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FRACTIONAL INTEGRATION AND FRACTIONAL DIFFERENTIATION OF THE M-SERIES

Manoj Sharma

Abstract

In this paper a new special function called as M-series is introduced. This series is a particular case of the $\bar{H}$-function of Inayat-Hussain. The M-series is interesting because the ${}_pF_q$-hypergeometric function and the Mittag-Leffler function follow as its particular cases, and these functions have recently found essential applications in solving problems in physics, biology, engineering and applied sciences. Let us note that the Mittag-Leffler function occurs as solution of fractional integral equations in those area. In this short note we have obtained formulas for the fractional integral and fractional derivative of the M-series.

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1. Introduction to the H-function

The $\bar{H}^{m,n}_{p,q}[z]$-function is a generalization of the familiar H-function of Fox [4], given by Inayat-Hussain [1]. He defined the $\bar{H}$-function, in terms of Mellin-Barness contour integral, as

$$\bar{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \theta(s)\, z^s\, ds, \qquad (1)$$

where the integrand (or the Mellin transform of the $\bar{H}$-function)

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j s) \prod_{j=1}^{n} [\Gamma(1-\alpha_j + A_j s)]^{a_j}}{\prod_{j=m+1}^{q} [\Gamma(1-\beta_j + B_j s)]^{b_j} \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j s)} \qquad (2)$$

contains fractional powers of some of the involved Γ-functions. Here $\alpha_j\,(j = 1,\dots,p)$ and $\beta_j\,(j = 1,\dots,q)$ are complex parameters; $A_j > 0\,(j = 1,\dots,p)$, $B_j > 0\,(j = 1,\dots,q)$; and the exponents $a_j\,(j = 1,\dots,n)$ and $b_j\,(j = m+1,\dots,q)$ can take *non-integer* values. Evidently, when all these exponents a_j and b_j take integer values only, the $\bar{H}$-function reduces to the familiar H-function of Fox, [4], see also [3], [7]. The sufficient conditions for the absolute convergence of the contour integral (1), as given by Buschman and Srivastava [6], are as follows:

$$\Omega := \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0 \text{ and } |\arg(z)| < \frac{1}{2}\pi\Omega.$$

2. The M-series

Here we give first the notation and *the definition of the M-series*, introduced by the author:

$$ {}_p\overset{\alpha}{M}_q\,(a_1,\dots,a_p;b_1,\dots,b_q;z) := {}_p\overset{\alpha}{M}_q\,(z),$$

$$ {}_p\overset{\alpha}{M}_q\,(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k}\,\frac{z^k}{\Gamma(\alpha k+1}\,. \tag{3}$$

Here, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $(a_j)_k, (b_j)_k$ are the Pochammer symbols. The series (3) is defined when none of the parameters b_js, $j = 1,2,,q$, is a negative integer or zero. If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. From the ratio test it is evident that the series in (3) is convergent for all z if $p \le q$, it is convergent for if $p = q+1$ and divergent, if $p > q+1$. When $p = q+1$ and $|z| = 1$, the series can converge in some cases. Let $\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$. It can be shown that when $p = q+1$ the series is absolutely convergent for $|z| = 1$ if $\Re(\beta) < 0$, conditionally convergent for $z = -1$ if $0 \le \Re(\beta) < 1$, and divergent for $|z| = 1$ if $1 \le \Re(\beta)$.

Some special cases of the ${}_p\overset{\alpha}{M}_q\,(z)$-function are the following:

(i) When there is no upper or lower parameters, we have

$$ {}_0\overset{\alpha}{M}_0\,(-;-;z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \tag{4}$$

and thus, the ${}_0\overset{\alpha}{M}_0\,(.)$-function reduces to the *Mittag-Leffler function*, [5].

(ii) When $\alpha = 1$, we have

$$ {}_p\overset{1}{M}_q(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!} = {}_pF_q(z), \tag{5} $$

so the series ${}_p\overset{1}{M}_q(z)$ becomes *the generalized hypergeometric function*, see [3],[7].

The M-series is a special case of the $\bar{H}$*-function*, by putting the following values in definition (1), $\forall j$: $a_j = 1, b_j = 1, B_j = 1, A_j = 1$, namely:

$$ \bar{H}^{m,n}_{p,q}\left[z \middle| \begin{matrix} (\alpha_j, 1; 1)_{1,n}, (\alpha_j, 1)_{n+1,p} \\ (\beta_j, 1)_{1,m}, (\beta_j, 1; 1)_{m+1,q} \end{matrix} \right] = {}_p\overset{\alpha}{M}_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z). \tag{6} $$

3. Fractional integral and fractional derivative of the M-series

Let us consider the fractional *Riemann-Liouville (R-L) integral operator*, see [7] (for lower limit $a = 0$, with respect to variable z), of the M-series (3):

$$ I_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{t^k}{\Gamma(\alpha k + 1)}\, dt. \tag{7} $$

The uniform convergence of the M-series follows from the properties of the $\bar{H}$-function, [1]. Then, using term by term integration we obtain

$$ I_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{1}{\Gamma(\alpha k + 1)} \int_0^z (z-t)^{\nu-1} t^k dt $$

$$ = \frac{1}{\Gamma(\nu)} \left\{ \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{1}{\Gamma(\alpha k + 1)} \right\} z^{\nu-1} \int_0^z (1 - \frac{t}{z})^{\nu-1} t^k dt. $$

Using the substitution $\frac{t}{z} := u$, finally (7) takes the form

$$ I_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) = \frac{1}{\Gamma(\nu)} \left\{ \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{1}{\Gamma(\alpha k+1)} \right\} z^{\nu+k} \int_0^1 (1-u)^{\nu-1} u^k du $$

$$ = \frac{z^{\nu}}{\Gamma(\nu)} \left\{ \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \right\} \frac{\Gamma(\nu)\Gamma(k+1)}{\Gamma(\nu + k + 1)} $$

$$ = \frac{z^{\nu}}{\Gamma(\nu+1)} \left\{ \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{(1)_k}{(\nu+1)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \right\}, \quad \text{or :} $$

$$ I_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) = \frac{z^{\nu}}{\Gamma(\nu+1)}\, {}_{p+1}\overset{\alpha}{M}_{q+1}(a_1, \dots, a_p, 1; b_1, \dots, b_q, \nu + 1; z), \tag{8} $$

that is, as naturally expected for fractional calculus operators of special functions being generalized hypergeometric functions, *a R-L fractional integral of an M-series is again M-series which indices p, q are increased to* $(p+1), (q+1)$.

Analogously, the *R-L fractional differential operator* (see [7]) of the M-series can be considered (with $a = 0$ and with respect to z):

$$D_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) = \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-\nu-1} \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{t^k}{\Gamma(\alpha k+1)}\, dt,$$

where $n = [\nu] + 1$. As before, term by term integration leads to

$$\begin{aligned} D_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) &= \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dz}\right)^n \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{1}{\Gamma(\alpha k+1)} \int_0^z (z-t)^{n-\nu-1} t^k dt \\ &= \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dz}\right)^n \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{1}{\Gamma(\alpha k+1)} B(n-k, k+1) \\ &= \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dz}\right)^n \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{1}{\Gamma(\alpha k+1)} z^{n-\nu+k} \frac{\Gamma(n-\nu)\Gamma(k+1)}{\Gamma(n-\nu+k+1)}, \end{aligned} \tag{9}$$

where $k+1 > 0, n-\nu > 0$ and we use the modified Beta-function:

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} = (b-a)^{\alpha+\beta-1} B(\alpha, \beta), \text{ for } \Re(\alpha) > 0, \Re(\beta) > 0.$$

Differentiation n times the term $z^{n-\nu+k}$ and using again $\Gamma(a+k) = (a)_k \Gamma(a)$, representation (9) reduces to

$$\begin{aligned} D_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) &= z^{-\nu} \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+1)} \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} \\ &= \frac{z^{\nu}}{\Gamma(1-\nu)} \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k (1)_k}{(b_1)_k \ldots (b_q)_k (1-\nu)_k} \frac{z^k}{\Gamma(\alpha k+1)}, \end{aligned}$$

which for $k+1 > 0, (k-\nu+1)_n \neq 0$, gives that a *R-L fractional derivative of an M-series is a M-series which indices p, q are increased to* $(p+1), (q+1)$:

$$D_z^{\nu}\, {}_p\overset{\alpha}{M}_q(z) = \frac{z^{-\nu}}{\Gamma(1-\nu)}\, {}_{p+1}\overset{\alpha}{M}_{q+1}(a_1, \ldots, a_p, 1; b_1, \ldots, b_q, 1-\nu; z). \tag{10}$$

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Department of Mathematics
Rustamji Institute of Technology
Border Security Force Academy
Tekanpur, Gwalior, Madhya Pradesh, INDIA

e-mail: manoj240674@yahoo.co.in

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