 1 < \alpha \leq 2. \tag{36}$$

The problem (35)–(36) is equivalent to

$$u = \sin(x_1)\cdot\sin(x_2) + 2I_t^{\alpha}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right). \tag{37}$$

In the view of the NIM,

$$u_j = \sin(x_1)\cdot\sin(x_2)\frac{(-4t^{\alpha})^j}{\Gamma(j\alpha+1)},\ j = 0,1,2,\cdots \tag{38}$$

Hence,

$$\begin{aligned} u(\bar{x},t) = \sum_{j=0}^{\infty} u_j &= \sin(x_1)\sin(x_2)\sum_{j=0}^{\infty}\frac{(-4t^{\alpha})^j}{\Gamma(j\alpha+1)} \\ &= \sin(x_1)\sin(x_2)E_{\alpha}\left(-4t^{\alpha}\right) \end{aligned} \tag{39}$$

is solution of (35)– (36).

Comment. In two-dimensions, since $\alpha > 1$, the solution is not necessarily positive hence does not represent diffusion (cf. Fig. 5) of any kind [4], [5].

Example 6. Consider the nonlinear time fractional diffusion equation

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) + 2u(x,t)^2, \quad t > 0,\ x \in \Re,\ 0 < \alpha \leq 1, \tag{40}$$

$$u(x,0) = e^{-x}. \tag{41}$$

The equivalent integral equation of the initial value problem (40)–(41) is

$$u = e^{-x} + I_t^{\alpha}\left(u_{xx}\right) + 2I_t^{\alpha}\left(u^2\right). \tag{42}$$

Equation (42) has the form (7) with $f = e^{-x}$, $L(u) = I_t^{\alpha}\left(u_{xx}\right)$, and $N(u) = 2I_t^{\alpha}\left(u^2\right)$. The algorithm (12) of NIM gives

$$\begin{aligned}
u_0 &= e^{-x},\\
u_1 &= e^{-x}\left(1+2e^{-x}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
u_2 &= e^{-x}\left(1+8e^{-x}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2\left(e^{-x}+2e^{-2x}\right)^2\frac{t^{3\alpha}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\\
&\quad +4e^{-2x}\left(1+2e^{-x}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},
\end{aligned}$$

and so on. Three term solution of (40)–(41) is $u = u_0 + u_1 + u_2$.

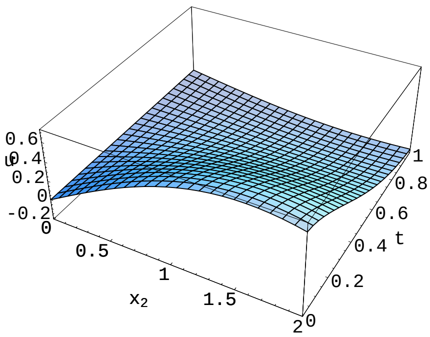

Fig. 5: (Ex. 5, $x_1 = \pi/4$, $\alpha = 1.5$)

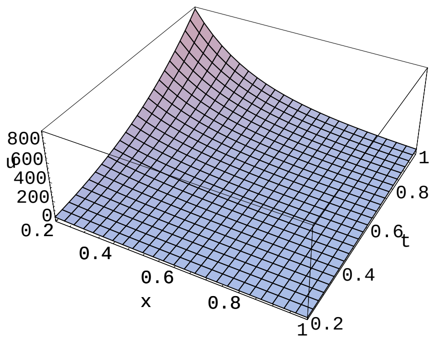

Fig. 6: (Ex. 6, $\alpha = 0.5$)

Example 7. Consider the nonlinear time fractional wave equation

$$D_t^{1.5}u(x,t) = u_{xx}(x,t) + u(x,t)^3, \quad t>0, \quad x\in\Re, \tag{43}$$

$$u(x,0) = 0,\ u_t(x,0) = x^2. \tag{44}$$

The initial value problem (43)–(44) can be written as

$$u = t\cdot x^2 + I_t^{1.5}\left(u_{xx}\right) + I_t^{1.5}\left(u^3\right). \tag{45}$$

The NIM algorithm (12) gives

$$\begin{aligned}
u_0 &= tx^2,\\
u_1 &= 1.12838\left(0.53333t^{2.5} + 0.101587t^{4.5}x^6\right),\\
u_2 &= 0.00842932t^9 + 0.05574t^{7.5}x^2 + 0.38125t^6x^4 + 0.00353593t^{11}x^6\\
&\quad +0.014726t^{9.5}x^8 + 0.0159598t^8x^{10} + 0.000521324t^{13}x^{12}\\
&\quad +0.00104532t^{11.5}x^{14} + 0.0000265981t^{15}x^{18},
\end{aligned}$$

and so on. Here, $u = u_0 + u_1 + u_2$ is three-term solution of (43)–(44), which has been plotted in Fig. 7.

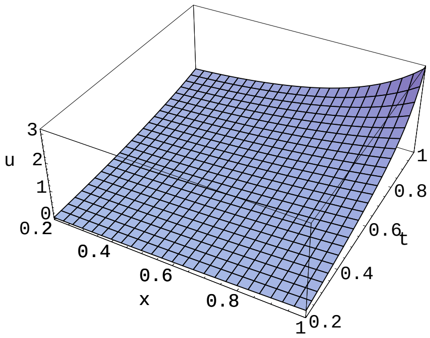

Fig. 7: (Ex. 7)

Mathematica 6 has been used for the computations in the present paper.

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