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Nonlinear Spectral Theories and Solvability of Nonlinear Hammerstein Equations

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We study some possibilities of nonlinear spectral theories for solving nonlinear operator equations. The main aim is to research a spectrum and establish some kind of nonlinear Fredholm alternative for Hammerstein operator KF.

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It is well-known that the major methods for studying solvability of linear or nonlinear equations in literature are: the variational method, the method of a vector field rotation and the fixed point methods.

"The brachistochrone problem" is usually considered to be the beginning of variational calculus and nonlinear analysis. It was first introduced by J. Bernoulli in the 17th century and was first solved by Isaac Newton. Another method for obtaining existence and uniqueness results has been built on a topological method known as the degree theory, the index theory or rotation of the vector fields. The founders of the index theory were: M. Atiyah, R. Bott, F. Hirzebruch and Is. Singer. The first claim, which is an equivalent to the Brouwer fixed-point theorem, was given and proved by H. Poincaré in 1883 and the next by P. Bole in 1904 (the first one-dimensional equivalent is the well-known Bolcano's theorem on the zeros of a continuous function, proved in 1817). In 1909 Brouwer proved the theorem on fixed points in the case of the three-dimensional space. In 1910 Adamar proved a similar statement for an arbitrary finite-dimensional space, by using a Cronecker index. The same statement was proved by Brouwer in 1912 by using simplex approximation and the notion of mapping degree.

Although Poincaré and Bole gave direct applications of their results in the theory of differential equations, as well as in spacial and analytical mechanics, since then there have not been serious applications of the Brouwer theorem in the mathematical analysis, except one Shauder's result in 1927 about the existence of solution of elliptic partial equations. The situation reversed when John von Neumann applied this theorem to proving the existence of solution for matrix games. These results, which present a base of the classic game theories, increased the mathematicians´s interest in studying the applications of this theorem in various areas of analysis.

Modern research in contractive type conditions started with the Banach fixed point theorem which is one of the classic statements in functional analysis. The following two facts have enabled broad applications of this theorem:

1. solving many kinds of numerical and functional equations can be done by finding fixed points of some mappings;

2. the Banach theorem provides effective calculation (construction) of fixed point and also gives the possibility for estimating error i.e. finding maximal distance from approximative to the accurate solution.

Given a Banach space X over the field $\mathbb{C}$ and a bounded linear operator $A : X \to X$. If some $x_0 \neq 0$ is a fix point of the operator A, i.e. $Ax = x$ has a nontrivial solution $x = x_0$, we can also say

$$(\exists x \neq 0)(I - A)x = 0.$$

It means that operator $I - A$ is not a bijection. More generally, we may consider whether the operator $\lambda I - A$ is a bijection and it leads us to the notion of spectrum which is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \quad \text{is not a bijection}\}.$$

The importance of the spectral theory for linear operators is well-known ([3],[5]). Various attempts have been made to define and study spectra also for nonlinear operators. In the very beginning, the term *spectrum* was used for nonlinear operators just in the sense of point spectrum (i.e. the set of eigenvalues) e.g. by Nemytskij [7] , [8] or Krasnosel'skij [6]. Later it became clear that a more complete description requires, as in the linear case, other ("non-discrete") spectral sets. Starting from the late sixties, this led to a number of definitions of nonlinear spectra which are all different. It was assumed that a reasonable definition of a spectrum of a continuous nonlinear operator should satisfy some minimal requirements, namely:

- It should be reduced to the familiar spectrum in case of linear operators
- It should share some of the usual properties with the linear spectrum (e.g. compactness)
- It should contain the eigenvalues of the operator involved

- It should have nontrivial applications, i.e. those which may not be obtained by other known means

From the viewpoint of these four requirements, any definition of a spectrum should focus on its analytical and topological properties and, of course on applications. Important special classes of continuous (nonlinear) operators are: the Fréchet differentiable operators, the Lipschitz continuous operators, the quasibounded operators and the linearly bounded operators. These are the operators for which many important results have been proved in the nonlinear spectral theory. Spectra have something to do with the "lack of invertibility" of operators. For some class of continuous nonlinear operators $\mathfrak{M}(X)$ we can define a resolvent set

$$\rho(F) = \left\{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{M}(X)\right\}$$

and a spectrum

$$\sigma(F) = \mathbb{K} \backslash \rho(F), \quad F \in \mathfrak{M}(X).$$

If we take $\mathfrak{M}(X) = \mathfrak{C}(X)$ we get Rhodius spectrum ([14]), and if we take $\mathfrak{M}(X) = \mathfrak{C}^1(X)$ we get Neuberger spectrum([15]). The Rhodius spectrum (for continuous operators) may be noncompact or empty, while the Neuberger spectrum(for $\mathfrak{C}^1$ operators) is always nonempty (in the complex case), but it needs be neither closed nor bounded. The Dörfner and Kachurovski spectrum is defined for Lipschitz continuous operators ([16],[17]). In contrast to the Neuberger spectrum, the Kachurovski spectrum is compact, but it may be empty. the Dörfner spectrum in turn is always closed, but it may be unbounded or empty. All four spectra considered so far are reduced to the familiar spectrum in the linear case, and they all contain the eigenvalues of the operator involved. A spectrum for certain special continuous operators (stably solvable operators) was introduced by Furi, Martelli and Vignoli in 1978 ([18]). That spectrum is always closed, sometimes even compact and it has many interesting applications. It need not contain the point spectrum in the nonlinear case. A certain modification of this spectrum has been recently given by Appell, Giorgieri and Väth. In 1997 Feng introduced a new spectrum of nonlinear operator (for epi and k-epi operators)([19],[2]). Roughly speaking, one may say that the Furi-Martelli-Vignoli spectrum takes into account the "asymptotic" properties of an operator, while the Feng spectrum reflects its "global" properties. This is also one reason why the latter contains the eigenvalues, but the former does not.

The range of applications of spectral theory is vast. It is a useful tool for solving nonlinear operator equations. When we want to apply some spectral theory to specific nonlinear problem, we have to choose carefully a spectrum which has at least some of the needed features for solving the problem.

For each spectral theory there is some associated eigenvalue theory dealing, to be more precise, with nontrivial solutions of the equation

$$F(x) = \lambda x.$$

The spectral theory for homogeneous nonlinear operators may be used to derive a certain *nonlinear Fredholm alternative* which provides existence and perturbation results for the p-Laplace equation ([1]).

Below we consider the nonlinear Hammerstein integral equations

$$x(t) = \int_{\Omega} k(s,t) f(s, x(s)) d\mu(s) + g(t), \qquad (t \in \Omega) \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $k(s,t) : \Omega \times \Omega \to \mathbb{R}$ is a measurable kernel and $f(s,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function (measurable in s, for all $u \in \mathbb{R}$ and continuous in u, for almost all $s \in \Omega$). A discrete analogon of (1) is nonlinear Hammerstein system of equations:

$$x(t) = \sum_{s=1}^{\infty} k(s,t) f(s, x(s)) + g(t), \qquad (t \in \mathbb{N}) \tag{2}$$

with kernel $k(s,t)(s,t \in \mathbb{N})$. In system (2) $k : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defines a linear bounded operator

$$Kx(t) = \sum_{s=1}^{\infty} k(s,t) x(s) \tag{3}$$

and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is a real function which generates a nonlinear operator superposition F

$$Fx(s) = f(s, x(s)) \quad (s \in \mathbb{N}) \tag{4}$$

The methods of studying solvability of nonlinear Hammerstein equations in the literature are: the variational method (for example, the Golomb functional $\Phi; grad\Phi = F$), the rotation of the vector field and the fixed point methods ([4],[6]). The necessary and sufficient conditions have been found for the nonlinear superposition operator F, for its action from one Banach space to another ($L_p \to L_q$; $l_p \to l_q$), as well as the conditions for its boundedness, continuity, absolute boundedness ([9]). Some results of the solvability of the Hammerstein integral equations (1) in Banach spaces L_p are given in [2]. The conditions of solvability and uniqueness of the Hammerstein system of equations (2) in Banach spaces l_p are given in [10]. For the nonlinear operator F the function

$$\mu_F(r) = \sup_{\|x\| \le r} \|Fx\|$$

is called the growth function of the operator F in normed space X. The function

$$\nu_F(r) = \inf\{\|a\|_q + br^{\frac{p}{q}} : |f(s,u)| \leq a(s) + b|u|^{\frac{p}{q}}, \quad |u| \leq r\}$$

also gives us some useful information about the operator F.

If F is a linear operator, then

$$\mu_F(r) = \|F\|r$$

Since the growth function $\mu_F(r)$ and $\nu_F(r)$ are logarithmically convex functions on the convex set $\mathcal{L}(F, act.)$([11]), we could define a norm of the superposition operator F as

$$\|F\| = \frac{\mu_F(r)}{r} \quad \text{or} \quad \|F\| = \frac{\nu_F(r)}{r}.$$

In that case, these norms would be topologically equivalent norms (see Theorem 3, in [9]). We note here that the reviewer of the paper [11] has given one counterexample that $\mathcal{L}(F, act.)$ is not the convex set. But in that example, the set

$$\mathcal{L}(F, act.) = \{(1/p, 1/q) : 4 \leq p < \infty, q \geq p/2\} \cup \{(1/p, 1/q) : 1 \leq p \leq 4, q \geq 2\}$$

is convex (see [13]), and therefore it is not in contradiction with the Theorem 2 given in [11].

The conception of the spectral radius for the linear operator L and relation $r_s(L) \leq \|L\|$, could be carried over onto the spectrum of the nonlinear superposition operator F with $r_s(F) \leq \|F\|$.

As for the equations (2), recently we have extended the results from [10] on the weighted Banach spaces $l_{p,\sigma}(1 \leq \sigma \leq \infty)$. Applying the fixed point theorem for monotone operators, we have gained (see [10],[12]) the conditions for the solvability of the system (2) in the spaces $l_{p,\sigma}$ and they are:

Theorem 1. *Let the operator K, defined by (3), be $\mathbb{P}$-positive. Suppose that the generator $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ of superposition operator F given by (4), satisfies condition $(u-v)(f(s,u) - f(s,v)) \leq c(u-v)^2$ for some $c_f > 0$ and $f(s,0) = 0$ for all $s \in \mathbb{N}$. If $c_f \mu_K < 1$ then, for arbitrary $g \in N(l_{2,\tau})$ the equation:*

$$x = KFx + g$$

has a solution $\hat{x} \in N(l_{2,\tau})$. If $g = Nl$ for some $l \in l_{2,\tau}$, then there exists $\hat{h} \in l_{2,\tau}$, such that $\hat{x} = N\hat{h}$, and

$$\|\hat{h}\| \leq \frac{\|l\|}{1 - c_f \mu_K}$$

Moreover,the solution $\hat{x}$ is unique in the space $l_{p,\sigma}$.

Theorem 2. *Let the operator K, defined by (3), be* $\mathbb{P}$*-quasi-positive in* $l_{2,\tau}$*. Suppose that the generator* $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ *of superposition operator F given by (4), satisfies* $(u-v)(f(s,u)-f(s,v)) \leq c(u-v)^2$ *for some* $c_f > 0$ *and* $f(s,0) = 0$ *for all* $s \in \mathbb{N}$*. If* $c_f \nu_K < -1$*, where*

$$\nu_K = sup\{\nu > 0, \quad \|Nh\| \geq \sqrt{\nu} Ph \quad (h \in l_{2,\tau})\},$$

then, for arbitrary $g \in N(l_{2,\tau})$ *the equation:*

$$x = KFx + g$$

has a solution $\hat{x} \in N(l_{2,\tau})$*. If* $g = Nl$ *for some* $l \in l_{2,\tau}$*, then there exists* $\hat{h} \in l_{2,\tau}$*, such that* $\hat{x} = N\hat{h}$*, and*

$$\|\hat{h}\| \leq -\frac{\|l\|}{1 + c_f \nu_K}$$

Moreover,the solution $\hat{x}$ *is unique in the space* $l_{p,\sigma}$*.*

The problem of solvability of the equations (1) or (2) is equivalent to the problem of solvability of the operator equation

$$x = KFx + g, \quad \text{or} \quad (I - KF)x = g. \tag{5}$$

Since the solvability of the equation (5) is closely related to the properties of the spectrum of the nonlinear Hammerstein operator KF, from the theorems that are given in [9] and [10], we can conclude that a spectrum of the operator KF is nonempty. On the other side, we could considered the spectral radius to be $r_s(F) \leq \frac{\mu_F(r)}{r} = \|F\|$.

We are looking forward to finding some kind of nonlinear Fredholm alternative for this Hammerstein operator. The aims of our research are: extending nonlinear spectral theories, finding new statements and new applications.

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