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Multiplicative Systems on Ultra-Metric Spaces

Nacima Memić

We perform analysis of certain aspects of approximation in multiplicative systems that appear as duals of ultrametric structures, e.g. in cases of local fields, totally disconnected Abelian groups satisfying the second axiom of countability or more general ultrametric spaces that do not necessarily possess a group structure. Using the fact that the unit sphere of a local field is a Vilenkin group, we introduce a new concept of differentiation in the field of p-adic numbers. Some well known convergence tests are generalized to unbounded Vilenkin groups, i.e. to the setting where the standard boundedness assumption related to the sequence of subgroups generating the underlying topology is absent. A new Fourier multiplier theorem for Hardy spaces on such locally compact groups is obtained. The strong L_q, $q > 1$, and weak L_1 boundedness of Fourier partial sums operators in the system constructed on more general ultrametric spaces is proved.

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1. **Introduction**

The field of reals $\mathbb{R}$, obtained as a completion of rationals $\mathbb{Q}$ with respect to the metric generated by the ordinary absolute value as the Archimedean norm on $\mathbb{Q}$, is just one among infinitely many completions of the rationals. According to the Ostrowski theorem, any nontrivial norm on $\mathbb{Q}$ is either the ordinary absolute value or a p-adic norm for some prime number p. The completion of the field $\mathbb{Q}$ with respect to the p-adic norm leads to the field $\mathbb{Q}_p$ of p-adic numbers. Any p-adic norm is non-Archimedean. Twenty years ago, when Volovich [16] explicitely stated the hypothesis of non-Archimedean structure of space-time at ultra-small distances, the already rich areas of applications of p-adic analysis to number theory or algebraic geometry were extended also into direction of the mathematical physics (see. e.g. [15]) .

Ultrametric analysis we are concerned with concentrates on complex valued functions of an argument belonging to an ultrametric space. (Tate's thesis [14] is the most remarkable representative in this interpretation of p-adic analysis.)

p-Adic differential calculus differs from the real case since piece-wise constant functions depending on a finite number of digits have vanishing derivative. All approaches to a p-adic derivative of complex valued functions have been based on the idea that additive characters should be eigenfunctions of the differentiation operator ([4], [11]). Our attempt is to use the multiplicative system of multiplicative characters on the unit sphere and then extend the p-adic derivative to the field $\mathbb{Q}_p$. The first of three parts of the dissertation is devoted to this goal.

The additive group of the ring of integers of a local field as well as the multiplicative group of units in such a field are particular examples of so-called Vilenkin groups ([1]). In the second part, we are concerned with certain aspects of harmonic analysis on Vilenkin groups that are present in the general setting, i.e. without the boundedness assumption related to the sequence of subgroups that determines the topology of a group under consideration.

Further generalizations to multiplicative systems constructed on compact ultra-metric spaces that need not have a group structure form the content of the concluding part of the dissertation.

2. **Differentiation on local fields**

Differentiation on totally disconnected local fields is based on the classical relation that the differentiation operator should be diagonalized by some given orthogonal system. We use this relation to define a derivative on the p-adic field and the dyadic field.

2.1. **Differentiation on the p-adic field.**

Let $\mathbb{Q}_p$ be the field of p-adic numbers endowed with the p-adic norm $\|.\|_p$. For every $\gamma \in \mathbb{Z}$, the sphere of radius p^γ is given by

$$S_\gamma = \{x \in \mathbb{Q}_p : \|x\|_p = p^\gamma\}.$$

Denote by $(\theta_n)_n$ the system of multiplicative characters on S_0.

We extend the characters θ_n to $\mathbb{Q}_p^*$ by the relation $\theta_n(x) = \frac{1}{\|x\|^{\frac{1}{2}}}\theta_n(\|x\|x)$.

We introduce a new definition of derivative on the field of p-adic numbers.

Definition 2.1. Let φ be a locally constant function. If the series $\sum_{n=0}^{\infty} n^{\alpha}\hat{\varphi}_{\gamma}(n)\theta_n(x)$ converges at a point $x \in S_{\gamma}$ for some $\alpha > 0$, then the function φ is said to be α-differentiable at x and $\varphi^{(\alpha)}(x) = \sum_{n=0}^{\infty} n^{\alpha}\hat{\varphi}_{\gamma}(n)\theta_n(x)$ is called its α-derivative at x.

The following properties of the α-derivative are proved.

Lemma 2.2. *Locally constant functions are infinitely differentiable.*

Lemma 2.3. *The α-derivative of any radial function is equal to 0. Every locally constant function whose derivatives of integer orders vanish is radial.*

Theorem 2.4. *Let φ be a locally constant function. For $\alpha > 0$, the α-derivative of φ has the form $\varphi^{(\alpha)}(x) = k^{\alpha}\varphi(x)$ if and only if $\varphi(x) = M(x)\theta_k(x)$ for some radial function $M(x)$.*

Proposition 2.5. *If a function φ is α-differentiable at some $x_0 \neq 0$, then the function $\varphi_z(x) = \varphi(\frac{x}{z})$ is α-differentiable at $x_0 z$ for every $z \neq 0$, and $\varphi_z^{(m)}(x_0 z) = (\varphi^{(m)})_z(x_0 z)$.*

Definition 2.6. f is a regular distribution if it is defined by

$$\langle f, \varphi\rangle = \int \psi(x)\overline{\varphi}(x)dx\,,$$

for any test function φ, where ψ is a fixed locally constant function.

Proposition 2.7. *The α-derivative of a regular distribution f defined by a locally constant function ψ is given by $\langle f^{(\alpha)}, \varphi\rangle = \langle \psi, \varphi^{(\alpha)}\rangle = \langle \psi^{(\alpha)}, \varphi\rangle$.*

Theorem 2.8. *Let f be a distribution on $\mathbb{Q}_p^*$ whose α-derivative is equal to $k^{\alpha} f$. Then $\langle f, \varphi\rangle = \int \theta_k(x)N(x)\overline{\varphi}(x)dx$ for every test function φ, where $N(x)$ is a fixed radial function .*

Definition 2.9. The multiplicative convolution of functions φ and ψ is given by the formula $(\varphi * \psi)(x) = \int_{S_0} \varphi(\frac{x}{t})\psi(t)dt, x \in \mathbb{Q}_p^*$, if the integral converges.

Notice that $\varphi * \psi \neq \psi * \varphi$.

The next proposition illustrates some properties of the multiplicative convolution.

Proposition 2.10.

(1) *On S_0, the relation $(\varphi * \psi)_\gamma = \varphi_\gamma * \psi$ holds.*
(2) *There is no identity element of the multiplicative convolution in the space of locally integrable functions.*

2.2. **Differentiation on the dyadic field.** On $\mathbb{R}_+$ endowed with the dyadic norm, the pseudo-differential operator with symbol x^α is defined as follows.

Definition 2.11. Let $\alpha \in \mathbb{R} \setminus \{-1\}$. We define a distribution $\Lambda^{\{\alpha\}}$ on $\mathbb{R}_+$ by

$$\langle \Lambda^{\{\alpha\}}, \varphi \rangle := \langle t^\alpha, F\varphi \rangle$$

where $\langle x^\alpha, \varphi \rangle := \int_0^{+\infty} x^\alpha \varphi(x)dx$, if $\alpha > -1$, and $\langle x^\alpha, \varphi \rangle := \int_0^{+\infty} x^\alpha(\varphi(x) - \varphi(0))dx$, when $\alpha < -1$.

For $f \in D'(\mathbb{R}_+)$, we put $D^\alpha f := f * \Lambda^{\{\alpha\}}$ if the convolution exists.

For all $x \in \mathbb{R}_+$, $n \in \mathbb{N}$, let $x_n = [2^n x](mod 2)$,and $x_{-n} = [2^{1-n}x](mod 2)$. As $x_{-n} = 0$ for n sufficiently large, the functions

$$t(x,y) = \sum_{n=1}^{\infty}(x_n y_{-n} + x_{-n} y_n) \text{ and } \psi(x,y) = (-1)^{t(x,y)}$$

are well defined on $\mathbb{R}_+ \times \mathbb{R}_+$.

Proposition 2.12. *Let $\alpha \in \mathbb{R}\setminus\{-1\}$, then we have $D^\alpha \psi(x,y)(y) = x^\alpha \psi(x,y)(y)$.*

3. Vilenkin groups

In this chapter we generalize Salem's and Lebesgue's tests for convergence of Fourier Vilenkin series to unbounded Vilenkin groups.

3.1. **A local Salem test on unbounded Vilenkin groups.** Theorem 1 obtained in [2] can be generalized to unbounded Vilenkin groups as follows.

Theorem 3.1. *Let G be a Vilenkin group, and f a continuous function at some point x satisfying:*

(1) $p_{k+1} \sup_{t \in U_k} |f(x+t) - f(x)| = o(1), \quad k \to \infty,$

(2) $$\lim_{l\to\infty} \lim_{k\to\infty} \sup_{t\in U_{k+1}} (C_k + 1) \sum_{\alpha=1}^{\frac{m_k}{m_l}-1} \frac{1}{\alpha} \left| \sum_{j=0}^{p_{k+1}-1} f(x - z_\alpha^{(k)} - jx_k - t)\zeta_k^{ja_k} \right| = 0,$$

uniformly in $1 \leq a_k < p_{k+1}$, *where* $C_k = \sup_{i \leq k} \frac{p_i}{p_{k+1}}$.

Then, $\lim_{n\to\infty} S_n(x,f) = f(x)$.

3.2. **Lebesgue test on unbounded Vilenkin groups.** We generalize the results of [7] to unbounded Vilenkin groups.

Definition 3.2. We introduce the function

$$f^\diamond(x) = lim_{k\to\infty} \frac{1}{m(G_k)} \int_{G_k} f(x-t) \sum_{r=0}^{a_k} \chi_{m_k}^r(t)dt\,,$$

if the limit exists uniformly with respect to $a_k \in \{0,1,2,\ldots,p_{k+1}-1\}$, at the point $x \in G$.

Definition 3.3. An element $x \in G$ is said to be a Lebesgue point of an integrable function f if

$$\frac{1}{m(G_k)} \int_{G_k} |f(x+t) - f(x)|dt = o(1)(k \to \infty).$$

Theorem 3.4. *If* G *is bounded and* x *is a Lebesgue point of* f*, then* $f^\diamond(x)$ *exists and is equal to* $f(x)$*. However, the existence of* $f^\diamond(x)$ *does not imply that* x *is a Lebesgue point. Moreover, if* G *is unbounded then* $f^\diamond(x)$ *need not exist even at Lebesgue points.*

Theorem 3.5. *Let* G *be any Vilenkin group, and* $f \in L^1(G)$*. Let* $n = a_k m_k + r$*, where* $1 \leq a_k < p_k$ *and* $r < m_k$*. Suppose that*

$$f^\diamond(x) = lim_{k\to\infty} \frac{1}{m(G_k)} \int_{G_k} f(x-t) \sum_{r=0}^{a_k} \chi_{m_k}^r(t)dt$$

exists uniformly in $a_k \in \{0,1,2,\ldots,p_{k+1}-1\}$ *at a point* $x \in G$*. Then*

$$S_n(f;x) - f^\diamond(x) = o(1) + \sum_{\alpha=1}^{m_k-1} \chi_{m_k}^{a_k}(z_\alpha^{(k)}) D_r(z_\alpha^{(k)}) \int_{G_k} f(x - z_\alpha^{(k)} - t)\chi_{m_k}^{a_k}(t)dt$$

as $n \to \infty$*. Thus the necessary and sufficient condition that the Fourier series of* f *converges at* x *is that*

$$(*) \sum_{\alpha=1}^{m_k-1} \chi_{m_k}^{a_k}(z_\alpha^{(k)}) D_r(z_\alpha^{(k)}) \int_{G_k} f(x - z_\alpha^{(k)} - t)\chi_{m_k}^{a_k}(t)dt = o(1)$$

uniformly in a_k *and* r *as* $k \to \infty$*.*

3.3. **Fourier multipliers on Hardy spaces.**

Definition 3.6. A complex function a is called an atom on G if

(1) $supp(a) \subset y + G_n$,
(2) $\|a\|_\infty \leq \frac{1}{\mu(G_n)}$,
(3) $\int_G a(x)dx = 0$.

If the group G is compact, then the function $a \equiv 1$ is also considered as an atom.

The atomic Hardy space H^1 consists of integrable functions f which can be represented as $f = \sum_{i=1}^{\infty} \lambda_i a_i$, where each a_i is an atom and $\sum_{i=1}^{\infty} |\lambda_i| < +\infty$.

The norm in H^1 is given by $\|f\|_{H^1} = \inf \sum_{i=1}^{\infty} |\lambda_i|$, where the infimum is taken over all such decompositions of f.

Definition 3.7. For any distribution f, let $Mf(x) = \sup_n |f * (\mu(G_n))^{-1} 1_{G_n}(x)|$.

The space H consists of all distributions f such that $Mf \in L^1$. The norm is given by $\|f\|_H = \|Mf\|_1$.

Onneweer and Quek [12] proved that $H = H^1$ on bounded locally compact Vilenkin groups.

G. Gat [8] established the strict inclusion $H^1 \subsetneq H$ on some compact unbounded Vilenkin groups.

There exists a maximal function that defines the space H^1 on compact unbounded Vilenkin groups constructed by P.Simon [13].

Our results obtained in [3] show that the situation remains the same on locally compact unbounded Vilenkin groups. Namely, the maximal function

$\tilde{M}f(x) = sup_{n,I_n} |f * (\mu(I_n))^{-1} 1_{I_n}(x)|$, generates the space H^1 on both bounded and unbounded locally compact Vilenkin groups, where I_n is an interval of the form $I_n = \biguplus_{i=\alpha}^{\beta} ix_n + G_{n+1}$ $0 \leq \alpha \leq \beta < p_{n+1}$.

Definition 3.8. Let $\phi \in L^\infty(\Gamma)$. ϕ is a multiplier in H^1 if the operator $Tf = (\phi f^\wedge)^\vee$ is bounded on H^1.

ϕ is a multiplier if and only if $Tf = (\phi f^\wedge)^\vee$ is bounded on the set of atoms.

Theorem 3.9.

In the next theorem, we generalize previous results of Kitada [10], Daly-Phillips [6] and Theorem 2(i) of [5] to the case of unbounded locally compact Vilenkin groups.

Theorem 3.10. *([3]) Let G be any Vilenkin group, $\phi \in L^\infty(\Gamma)$ and*
$\sup_N \int_{G_N^c} |(\phi - \phi_{N+1})^\vee(y)|dy = O(1)$,
where $\phi_{N+1} = \phi 1_{\Gamma_{N+1}}$ and $\wedge, \vee$ denote respectively the Fourier transform and the inverse Fourier transform. Then ϕ is a multiplier on H^1.

An example has been constructed to show that our estimate is sharper in comparison with Daly and Phillips' results.

The following result is an extension of the Marcinkiewicz multiplier theorem for Hardy spaces in the bounded case.

Corollary 3.11. *([3]) If $\phi \in L^\infty(\Gamma)$ on a bounded compact Vilenkin group G fulfills the requirement*

$$m_N^{p-1} \sum_{k=m_{N+1}}^{m_{N+2}-1} |\triangle\phi(k)|^p = O(1) \text{ for some } p \in (1,2]\,,$$

where $\triangle\phi(k) = \phi(k) - \phi(k+1)$, then $\phi \in m(H^1)$.

4. Multiplicative systems on ultra-metric spaces

Our setting in this chapter are ultrametric spaces that need not possess a group structure. We construct a multiplicative system $(\chi_n)_n$ on a given space X and deduce some of its basic properties.

Definition 4.1. Let G be a compact, 0-dimensional metric space. Suppose that $(C_n)_n$ is a sequence of covers of G with the following properties:

(1) Elements of a given C_n are disjoint and clopen.
(2) Each element of C_n is properly contained in some element of C_{n-1}.
(3) $C_0 = G$.
(4) $\bigcup_n C_n$ is a base for the topology of G.

Proposition 4.2. *If for every n, all elements of C_n are given the same measure, then X is homeomorphic to some additive Vilenkin group G determined by the sequence $(p_n)_n$, where p_n is the number of elements from C_{n+1} contained in one element from C_n.*

In the general case, when elements from C_n need not have the same measure, except for those contained in the same element of C_{n-1}, a multiplicative system is constructed with the following properties.

Proposition 4.3.

(1) $\int \chi_n = 0$ *for every* $n \neq 0$.
(2) *The family* $(\chi_n)_n$ *is orthonormal.*

If $U_n(x)$ is the unique element from C_n that contains the point x, then $m_n(x)$ will denote the measure of $U_n(x)$. The analogue of the property $D_{m_n}(t) = m_n 1_{G_n}(t)$ of the Dirichlet kernel is given in the following proposition.

Proposition 4.4. *Let* $x, t \in X$, *and* $n \geq 0$. *Then we have*

$$\sum_{k=0}^{m_n(x)-1} \chi_k(x)\overline{\chi}_k(t) = m_n(x) 1_{U_n(x)}(t)\,.$$

An analogue of the Calderon-Zygmund decomposition is also proved.

Lemma 4.5. *Let* $f \in L^1(X)$, $y > 0$ *and* $(\alpha_n)_n$ *be a sequence of integers. Suppose* $\|f\|_1 \leq y$. *Then, there exist functions* g, b *and a sequence* $B = \{\omega_j\}$ *of disjoint intervals of* X *such that:*

(1) $f = g + b$.
(2) $|g| \leq Cy$, *a.e.*
(3) $\|g\|_1 \leq \|f\|_1$
(4) $B = \bigcup_{n=0}^{\infty} B_n$, *where every* $\omega_j \in B_n$ *is strictly contained in some element from* C_n, *and forms a union of elements of* C_{n+1}.
(5) b *is supported in* $\bigcup_j \omega_j$.
(6) $\int_{\omega_j} b = 0$ *for every* $\omega_j \in B$ *and* $\int_{\omega_j} b\theta_n^{\alpha_n} = 0$, *if* $\omega_j \in B_n$.
(7) $\int_{\omega_j} |b| \leq C \int_{\omega_j} |f|$ *for every* $\omega_j \in B$.
(8) $\sum_j \mu(\omega_j) \leq y^{-1}\|f\|_1$.

If S_n is the n-th partial sum with respect to the system $(\chi_k)_k$, then using the previous lemma we obtain the following result on boundedness of the respective operators.

Theorem 4.6. *There exist constants C_p, $p \geq 1$, such that*

(1) $\|S_n f\|_p \leq C_p \|f\|_p$, *and*

(2) $\mu(\{|S_n f| > y\}) \leq C_1 y^{-1} \|f\|_1$.

Finally, on a class of locally compact ultra-metric spaces, we construct a wavelet system of a Khrennikov-Kozyrev type ([9]) and prove that it is a basis of eigenfunctions of an ultra-metric diffusion operator.

The operator has the following form:

$$Tf(x) = \int T(x,y)(f(x) - f(y))dy\,,$$

where the kernel $T(x,y)$ is symmetric, positive, locally constant and only depends on the distance $\|x-y\|_p$.

For the system of functions $\psi_{n,k,j}(x) = \dfrac{1_{U_n^k}(x) e^{\frac{2\pi i l j}{p_n^k}}}{\sqrt{\mu(U_n^k)}}$, where U_n^k is a basis for the topology of the space, and the numbers p_n^k, l depend on U_n^k and x respectively, we prove the following:

Theorem 4.7. *The system* $\psi_{n,k,j}$ *is an orthonormal total basis for the operator* T.

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Department of Mathematics,
University of Sarajevo,
Zmaja od Bosne 33-35,
71000 Sarajevo,
Bosnia and Herzegovina

E-Mail: nacima.o@gmail.com