

FUNDAMENTAL INVESTIGATIONS ON THE UNIT GROUPS OF COMMUTATIVE GROUP ALGEBRAS IN BULGARIA

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Abstract. *In this paper we give the first investigations and also some basic results on the unit groups of commutative group algebras in Bulgaria. These investigations continue some classical results. Namely, it is supposed that the cardinality of the starting group is arbitrary.*

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1. Introduction

Everywhere in this paper RG will be the group algebra of an abelian group G over a commutative ring R with identity. Denote by tG the torsion subgroup of G , by G_p be the p -component of G , by $U(RG)$ the multiplicative group of RG and by $S(RG)$ the Sylow p -subgroup of the group $V(RG)$ of normalized units of RG , i.e. the p -component of $V(RG)$. The investigations of the group $S(RG)$ begin with the fundamental paper of Berman [1] in which a complete description of $S(RG)$, up to isomorphism, is given, when G is a countable abelian p -group and R is a countable field of positive characteristic p such that if G is not a restricted direct product of cyclic groups, then the field R is perfect.

Let G be an abelian p -group and let K be a field of characteristic p . May [6] proves that $S(KG)$ is simply presented if and only if G is a simply presented abelian p -group provided the field R is perfect. Therefore, if K is a perfect field of characteristic p and G is a simply presented abelian p -group, then the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(KG)$ together with the description of the maximal divisible subgroup of $S(RG)$ give a full description, up to isomorphism, of the group $S(KG)$.

The abelian group terminology is in agreement with [4].

2. On the unit groups of modular group algebras

The group algebra RG is called modular if the characteristic of R is a prime number p . For any ordinal α we define G^{p^α} and R^{p^α} inductively by the following way: $G^{p^0} = G$, $G^{p^{\alpha+1}} = (G^{p^\alpha})^p$ and $G^{p^\alpha} = \bigcap_{\beta < \alpha} G^{p^\beta}$ if α is a limit ordinal. Analogously, $R^{p^0} = R$, $R^p = \{r^p | r \in R\}$, $R^{p^{\alpha+1}} = (R^{p^\alpha})^p$ and

$R^{p^\alpha} = \bigcap_{\beta < \alpha} R^{p^\beta}$ if α is a limit ordinal. We denote $G[p^n] = \{g \in G \mid g^{p^n} = 1\}$, $n \in \mathbb{N}$.

We use the signs \sum and \coprod for a mark of direct sums and coproduct of groups (algebras), respectively. Denote by $\coprod_{\alpha} G$ a coproduct of α copies of G , where α is a cardinal number. Let $r(G)$ be the rank of an abelian group G [4, p.103]. For any ordinal α the Ulm-Kaplansky invariant $f_\alpha(G)$ of G is defined in the following way: $f_\alpha(G) = r(G^{p^\alpha}[p]/G^{p^{\alpha+1}}[p])$ [4, p.182].

Mollov [7 and 8] calculates the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(RG)$ when G is an arbitrary abelian group and R is a field of positive characteristic p and gives a full description, up to isomorphism, of the maximal divisible subgroup $dS(RG)$ of $S(RG)$.

Theorem 2.1. ([7 and 8]) *Let G be an abelian group, L be a field of positive characteristic p , K be the maximal perfect subfield of L and α be an arbitrary ordinal. If the group G^{p^α} is p -divisible and $L^{p^\alpha} = K$, then $f_\alpha(S) = 0$. Let at least one of these conditions is not fulfilled. If $G_p^{p^\alpha} = 1$, then $f_\alpha(S) = 0$. Let $G_p^{p^\alpha} \neq 1$. If L^{p^α} and G^{p^α} are infinite, then*

$$f_\alpha(S) = \begin{cases} \max(|L|, |G^{p^\alpha}|), & \text{if } \alpha < \omega, \\ \max(|K|, |G^{p^\alpha}|), & \text{if } \alpha \geq \omega, \end{cases}$$

where ω is the least infinite ordinal and if L^{p^α} and G^{p^α} are finite, then

$$f_\alpha(S) = (|G_p^{p^\alpha}| - 2|G_p^{p^{\alpha+1}}| + |G_p^{p^{\alpha+2}}|)|G^{p^\alpha}/G_p^{p^{\alpha+1}}| \log_p |L^{p^\alpha}|.$$

The maximal divisible subgroup of $S(LG)$ is $S(KG^*)$, where G^* is the maximal p -divisible subgroup of G . If $dG_p = 1$, then $S(KG^*) = 1$ and if $dG_p \neq 1$, then $S(KG^*) \cong \sum_{\lambda}^{\bullet} Z(p^\infty)$ where $\lambda = \max(|K|, |G^*|)$.

Let R be a commutative ring with identity of prime characteristic p . Nachev and Mollov [19] calculate the invariants $f_\alpha(S)$ of $S(RG)$ when G is G an abelian p -group. We set $R(p) = \{x \in R \mid x^p = 0\}$.

Theorem 2.2. (Nachev [18]) *Let R be a commutative ring with identity of prime characteristic p , let G be an abelian group and let α be any ordinal. Then the Ulm-Kaplansky invariants $f_\alpha S = f_\alpha S(RG)$ of $S(RG)$ are determined as follows:*

1) *If $G^{p^\alpha}[p] \neq 1$ and $G^{p^\alpha} \neq G^{p^{\alpha+1}}$, then*

$$f_\alpha S = \begin{cases} \max(|R^{p^\alpha}|, |G^{p^\alpha}|), & \text{if } |R^{p^\alpha}| \geq \aleph_0 \text{ or } |G^{p^\alpha}| \geq \aleph_0, \\ (|G^{p^\alpha}| - |G^{p^{\alpha+1}}|) \log_p |R^{p^\alpha}| + (|G^{p^{\alpha+1}}| - 1) \log_p |R^{p^\alpha}(p)| - \\ - (|G^{p^{\alpha+1}}| - |G^{p^{\alpha+2}}|) \log_p |R^{p^{\alpha+1}}| - (|G^{p^{\alpha+2}}| - 1) \log_p |R^{p^{\alpha+1}}(p)|, \\ \text{if } |R^{p^\alpha}| < \aleph_0 \text{ and } |G^{p^\alpha}| < \aleph_0. \end{cases}$$

2) *If $G^{p^\alpha}[p] \neq 1$ and $G^{p^\alpha} = G^{p^{\alpha+1}}$, then*

$$f_\alpha S = \begin{cases} 0, & \text{if } R^{p^\alpha} = R^{p^{\alpha+1}} \\ \max(|R^{p^\alpha}/R^{p^{\alpha+1}}|, |G^{p^\alpha}|), & \text{if } R^{p^\alpha} \neq R^{p^{\alpha+1}} \end{cases}$$

where $R^{p^\alpha}/R^{p^{\alpha+1}}$ is considered as additive group.

3) If $G^{p^\alpha}[p] = 1$ and $G^{p^\alpha} \neq G^{p^{\alpha+1}}$, then

$$f_\alpha S = \begin{cases} 0, & \text{if } R^{p^\alpha}(p) = 0, \\ \max(|R^{p^\alpha}(p)|, |G^{p^\alpha}|), & \text{if } R^{p^\alpha}(p) \neq 0. \end{cases}$$

4) If $G^{p^\alpha}[p] = 1$ and $G^{p^\alpha} = G^{p^{\alpha+1}}$, then

$$f_\alpha S = \begin{cases} 0, & \text{if } R^{p^\alpha}(p) = R^{p^{\alpha+1}}(p), \\ (|G^{p^\alpha}| - 1) \log_p |R^{p^\alpha}(p)/R^{p^{\alpha+1}}(p)|, & \text{if } |R^{p^\alpha}(p)/R^{p^{\alpha+1}}(p)| < \aleph_0 \\ & \text{and } |G^{p^\alpha}| < \aleph_0, \\ \max(|R^{p^\alpha}(p)/R^{p^{\alpha+1}}(p)|, |G^{p^\alpha}|), & \text{if 1) } |R^{p^\alpha}(p)/R^{p^{\alpha+1}}(p)| \geq \aleph_0 \\ & \text{or 2) } |G^{p^\alpha}| \geq \aleph_0 \text{ and} \\ & R^{p^\alpha}(p) \neq R^{p^{\alpha+1}}(p). \end{cases}$$

Let α be the least ordinal such that $G^{p^\alpha} = G^{p^{\alpha+1}}$ and $R^{p^\alpha} = R^{p^{\alpha+1}}$. Then for the maximal divisible subgroup $dS(RG)$ of $S(RG)$ the following holds: $dS(RG) \cong \sum_{\lambda}^{\bullet} Z(p^\infty)$ where

$$\lambda = \begin{cases} \max(|R^{p^\alpha}|, |G^{p^\alpha}|), & \text{if } G^{p^\alpha}[p] \neq 1 \\ \max(|R^{p^\alpha}(p)|, |G^{p^\alpha}|), & \text{if } G^{p^\alpha}[p] = 1, G^{p^\alpha} \neq 1 \text{ and } R^{p^\alpha}(p) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Molloy and Nachev [16] give a full description, up to isomorphism, of the group $V(RG)$ of normalized units in RG when

(i) R is a direct product of m perfect fields of characteristic p , $m \in \mathbb{N}$, G is p -mixed abelian group, the p -component G_p of G is simply presented and either G is splitting or G is of countable torsion free rank and

(ii) R is an infinite direct product of commutative indecomposable rings with identity, G is a finite abelian group and the exponent of G is an invertible element in R .

3. On the unit groups of semisimple group algebras

Let G be an abelian group and let K be a field such that the characteristic of K does not divide the orders of the elements of G . Berman and Rossa [2] have described the p -torsion subgroup $S(KG)$ of $V(KG)$ when G is a countable abelian p -group. Karpilovsky [5, 2.5 Theorem, p.126] has determined the isomorphism class of $U(\mathbb{Q}G)$, i.e. he has given a description of $U(\mathbb{Q}G)$, up to isomorphism, when G is a finitely generated abelian group. Molloy [9] has described $V(KG)$, up to isomorphism, when (a) G is an infinite direct sum of cyclic p -groups and $K = \mathbb{Q}$ and (b) G is an abelian p -group and $K = \mathbb{R}$.

Let G be an abelian p -group and $G^1 = \bigcap_{n=1}^{\infty} G^{p^n}$. Let ε_i , $i \in \mathbb{N}$, be a primitive p^i -th root of the identity. Following Berman [1] the field K of characteristic not equal to p is called a field of the first kind with respect to p if

there is a natural i , $i \geq 2$, such that $K(\varepsilon_i) \neq K(\varepsilon_{i+1})$. Otherwise it is called a field of *the second kind with respect to p* .

Theorem 3.1. (Mollov [10]). *If G is an infinite abelian p -group and K is a field of the first kind with respect to p , then*

$$S(KG) \cong S^1(KG) \times S(K(G/G^1)),$$

where $S(K(G/G^1))$ is a separable group. If $G^1 = 1$, then $S^1(KG) = 1$ and if $G^1 \neq 1$, then $S^1(KG) \cong \coprod_{|G^1|} Z(p^\infty)$.

This theorem show that for full description of $S(KG)$ in the case of abelian p -group G it is enough to consider the case when G is a separable group. In this case in [12 and 13] the Ulm- Kaplansky invariant of $S(KG)$ are calculated. For this aim Mollov [10] introduces the set

$$s_p(K) = \{i \in \mathbb{N}_0 | K(\varepsilon_i) \neq K(\varepsilon_{i+1})\}.$$

This set is said to be a *spectrum of the field K with respect to p* [10].

Theorem 3.2. (Mollov [12 and 13]) *Let G be an unbounded separable abelian p -group, let B be its basic subgroup and let K be a field of the first kind with respect to p . Then the Sylow p -subgroup $S(KG)$ of the group $V(KG)$ of the normalized units of the algebra KG is separable and if i is a non-negative integer, then*

$$f_i(S) = \left\{ \begin{array}{ll} |B|, & \text{if } i+1 \in s_p(K); \\ 0, & \text{if } i+1 \notin s_p(K). \end{array} \right\}.$$

The following theorem gives a complete description, up to isomorphism, of $S(RG)$.

Theorem 3.3. (Mollov [10]) *Let G be an infinite abelian p -group and let K be a field of the second kind with respect to p . Then*

- (i) if either $p \neq 2$ or
- (ii) $p = 2$ and $K \neq K(\varepsilon_2)$, then

$$S(KG) \cong \coprod_{|G|} Z(p^\infty);$$

- (iii) if $p = 2$ and $K = K(\varepsilon_2)$, then $S(KG) \cong dS(KG) \times S(K(G/G^2))$.

If $G^2 = 1$, then $dS(KG) = 1$ and if $G^2 \neq 1$, then $dS(KG) \cong \coprod_{|G^2|} Z(p^\infty)$.

Besides $S(K(G/G^2)) \cong \coprod_{|G/G^2|-1} (2)$.

Definition. (Mollov [11]) The field K is said to be *p -trivial with respect to prime q* , $q \neq p$, if the q -component $K(\varepsilon_i)_q$ of the unit group of $K(\varepsilon_i)$ is identity for every non-negative integer i . Otherwise the field K is said to be *p -nontrivial with respect to prime q* . Let $S_q(KG)$ is the q -component of $S(KG)$. We say that an abelian group G has a *finite exponent n* and we write $\exp(G) = n$ if $G^n = 1$ and n is the least natural with this property.

Theorem 3.4. (Molloy [11]) *Let G be an infinite abelian p -group, let q be a prime, $q \neq p$ and let K be a field such that $\text{char}K \neq p$. If K is p -trivial with respect to prime q , then $S_q(KG) = 1$. Let K be p -nontrivial with respect to prime q and let t be the least natural, such that $K(\varepsilon_t)_q \neq 1$. If $\exp(G) < p^t$, then $S_q(KG) = 1$. If $\exp(G) \geq p^t$, then*

$$S_q(KG) \cong \prod_{|G|} K(\varepsilon_t)_q.$$

What is more, if K is a field of the second kind with respect to $p = 2$ and (i) if $K_q = 1$ and $\exp(G) \geq 4$, then

$$S_q(KG) \cong \prod_{|G|} K(\varepsilon_2)_q$$

and (ii) if either $K_q \neq 1$ or $\exp(G) = 2$, then $S_q(KG) \cong \prod_{|G|} K_q$.

Nachev [17] has given a description of the torsion subgroup $tV(LG)$ of $V(LG)$ when G is an abelian p -group and L is a commutative ring with identity of characteristic not equal to p such that L contains the p^n -th roots of unity, $n \in \mathbb{N}$. Chatzidakis and Pappas [3] have described $U(KG)$ when the torsion abelian group G is a direct sum of countable groups.

Nachev and Molloy [20] describe $U(RG)$, up to isomorphism, when G is an abelian p -group and at least one of the following conditions (a) or (b) is fulfilled:

(a) the first Ulm factor G/G^1 of G is a direct sum of cyclic groups and R is a field of the first kind with respect to p ;

(b) R is a field of the second kind with respect to p .

Let R^* be the multiplicative group of the ring R . If R is a direct product of m indecomposable rings R_i , $m \in \mathbb{N}$, Molloy and Nachev [14] give a description of the unit group $U(RG)$ of RG in the following cases:

(a) when R_i is a ring of prime characteristic p_i , tG/G_{p_i} is finite and the exponent of tG/G_{p_i} belongs to R_i^* ;

(b) when R_i is of characteristic zero, R_i has no nilpotents, tG is finite of exponent n and $n \in R_i^*$.

Molloy and Nachev [15] give four conditions and prove that $V(RG) = GS(RG)$ if and only if exactly one of them is fulfilled.

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