# NONLINEAR BIHARI TYPE INTEGRAL INEQUALITIES WITH MAXIMA

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***Abstract.*** *Some new nonlinear integral inequalities that involve the maximum of the unknown scalar function of one variable are solved. The considered inequalities are generalizations of the classical nonlinear integral inequality of Bihari. The importance of these integral inequalities is defined by their wide applications in qualitative investigations of differential equations with "maxima" and it is illustrated by some direct applications.*

**Keywords:** integral inequalities, maxima, scalar functions of one variable, differential equations with "maxima"

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## 1. Introduction

The integral inequalities that provide explicit bounds on unknown functions play an important role in the development of the theory of differential and integral equations. For instance, the explicit bounds given by the well-known Gronwall−Bellman inequality and its nonlinear generalization due to Bihari ([2], [5]) are used to a considerable extent in the literature. However, in the situations of qualitative investigations of differential equations with "maxima" ([1], [3], [6]) totally different types of integral inequalities are requaired ([4]).

The main purpose of this paper is to solve some nonlinear Bihari-like inequalities that can be used to study the qualitative behavior of the solutions of differential equations with "maxima". Some applications of the obtained results are also given.

## 2. Main Results

Let $t_0$, $T$ be fixed points such that $0 \leq t_0 < T \leq \infty$ and $h = const \geq 0$.

**Definition 1.** We will say that a function $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is from the class $\Omega$ if :

$(i)$ $g$ is a nondecreasing function and $g(x) > 0$ for $x > 0$;
$(ii)$ $g(tx) \geq tg(x)$ for $0 \leq t \leq 1$ and $x \geq 0$;
$(iii)$ $g(x) + g(y) \geq g(x + y)$ for $x,\ y \geq 0$;
$(iv)$ $\int_1^{\infty} \frac{dx}{g(x)} = \infty$.

**Remark 1.** Note the functions $g(x) = \sqrt{x}$ and $g(x) = x$ are from the class $\Omega$.

**Theorem 1.** Let the following conditions be fulfilled:
1. The function $\alpha \in C^1([t_0, T), \mathbb{R}_+)$ is nondecreasing and $\alpha(t) \leq t$.
2. The functions $p,\ q \in C([t_0, T), \mathbb{R}_+)$ and $a,\ b \in C([\alpha(t_0), T), \mathbb{R}_+)$.
3. The function $k \in C([\alpha(t_0) - h, T), \mathbb{R}_+)$.
4. The function $g \in \Omega$.
5. The function $u \in C([\alpha(t_0) - h, T), \mathbb{R}_+)$ and satisfies the inequalities

$$\begin{aligned} u(t) \leq k(t) + \int_{t_0}^{t} \Big[p(s)g\big(u(s)\big) + q(s)g\Big(\max_{\xi\in[s-h,\, s]} u(\xi)\Big)\Big]ds \\ + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(u(s)\big) + b(s)g\Big(\max_{\xi\in[s-h,\, s]} u(\xi)\Big)\Big]ds \quad \text{for } t \in [t_0, T), \end{aligned} \tag{1}$$

$$u(t) \leq k(t) \qquad \text{for } t \in [\alpha(t_0) - h, t_0]. \tag{2}$$

Then for $t_0 \leq t \leq t_1$ the inequality

$$u(t) \leq k(t) + e(t)G^{-1}\Big(G(1) + A(t)\Big) \tag{3}$$

holds, where $G^{-1}$ is the inverse function of

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}\ , \quad 0 < r_0 < 1 \tag{4}$$

$$A(t) = \int_{t_0}^{t} \Big[p(s) + q(s)\Big]ds + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s)\Big]ds, \tag{5}$$

$$t_1 = \sup\Big\{\tau \in [t_0, T)\ :\ G(1) + A(t)\ \in Dom\big(G^{-1}\big)\ \text{ for }\ t \in [t_0, \tau]\Big\}, \tag{6}$$

$$\begin{aligned} e(t) = 1 + \int_{t_0}^{t} \Big[p(s)g\big(k(s)\big) + q(s)g\Big(\max_{\xi\in[s-h,\, s]} k(\xi)\Big)\Big]ds \\ + \int_{t_0}^{\max(\alpha(t),\, t_0)} \Big[a(s)g\big(k(s)\big) + b(s)g\Big(\max_{\xi\in[s-h,\, s]} k(\xi)\Big)\Big]ds,\ t \in [t_0, T). \end{aligned} \tag{7}$$

P r o o f: Define a function $z : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by the equalities

$$z(t) = \begin{cases} \int_{t_0}^{t} \Big[p(s)g\big(u(s)\big) + q(s)g\big(\max_{\xi\in[s-h,\, s]} u(\xi)\big)\Big]ds \\ \ + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(u(s)\big) + b(s)g\big(\max_{\xi\in[s-h,\, s]} u(\xi)\big)\Big]ds, & t \in [t_0, T) \\ 0, & t \in [\alpha(t_0) - h, t_0]. \end{cases}$$

From inequality (1) and the definition of $z(t)$ we have for $t \in [\alpha(t_0) - h, T)$

$$u(t) \leq k(t) + z(t). \tag{8}$$

Let $t \in [t_0, T) : \alpha(t) \geq t_0$. Then from inequality (8), the definition of the function $z(t)$, and condition 4 of Theorem 1 it follows the inequality

$$
\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(u(s)\big) + b(s)g\Big(\max_{s\in[s-h,s]} u(\xi)\Big)\Big]ds \\
(9) \qquad &\le \int_{t_0}^{\max(\alpha(t),\, t_0)} \Big[a(s)g\big(k(s)\big) + b(s)g\Big(\max_{s\in[s-h,\, s]} k(\xi)\Big)\Big]ds \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(z(s)\big) + b(s)g\Big(\max_{s\in[s-h,\, s]} z(\xi)\Big)\Big]ds.
\end{aligned}
$$

Let $t \in [t_0, T) : \alpha(t) < t_0$. Then from the definition of function $z(t)$ we get

$$
\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(u(s)\big) + b(s)g\Big(\max_{s\in[s-h,\, s]} u(\xi)\Big)\Big]ds \\
(10) \qquad &= \int_{t_0}^{\max(\alpha(t),\, t_0)} \Big[a(s)g\big(k(s)\big) + b(s)g\Big(\max_{s\in[s-h,\, s]} k(\xi)\Big)\Big]ds \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(z(s)\big) + b(s)g\Big(\max_{s\in[s-h,\, s]} z(\xi)\Big)\Big]ds.
\end{aligned}
$$

From the definition of function $z(t)$ and inequalities (8), (9), (10) it follows

$$
\begin{aligned}
(11) \qquad z(t) &\le e(t) + \int_{t_0}^{t} \Big[p(s)g\big(z(s)\big) + q(s)g\Big(\max_{\xi\in[s-h,\, s]} z(\xi)\Big)\Big]ds \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(z(s)\big) + b(s)g\Big(\max_{\xi\in[s-h,\, s]} z(\xi)\Big)\Big]ds, \qquad t \in [t_0, T),
\end{aligned}
$$

$$
(12) \quad z(t) \le k(t) \qquad \text{for} \quad t \in [\alpha(t_0) - h, t_0),
$$

where function $e(t)$ is defined by equality (7). Note the function $e(t)$ is nondecreasing for $t \in [t_0, T)$ and $e(t_0) = 1$.

From inequalities (11), (12), condition 4 of Theorem 1 and $\frac{1}{e(t)} \le 1$ we obtain for $t \in [t_0, T)$ the inequality

$$
\begin{aligned}
(13) \qquad \frac{z(t)}{e(t)} &\le 1 + \int_{t_0}^{t} \Big[p(s)g\Big(\frac{z(s)}{e(s)}\Big) + q(s)g\Big(\frac{\max_{\xi\in[s-h,\, s]} z(\xi)}{e(t)}\Big)\Big]ds \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\Big(\frac{z(s)}{e(s)}\Big) + b(s)g\Big(\frac{\max_{\xi\in[s-h,\, s]} z(\xi)}{e(t)}\Big)\Big]ds.
\end{aligned}
$$

From the monotonicity of $e(t)$ we obtain for $t \in [t_0, T)$ and $s \in [\alpha(t_0), t]$

$$
(14) \quad \frac{\max_{\xi\in[s-h,\, s]} z(\xi)}{e(t)} \le \frac{\max_{\xi\in[s-h,\, s]} z(\xi)}{\hat{e}(s)} = \max_{\xi\in[s-h,\, s]} \frac{z(\xi)}{\hat{e}(s)} \le \max_{\xi\in[s-h,\, s]} \frac{z(\xi)}{\hat{e}(\xi)},
$$

where the continuous nondecreasing function $\hat{e} : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ is defined by

$$
\hat{e}(t) = \begin{cases} e(t) & \text{for} \quad t \in [t_0, T) \\ e(t_0) & \text{for} \quad t \in [\alpha(t_0) - h, t_0] \end{cases}.
$$

From (13) and (14) follows that for $t \in [t_0, T)$ the following inequality holds

$$
\begin{aligned}
(15)\quad \frac{z(t)}{\hat{e}(t)} \le 1 &+ \int_{t_0}^{t} \Big[ p(s) g\Big(\frac{z(s)}{\hat{e}(s)}\Big) + q(s) g\Big(\max_{\xi \in [s-h,\, s]} \frac{z(\xi)}{\hat{e}(\xi)}\Big)\Big] ds \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[ a(s) g\Big(\frac{z(s)}{\hat{e}(s)}\Big) + b(s) g\Big(\max_{\xi \in [s-h,\, s]} \frac{z(\xi)}{\hat{e}(\xi)}\Big)\Big] ds.
\end{aligned}
$$

Define a function $U : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by $U(t) = \frac{z(t)}{\hat{e}(t)}$ . Set the right part of inequality (15) by the function $Z : [t_0, T) \to \mathbb{R}_+$. Note the function $Z(t)$ is increasing, $Z(t_0) = 1$ and for $t \in [t_0, T)$ the inequality $U(t) \le Z(t)$ holds. Differentiate the function $Z(t)$, use its monotonicity, condition 1 of Theorem 1, and obtain

$$
\begin{aligned}
(16)\quad \big(Z(t)\big)' &\le [p(t) + q(t)] g\big(Z(t)\big) + \Big[a\big(\alpha(t)\big) + b\big(\alpha(t)\big)\Big] g\big(Z\big(\alpha(t)\big)\big)\big(\alpha(t)\big)' \\
&\le g\big(Z(t)\big)\Big[p(t) + q(t) + \big[a\big(\alpha(t)\big) + b\big(\alpha(t)\big)\big]\big(\alpha(t)\big)'\Big].
\end{aligned}
$$

From definition (4) and inequality (16) it follows that

$$
(17)\quad \frac{d}{dt} G\Big(Z(t)\Big) = \frac{\big(Z(t)\big)'}{g\Big(Z(t)\Big)} \le p(t) + q(t) + \Big[a\big(\alpha(t)\big) + b\big(\alpha(t)\big)\Big]\big(\alpha(t)\big)'.
$$

Integrate inequality (17) from $t_0$ to $t$ for $t \in [t_0, T)$, change the variable $\eta = \alpha(s)$ and obtain

$$
(18)\quad G\big(Z(t)\big) \le G(1) + \int_{t_0}^{t} \Big[p(\eta) + q(\eta)\Big] d\eta + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\eta) + b(\eta)\Big] d\eta.
$$

Since $G^{-1}(t)$ is an increasing function, from inequalities (8), (18) and $U(t) \le Z(t)$, the definitions of the functions $U(t)$ and $\hat{e}(t)$ we obtain the inequality (3).

□

In the case when the function $k(t)$ into the right part of inequality (1) is replaced by a constant we will obtain the following bound for $u(t)$:

**Corollary 1.** Let the following conditions be fulfilled:
1. The conditions 1 and 2 of Theorem 1 are satisfied.
2. The function $\phi \in C([\alpha(t_0) - h, t_0], \mathbb{R}_+)$ and $\phi(t) \ge k$, $k = const \ge 0$.
3. The function $g \in C(\mathbb{R}_+, (0, \infty))$ is increasing.
4. The function $u \in C([\alpha(t_0) - h, T), \mathbb{R}_+)$ and satisfies the inequalities

$$
\begin{aligned}
(19)\quad u(t) \le k &+ \int_{t_0}^{t} \Big[p(s) g\big(u(s)\big) + q(s) g\Big(\max_{\xi \in [s-h,\, s]} u(\xi)\Big)\Big] ds \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) g\big(u(s)\big) + b(s) g\Big(\max_{\xi \in [s-h,\, s]} u(\xi)\Big)\Big] ds \quad \text{for } t \in [t_0, T),
\end{aligned}
$$

$$
(20)\quad u(t) \le \phi(t) \qquad \text{for } t \in [\alpha(t_0) - h, t_0].
$$

Then for $t_0 \le t \le t_2$ the inequality

$$u(t) \le G^{-1}\Big(G(k) + A(t)\Big) \tag{21}$$

holds, where $A(t)$ is defined by (5), $G^{-1}$ is the inverse function of $G$, which is defined by (4), $0 < r_0 < k$,

$$t_2 = \sup\Big\{\tau \in [t_0, T)\ :\ G(k) + A(t)\ \in Dom\big(G^{-1}\big)\ \text{ for }\ t \in [t_0, \tau]\Big\}.$$

**Remark 2.** In the case when $h \equiv 0$ and $\alpha(t) \equiv t$ the result of the Corollary 1 reduces to the classical Bihari inequality.

In the nonlinear case when the unknown function is in a power the following result is valid:

**Theorem 2.** Let the following conditions be fulfilled:

1. The conditions 1, 2 and 4 of Theorem 1 are satisfied.
2. The function $\phi \in C([\alpha(t_0) - h, t_0], \mathbb{R}_+)$.
3. The function $k \in C([t_0, T), (0, \infty))$ is nondecreasing and the inequality $M = \max_{s\in[\alpha(t_0)-h,\, t_0]} \phi(s) \le \sqrt[n]{k(t_0)}$ holds.
4. The function $u \in C([\alpha(t_0) - h, T), \mathbb{R}_+)$ and satisfies the inequalities

$$\begin{aligned}\big(u(t)\big)^n \le k(t) &+ \int_{t_0}^{t} \Big[p(s)g\big(u(s)\big) + q(s)g\Big(\max_{\xi\in[s-h,\, s]} u(\xi)\Big)\Big] ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(u(s)\big) + b(s)g\Big(\max_{\xi\in[s-h,\, s]} u(\xi)\Big)\Big] ds, \quad t \in [t_0, T)\end{aligned} \tag{22}$$

$$u(t) \le \phi(t) \qquad \text{for } t \in [\alpha(t_0) - h, t_0], \tag{23}$$

where $n = const > 1$.

Then for $t_0 \le t \le t_3$ the inequality

$$u(t) \le \sqrt[n]{k(t)} + e_1(t)\Big\{\frac{1}{n}\Big(k(t)\Big)^{\frac{1-n}{n}} + G^{-1}\Big(G(1) + B_1(t) + B_2(t)\Big)\Big\} \tag{24}$$

holds, where

$$\begin{aligned}e_1(t) = 1 &+ \int_{t_0}^{t} \Big[p(s)g\big(\psi(s)\big) + q(s)g\Big(\max_{\xi\in[s-h,\, s]} \psi(\xi)\Big)\Big] ds \\ &+ \int_{t_0}^{\max(\alpha(t),\, t_0)} \Big[a(s)g\big(\psi(s)\big)ds + b(s)g\Big(\max_{\xi\in[s-h,\, s]} \psi(s)\Big)\Big] ds,\end{aligned} \tag{25}$$

$$B_1(t) = \frac{1}{n}\int_{t_0}^{t} \Big[p(s)\Big(k(s)\Big)^{\frac{1-n}{n}} + q(s) \max_{\xi\in[s-h,\, s]} \Big(k(\xi)\Big)^{\frac{1-n}{n}}\Big] ds, \tag{26}$$

$$B_2(t) = \frac{1}{n}\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\Big(K(s)\Big)^{\frac{1-n}{n}} + b(s) \max_{\xi\in[s-h,\, s]} \Big(K(\xi)\Big)^{\frac{1-n}{n}}\Big] ds, \tag{27}$$

$$K(t) = \begin{cases} k(t), & t \in [t_0, T) \\ k(t_0), & t \in [\alpha(t_0), t_0) \end{cases}, \quad \psi(t) = \begin{cases} \sqrt[n]{k(t)}, & t \in (t_0, T) \\ M, & t \in [\alpha(t_0) - h, t_0] \end{cases}.$$

$G^{-1}$ is the inverse function of $G$, defined by equality (4), and

$t_3 = \sup\Big\{\tau \in [t_0, T):\ G(1) + B_1(t) + B_2(t) \in Dom\big(G^{-1}\big) \text{ for } t \in [t_0, \tau]\Big\}.$

P r o o f: Define a function $z : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by the equalities

$$z(t) = \begin{cases} \frac{\sqrt[n]{k(t)}}{n\ k(t)}\Big( \int_{t_0}^{t} \Big[p(s)g\big(u(s)\big) + q(s)g\Big(\max_{\xi\in[s-h,\,s]} u(\xi)\Big)\Big]ds \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)g\big(u(s)\big) + b(s)g\Big(\max_{\xi\in[s-h,\,s]} u(\xi)\Big)\Big]ds\Big), & t \in [t_0, T) \\ 0, & t \in [\alpha(t_0) - h, t_0]. \end{cases}$$

From inequality (22) and the definition of $z(t)$ we have for $t \in [t_0, T)$

$$\big(u(t)\big)^n \leq\ k(t)\Big(1 + n\frac{z(t)}{\sqrt[n]{k(t)}}\Big) \qquad \text{or} \qquad u(t) \leq \sqrt[n]{k(t)}\Big(1 + n\frac{z(t)}{\sqrt[n]{k(t)}}\Big)^{\frac{1}{n}}.$$

Apply Bernoulli's inequality $(1 + x)^a \leq 1 + ax$ where $0 < a < 1$ and $-1 < x$, and observe that

$$(28)\quad u(t) \leq \sqrt[n]{k(t)}\Big(1 + \frac{z(t)}{\sqrt[n]{k(t)}}\Big) = \sqrt[n]{k(t)} + z(t) = \psi(t) + z(t), \qquad t \in [t_0, T),$$

$$(29)\quad u(t) \leq \phi(t) \leq \phi(t) + z(t) = \psi(t) + z(t), \qquad t \in [\alpha(t_0) - h, t_0],$$

$$(30)\quad \max_{\xi\in[s-h,\,s]} u(\xi) \leq \max_{\xi\in[s-h,\,s]} \psi(\xi) + \max_{\xi\in[s-h,\,s]} z(\xi), \qquad s \in [\alpha(t_0), T).$$

Similarly to the proof of Theorem 1 we obtain inequalities (9) and (10) where $k(s)$ is replaced by $\sqrt[n]{\psi(s)}$.

Then for $t \in [t_0, T)$ we get

$$\begin{aligned} z(t) \leq\ & \frac{\sqrt[n]{k(t)}}{n\ k(t)} e_1(t) \\ (31)\qquad & + \frac{1}{n}\int_{t_0}^{t} \Big[p(s)\big(k(s)\big)^{\frac{1-n}{n}} g\big(z(s)\big) + q(s)\big(k(s)\big)^{\frac{1-n}{n}} g\Big(\max_{\xi\in[s-h,\,s]} z(\xi)\Big)\Big]ds \\ & + \frac{1}{n}\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\big(K(s)\big)^{\frac{1-n}{n}} g\big(z(s)\big) + b(s)\big(K(s)\big)^{\frac{1-n}{n}} g\Big(\max_{\xi\in[s-h,\,s]} z(\xi)\Big)\Big]ds. \end{aligned}$$

According to Theorem 2 from (31) and $z(t) \leq \phi(t)$ for $t \in [\alpha(t_0) - h, t_0]$ we get

$$(32)\qquad z(t) \leq e_1(t)\Big\{\frac{\sqrt[n]{k(t)}}{n\ k(t)} + G^{-1}\Big(G(1) + B_1(t) + B_2(t)\Big)\Big\}.$$

Substitute bound (32) for $z(t)$ into the right part of (28) and obtain the required inequality (24).

□

## 3. Applications

We will apply some of the obtained above results to the following system of differential equations with "maxima"

$$x' = f\Big(t,\ x(t),\ \max_{s\in[\beta(t),\,\alpha(t)]} x(s)\Big) \qquad \text{for} \quad t \geq t_0, \tag{33}$$

with an initial condition

$$x(t) = \varphi(t) \qquad\qquad \text{for} \quad t \in [\alpha(t_0) - h, t_0], \tag{34}$$

where $x \in \mathbb{R}^n$, $h > 0$ is a constant and $t \geq t_0$.

**Theorem 3.** (Bounds). Let the following conditions be fulfilled:

1. The functions $\alpha,\ \beta \in C^1([t_0,\infty),\mathbb{R}_+)$, $\alpha(t)$ is a nondecreasing function, $\beta(t) \leq \alpha(t) \leq t$ and $0 < \alpha(t) - \beta(t) \leq h$ for $t \geq t_0$.

2. The function $f \in C([t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and satisfies for $t \geq t_0$ and $x,\ y \in \mathbb{R}^n$ the condition

$$\Big|\Big|f(t,\ x,\ y)\Big|\Big| \leq P(t)\sqrt{||x||} + Q(t)\sqrt{||y||},$$

where $P,\ B \in C([t_0,\infty),\mathbb{R}_+)$.

3. The function $\varphi \in C([\alpha(t_0) - h, t_0], \mathbb{R}_+)$.

4. The function $x(t; t_0, \varphi)$ is a solution of the initial value problem (33),(34) defined for $t \geq \alpha(t_0) - h$.

Then the solution $x(t; t_0, \varphi)$ satisfies for $t \geq t_0$ the inequality

$$\big|\big|x(t; t_0, \varphi)\big|\big| \leq \frac{1}{4}\bigg(2\sqrt{\big|\big|\varphi(t_0)\big|\big|} + \int_{t_0}^{t} \Big[P(s) + Q(s)\Big] ds\bigg)^2. \tag{35}$$

P r o o f: The function $x(t) = x(t; t_0, \varphi)$ satisfies the following integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f\Big(s,\ x(s),\ \max_{\xi\in[\beta(s),\,\alpha(s)]} x(\xi)\Big) ds \qquad \text{for} \quad t \geq t_0.$$

Then for the norm of the solution $x(t)$ we obtain for $t \geq t_0$

$$\begin{aligned} ||x(t)|| &\leq ||\varphi(t_0)|| + \int_{t_0}^{t} \Big|\Big| f\Big(s,\ x(s),\ \max_{\xi\in[\beta(s),\,\alpha(s)]} x(\xi)\Big)\Big|\Big|\, ds \\ &\leq ||\varphi(t_0)|| + \int_{t_0}^{t} P(s)\sqrt{||x(s)||} ds + \int_{t_0}^{t} Q(s)\sqrt{\max_{\xi\in[\beta(s),\,\alpha(s)]} ||x(\xi)||}\, ds. \end{aligned} \tag{36}$$

Set $u(t) = ||x(t)||$ for $t \in [\alpha(t_0) - h, \infty)$. Then from (36) we get for $t \geq t_0$

$$u(t) \leq ||\varphi(t_0)|| + \int_{t_0}^{t} P(s)\sqrt{u(s)}\, ds + \int_{t_0}^{t} Q(s)\sqrt{\max_{\xi\in[\beta(s),\alpha(s)]} u(\xi)}\, ds. \tag{37}$$

Change the variable $s = \alpha^{-1}(\rho)$ in the second integral of (37), use the inequality $\max_{\xi\in[\beta(t),\,\alpha(t)]} u(\xi) \leq \max_{\xi\in[\alpha(t)-h,\,\alpha(t)]} u(\xi)$ for $t \in [t_0, T)$ and obtain

$$(38)\quad u(t) \leq ||\varphi(t_0)|| + \int_{t_0}^{t} P(\rho)\sqrt{u(\rho)}d\rho + \int_{\alpha(t_0)}^{\alpha(t)} Q(\alpha^{-1}(\rho))(\alpha^{-1}(\rho))' \sqrt{\max_{\xi\in[\rho-h,\rho]} u(\xi)}d\rho.$$

Note the conditions of Corollary 1 are satisfied for $k = ||\varphi(t_0)||$, $q(t) \equiv 0$ for $t \in [t_0, \infty)$, $p(t) \equiv P(t)$, $b(s) \equiv Q(\alpha^{-1}(s))(\alpha^{-1}(s))'$ for $t \in [\alpha(t_0), \infty)$, $a(t) \equiv 0$, $g(u) = \sqrt{u}$, $G(u) = 2\sqrt{u}$, $G^{-1}(u) = \frac{1}{4}u^2$, $Dom\big(G^{-1}\big) = \mathbb{R}_+$ and $t_2 = \infty$.

According to Corollary 1 from inequality (38) we obtain for $t \geq t_0$ the required inequality (35).

□

## References


[1] V. G. Angelov, D. D. Bainov: On the functional differential equations with "maximums", *Appl. Anal.*, **16**, (1983), 187–194.

[2] D. D. Bainov, , P. S. Simeonov: *Integral inequalities and applications*, Kluwer, (1989).

[3] S. G. Hristova, L. F. Roberts: Boundedness of the solutions of differential equations with "maxima", *Int. J. Appl. Math.*, **4** , No. 2, (2000), 231–240.

[4] S. G. Hristova, K. V. Stefanova: Linear integral inequalities involving maxima of the unknown scalar functions, *Funkcialaj Ekvacioj*, (2010) (accepted).

[5] B. G. Pachpatte: Explicit bounds on certain integral inequalities, *J. Math. Anal. Appl.*, **267**, (2002), 48–61.

[6] E. P. Popov: *Automatic regulation and control*, Moscow, (1966) (in Russian).



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