FUNDAMENTAL INVESTIGATIONS ON THE ISOMORPHISM OF COMMUTATIVE GROUP ALGEBRAS IN BULGARIA

Nako Nachev

Abstract. *The isomorphism problem of arbitrary algebraic structures plays always a central role in the study of a given algebraic object. In this paper we give the first investigations and also some basic results on the isomorphism problem of commutative group algebras in Bulgaria.*

Keywords: group algebras, isomorphism, modular and semisimple group algebras, characteristic function of group algebra, field of the first and of the second kind, invariants of Ulm-Kaplansky.

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1. Introduction

The group rings and algebras have interesting properties which combine the properties of the groups and the rings. Investigation of these objects has a fundamental value in the algebra. Let RG be a group algebra of a group G over a ring R. The investigation of the unit group of the group algebra RG represents a hight interest. However, the question for the isomorphism of two group algebras as R-algebras in more important. It can be formulated in the following way: if G is an abelian group, H is any group and R is a ring with identity then find necessary and sufficient conditions for the isomorphism $RH \cong RG$ as R-algebras, that is find a full system of invariants of RG in terms of R and G which determined RG up to isomorphism.

The important of the isomorphism problem is underlined in many algebraic forums. This problem is raised in 1947 year in the Michigan algebraic conference from R. Trell. More specially for the group algebras of the crystallographic groups this problem is raised in the session of AMS in 1979 year in Washington from Farkash [5]. In connection with the isomorphism problem there exists the following wide known conjecture from “Group rings” of Zalesky and Mikhalev [15, conjecture 9.4, page 61]: if K is a field of characteristic p, G is a p-group and H is an arbitrary group, then $KG \cong KH$ as K-algebras if and only if $G \cong H$. This conjecture is not proved even in a partial case when if G is an abelian p-group. However, for a wide class of abelian p-groups it obtains a positive solution, namely for the class of totally projective p-groups. For this class of groups the conjecture is solved from Berman and Mollov [2] and independently from May [7]. For finite abelian groups the solution of the problem is obtained from Jennings [6] in 1941 year, from Deskins [4] in 1956 year and from Coleman [3] in 1964 year. In the fundamental paper [1] Berman give a positive answer of this conjecture in 1967 year, when the abelian p-group is countable. May [9] gives a positive answer of the cited conjecture when G is a p-local Warfield group and K is a perfect field of prime characteristic p.

2. Isomorphism of modular group algebras

The investigation of the isomorphism problem of infinite commutative modular group algebras begins in 1967 year from Berman [1]. Berman proved that if K is a field of characterisitic p, G is a countable abelian p-group and H is an arbitrary abelian group then $KG \cong KH$ as K-algebras implies $G \cong H$. In 1969 year Berman and Mollov [2] prove that if $KG \cong KH$ as K-algebras for any group H where K is a field of characteristic p and G is an abelian p-group, then

1) $G \cong H$ if G is a direct sum of cyclic groups,

2) $dGp \cong dHp$, where dGp is the maximal divisible subgroup of the p-component Gp of G and

3) H is an abelian p-group and the Ulm-Kaplansky invariants of G and H are correspondingly equal. The some result obtains independently the American mathematical Warren May [7]. Later May proves that if G is a totally projective p-grpoup, K is a field of characteristic p and $KG \cong KH$ as K-algebras, then H is also totally projective p-group. This result implies that if $KG \cong KH$ then $G \cong H$ in the totally projective case. These results give a positive answer of the conjecture 9.4, from the indicated paper of Zalesky and Mikhalev in the considered case. (It is well known, that the class of totally projective p-groups is enough wide and possesses entirely natural properties). For arbitrary abelian groups and a field of characteristic p May [7] proved that if $KG \cong KH$, then $G/G_0 \cong H/H_0$, $dG_p \cong dH_p$ and $f_\alpha(G_p) = f_\alpha(H_p)$ for every ordinal α, where G_0 and H_0 are the torsion subgroups of G and H and $f_\alpha(G_p)$ and $f_\alpha(H_p)$ are the α^{-th} Ulm-Kaplansky invariants of G and H, respectively. May, Mollov and Nachev [10] prove a new result for isomorphism of modular group algebras of p-mixed abelian groups.

3. Isomorphism of semisimple group algebras

In 1967 Berman [1] give necessary and sufficient conditions for the isomorphism $KG \cong KH$ when G and H are countable abelian p-groups and K

is a field of characteristic not equal to p. In 1986 year Mollov [11] gives such conditions for the indicated case when G is a finite abelian group.

May [8] obtains the following result.

Theorem. *Let K is an algebraic closed field such that the characterisitic of K does not divide the orders of the torsion elements of G. Then $KG \cong KH \Leftrightarrow G/G_0 \cong H/H_0$ and $|G_0| = |H_0|$.*

In 1990 year May, Karpilovsky and Ullery raise the following problem. Let $\mathbb{Z}\left[\frac{1}{p}\right]$ is the ring of every rational numbers which denominators are degrees of the prime p. Let G be a direct product of two quasi cyclic groups and let H be a direct product of three quasi cyclic groups. Does hold $\mathbb{Z}\left[\frac{1}{p}\right]G \cong \mathbb{Z}\left[\frac{1}{p}\right]H$ as $\mathbb{Z}\left[\frac{1}{p}\right]$-algebras.A positive answer of this question give Nachev [12] and he solves this problem in more general form.

Theorem (Nachev [12]). *Let G and H are arbitrary abelian p-groups.Then $\mathbb{Z}\left[\frac{1}{p}\right]G \cong \mathbb{Z}\left[\frac{1}{p}\right]H$ as $\mathbb{Z}\left[\frac{1}{p}\right]$-algebras if and only if $QG \cong QH$ as Q-algebras.*

This theorem implies that the soluition of the indicated conjecture is reduced to the isomorphism over the field Q of rational numbers and furthermore one can use the result of Berman [1]. Nachev [13] in 1995 year proves one result of isomorphism of group algebras over a primarily neat field. There are obtained some result for group algebras over fields of the first kind with respect to prime p. These results one given preliminarily from Berman and Mollov and are extended from Nachev and Mollov [14].

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Nako Nachev
Faculty of Mathematics and Informatics
University of Plovdiv
236 Bulgaria Blvd.
4003 Plovdiv, Bulgaria
e-mail: nachev@uni-plovdiv.bg