PROPERTIES OF $L_P(K)$-SOLUTIONS OF LINEAR NONHOMOGENEOUS IMPULSIVE DIFFERENTIAL EQUATIONS WITH UNBOUNDED LINEAR OPERATOR

Atanaska Georgieva

Abstract. *Sufficient conditions for the existence of $L_p(k)$-solutions of linear nonhomogeneous impulsive differential equations with unbounded linear operator are found. An example of the theory of the linear nonhomogeneous partial impulsive differential equations of parabolic type is given.*

Keywords: Linear nonhomogeneous impulsive differential equations, $L_p(k)$-solutions, partial impulsive differential equations of parabolic type
2010 Mathematics Subject Classification: 34A37, 47H10

1. Introduction

We study the existence of solutions in the spaces $L_p(k)$ $(1 \leq p \leq \infty)$ of linear nonhomogeneous impulsive differential equations with unbounded linear operator. In Theorem 1 we prove the existence of $L_p(k)$-solutions. Further we give an example. We consider a partial impulsive differential equation with elliptic linear part and reduce it to an ordinary impulsive differential equation. This ordinary equation satisfy the conditions of Theorem 1 and therefore there exist $L_p(k)$-solutions of the considered ordinary equation. By this way, we establish $L_p(k)$-solutions also of the given partial impulsive equation.

2. Statement of the problem

Let X be a Banach space with norm $\|.\|$ and identity I. By $D(T) \subset X$ we will denote the domain of the operator $T : D(T) \to X$.

We consider the following linear nonhomogeneous impulsive differential equation

$$\frac{du}{dt} = A(t)u + f(t) \quad \text{for} \ \ t \neq t_n \tag{1}$$

$$u(t_n^+) = Q_n(u(t_n)) + h_n \quad \text{for} \ \ n = 1, 2, \dots , \tag{2}$$

where $A(t)$: $D(A(t)) \to X$ $(t \in \mathbb{R}_+)$ and Q_n: $D(Q_n) \to D(A(t_n))$ $(n = 1, 2, ...)$ are linear unbounded operators. The sets $D(A(t))$ and $D(Q_n)$ $(t \geq 0,\ n = 1, 2, ...)$ are dense in X. The function $f(.) : \mathbb{R}_+ \to X$ is continuous and $h = \{h_n\}_{n=1}^{\infty}$ is a sequence of elements of X. The points of jump t_n satisfy the following conditions $0 = t_o < t_1 < ... < t_n < ...,\ \lim\limits_{n\to\infty} t_n = \infty$. We set $Q_0 = I,\ h_0 = 0$.

Furthermore, we assume that all considered functions are left continuous and there exist the Cauchy operator $U(t,s)$ $(0 \leq s \leq t)$ of the linear ordinary equation

$$\frac{du}{dt} = A(t)u. \tag{3}$$

Remark 1. . Sufficient conditions for the existence of $U(t,s)$ can be found in ([2], [3], [4]).

It is easy to prove that the functions $u(t) = V(t,s)\xi$ for $\xi \in D(A(s))$ with

$$V(t,s) = U(t,t_n)Q_nU(t_n,t_{n-1})Q_{n-1}\,\ldots\,Q_kU(t_k,s) \tag{4}$$

$(0 \leq s \leq t_k \leq t_n < t)$ satisfy the linear impulsive Cauchy problem

$$\frac{du}{dt} = A(t)u \quad \text{for} \;\; t \neq t_n \tag{5}$$

$$u(t_n^+) = Q_n(u(t_n)) \quad \text{for} \;\; n = 1,\, 2,\, ... \tag{6}$$

$$u(s) = \xi. \tag{7}$$

Let us note that the operator $V(t,s)$ is bounded if one of the following conditions holds

$(B1)$ $Q_nU(t_n,t_{n-1})$ are bounded operators $(n = 1,\, 2,\, ...)$.

$(B2)$ $U(t_{n+1},t_n)Q_n$ are bounded operators $(n = 1,\, 2,\, ...)$.

Let the following condition be fulfilled.

(H) There exists a continuous function $k(\,.\,,\,.\,) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|V(t,s)\xi\| \leq k(t,s)\|\xi\|$, where $0 \leq s < t$ and $\xi \in D(A(s))$.

We introduce the following spaces

$$L_p(k) = \left\{ g(.) : \mathbb{R}_+ \to X : \sup_{t\in\mathbb{R}_+} \int_0^t k(t,s)\|g(s)\|^p ds < \infty \right\}$$

$$l_p(k) = \left\{ g = \{g_n\}_{n=1}^{\infty} \subset X : \sup_{t\in\mathbb{R}_+} \sum_{0<t_n<t} k(t,t_n^+)\|g_n\|^p < \infty \right\}$$

with norms

$$\|g\|_{L_p(k)} = \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s)\|g(s)\|^p ds \right)^{\frac{1}{p}}, \quad \|g\|_{l_p(k)} = \sup_{t\in\mathbb{R}_+} \left(\sum_0^t k(t,t_n^+)\|g_n\|^p \right)^{\frac{1}{p}}.$$

We introduce the following conditions.

$(H1)$ There exists constant $M_1 > 0$ such that $\int_0^t k(t,s)ds \leq M_1$.

$(H2)$ There exists constant $M_2 > 0$ such that $\sum_{0<t_n<t} k(t,t_n^+) \leq M_2$.

3. Main results

Lemma 1. . Let the following conditions be fulfilled:
1. Condition $(B1)$ or $(B2)$ holds.
2. Conditions (H), $(H1)$ and $(H2)$ hold.

Then for any function $f\in L_p(k)$ and for any sequence $h=\{h_n\}_{n=1}^{\infty}\in l_p(k)$ the linear nonhomogeneous impulsive equation (1), (2) has a bounded solution $u(t)$ $(t\in\mathbb{R}_+)$ such that

$$u(t)=V(t,0)u(0)+\int_0^t V(t,s)f(s)ds+\sum_{0<t_n<t}V(t,t_n^+)h_n. \tag{8}$$

Proof: It is immediately verified that the function $u(t)$ is a solution of the linear nonhomogeneous impulsive equation (1), (2).

We shall estimate the norm of the integral and the sum in (8).

Let $q=\dfrac{p}{p-1}$. We use Holder's inequality. For the norm of the integral in (8) we obtain the estimate

$$\left\|\int_0^t V(t,s)f(s)ds\right\|\le\int_0^t k(t,s)\|f(s)\|ds\le\int_0^t k^{\frac{1}{q}}(t,s)k^{\frac{1}{p}}(t,s)\|f(s)\|ds\le$$

$$\le\left(\int_0^t k(t,s)ds\right)^{\frac{1}{q}}\left(\int_0^t k(t,s)\|f(s)\|^p ds\right)^{\frac{1}{p}}\le M_1^{\frac{1}{q}}\|f\|_{L_p(k)}.$$

For the norm of the sum in (8) we obtain the estimate

$$\left\|\sum_{0<t_n<t}V(t,t_n^+)h_n\right\|\le\sum_{0<t_n<t}k(t,t_n^+)\|h_n\|\le\sum_{0<t_n<t}k^{\frac{1}{q}}(t,t_n^+)k^{\frac{1}{p}}(t,t_n^+)h_n\le$$

$$\le\left(\sum_{0<t_n<t}k(t,t_n^+)\right)^{\frac{1}{q}}\left(\sum_{0<t_n<t}k(t,t_n^+)\|h_n\|^p\right)^{\frac{1}{p}}\le M_2^{\frac{1}{q}}\|h\|_{l_p(k)}.$$

□

Lemma 2. . Let the following conditions be fulfilled:
1. Condition $(B1)$ or $(B2)$ holds.
2. Conditions (H) and $(H1)$ hold.

Then the operator G_1, defined by the formula

$$G_1f(t)=\int_0^t V(t,s)f(s)ds \tag{9}$$

maps $L_p(k)$ into $L_p(k)$ and the following estimate is valid

$$\|G_1f\|_{L_p(k)}\le M_1\|f\|_{L_p(k)}, \tag{10}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof: Let $f \in L_p(k)$. From Lemma 1 we have the estimate

$$\|G_1 f(t)\| \leq M_1^{\frac{1}{q}} \|f\|_{L_p(k)}. \tag{11}$$

We shall prove $G_1 f \in L_p(k)$. From (11) and condition $(H1)$ we obtain

$$\|G_1 f\|_{L_p(k)} = \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s)\|G_1 f(s)\|^p ds \right)^{\frac{1}{p}} \leq \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s) M_1^{\frac{p}{q}} \|f\|^p_{L_p(k)} ds \right)^{\frac{1}{p}} =$$

$$= M_1^{\frac{1}{q}} \|f\|_{L_p(k)} \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s) ds \right)^{\frac{1}{p}} \leq M_1 \|f\|_{L_p(k)}.$$

Hence inequality (10) holds.

□

Lemma 3. . Let the following conditions be fulfilled:
1. Condition $(B1)$ or $(B2)$ holds.
2. Conditions (H) and $(H2)$ hold.

Then the operator G_2, defined by the formula

$$G_2 h(t) = \sum_{0<t_n<t} V(t, t_n^+) h_n \tag{12}$$

maps $l_p(k)$ into $L_p(k)$ and the following estimate is valid

$$\|G_2 h\|_{L_p(k)} \leq M_1^{\frac{1}{p}} M_2^{\frac{1}{q}} \|h\|_{l_p(k)}, \tag{13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Let $h = \{h_n\}_{n=1}^{\infty}$ be an arbitrary secuence of $l_p(k)$. From Lemma 1 we have the estimate

$$\|G_2 h(t)\| \leq M_2^{\frac{1}{q}} \|h\|_{l_p(k)}. \tag{14}$$

We shall prove $G_2 h \in L_p(k)$. From (14) and condition $(H2)$ we obtain

$$\|G_2 h\|_{L_p(k)} = \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s)\|G_2 h(s)\|^p ds \right)^{\frac{1}{p}} \leq \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s) M_2^{\frac{p}{q}} \|h\|^p_{l_p(k)} ds \right)^{\frac{1}{p}} =$$

$$= M_2^{\frac{1}{q}} \|h\|_{l_p(k)} \sup_{t\in\mathbb{R}_+} \left(\int_0^t k(t,s) ds \right)^{\frac{1}{p}} \leq M_2^{\frac{1}{q}} M_1^{\frac{1}{p}} \|h\|_{l_p(k)}.$$

Hence inequality (13) holds.

□

Theorem 1. . Let the following conditions be fulfilled:
1. Condition $(B1)$ or $(B2)$ holds.
2. Conditions (H), $(H1)$ and $(H2)$ hold.
3. The function $V(t,0)\xi \in L_p(k)$ $(t \in \mathbb{R}_+,\ \xi \in D(A(0)))$.

Then for any function $f \in L_p(k)$ and for any sequence $h = \{h_n\}_{n=1}^{\infty} \in l_p(k)$ the linear nonhomogeneous impulsive equation (1), (2) has in $L_p(k)$ a unique solution and this solution is bounded.

Proof: Let the function $f \in L_p(k)$ and the sequence $h = \{h_n\}_{n=1}^{\infty} \in l_p(k)$. Then we write down equality (8) in the form

$$u(t) = V(t,0)u(0) + G_1 f(t) + G_2 h(t), \tag{15}$$

where the operators G_1 and G_2 are defined by (9), (12). From (15), Lemma 2, Lemma 3 and condition 3 of Theorem 1 it follows that the solution of the linear nonhomogeneous impulsive equation (1), (2) is in the space $L_p(k)$.

□

We shall illustrate Theorem 1 by an example from the qualitative theory of the linear nonhomogeneous partial impulse differential equations.

Example. In this example we consider a partial impulse differential equation and reduce it to an ordinary impulse differential equation. For this ordinary impulsive differential equation, the conditions of Theorem 1 are fulfilled. Several notations and results for ordinary differential equations, used in the example, are given in capite $5 - 7$ of [4]. Note the introdution to the theory of partial impulse differential equations is considered in [1].

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $Q = (0,\infty) \times \Omega$ and $\Gamma = (0,\infty) \times \partial\Omega$. We denote

$$P_n = \{(t_n, x) :\ x \in \Omega\}, \quad P = \bigcup_{n=1}^{\infty} P_n,$$

$$\Lambda_n = \{(t_n, x) :\ x \in \partial\Omega\}, \quad \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n.$$

Consider the linear nonhomogeneous impulse parabolic equation with initial and smooth conditions

$$\frac{\partial u}{\partial t} = \tilde{A}(t,x,D)u + \tilde{f}(t,x), \quad (t,x) \in Q \setminus P \tag{16}$$

$$D^{\alpha}u(t,x) = 0, \quad |\alpha| < m, \ (t,x) \in \Gamma \setminus \Lambda \tag{17}$$

$$u(0,x) = v(x), \quad x \in \Omega \tag{18}$$

$$u(t_n^+, x) = \tilde{Q}_n(u(t_n,x)) + \tilde{h}_n(x)), \quad x \in \overline{\Omega}, \ n = 1,\, 2,\, \dots, \tag{19}$$

where

$$\tilde{A}(t,x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha,$$

$\tilde{Q}_n : D(\tilde{Q}_n) \to D(\tilde{A}(t_n,x,D))$ $(n = 1, 2, ...)$ are linear operators, $\tilde{f}(\,.\,,\,.\,) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ and $\tilde{h}_n(.) : \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

Let $X = L_p(\Omega,\mathbb{R})$ $(1 < p < \infty)$, where

$$L_p(\Omega,\mathbb{R}) = \left\{ v : \Omega \to \mathbb{R};\ \int_\Omega |v(x)|^p dx < \infty \right\}$$

with norm $|v|_p = \left(\int_\Omega |v(x)|^p dx \right)^{\frac{1}{p}}$.

With the family $\tilde{A}(t,x,D)$, $(t \in \mathbb{R}_+)$ of strongly elliptic operators we associate a family of linear operators $A(t)$, $(t \in \mathbb{R}_+)$ acting in X by

$$A(t)u = \tilde{A}(t,x,D)u, \ \text{ for } u \in D.$$

This is done as follows $D = D(A(t)) = W^{2m,p}(\Omega) \bigcap W_0^{m,p}(\Omega)$, $(t \in \mathbb{R}_+)$.

Let $v \in X$. We set

$$f(t)(x) = \tilde{f}(t,x), \quad Q_n(u(t_n))(x) = \tilde{Q}_n(u(t_n,x)),$$
$$h_n(x) = \tilde{h}_n(x), \quad (t \in \mathbb{R}_+,\ x \in \overline{\Omega}),$$

where $Q_n\colon D(Q_n) \to D$ are linear operators, the sets $D(Q_n) \subset X$ lie dense in X, the function $f(.) : \mathbb{R}_+ \to X$ is continuous and $\{h_n\}_{n=1}^\infty \subset X$.

Let $U(t,s)$ is the Cauchy operator of the linear equation

$$\frac{du}{dt} = A(t)u.$$

Sufficient conditions for the validity of the estimate

$$|U(t,s)|_{p\to p} \leq Ce^{-k(t-s)} \ \ (0 \leq s \leq t;\ C,k > 0 \text{ constants})$$

are given in [4].

We shall consider the concrete case when $t_n = n$ $(n = 1, 2, ...)$,

$$\tilde{f}(t,x) = e^{-\gamma t}\psi(x), \quad \tilde{Q}_n\xi = \frac{kn}{C(1+n^2)e^{C+k}}\,\xi, \quad \tilde{h}_n(x) = e^{-\alpha n}\varphi(x),$$

where the functions $\psi,\ \varphi \in X$ and $\alpha,\ \gamma$ are positive constants.
Then for $\xi \in X$

$$f(t) = e^{-\gamma t}\xi, \quad \tilde{Q}_n\xi = \frac{kn}{C(1+n^2)e^{C+k}}\,\xi, \quad h_n = e^{-\alpha n}\xi.$$

Let $V(t,s)$ $(0 \leq s \leq t)$ is the Cauchy operator of the linear impulse equation

$$\frac{du}{dt} = A(t)u \quad \text{for } t \neq t_n$$
$$u(t_n^+) = Q_n(u(t_n)) \ \text{ for } n = 1,\, 2,\, ...$$

Then for $0 < s \leq k < n < t$, $\xi \in D$ the following estimate is valid

$$|V(t,s)\xi|_p \leq kte^{-k(t-s)}|\xi|_p \,.$$

We set $k(t,s) = kte^{-k(t-s)}$.

In this case the conditions of Theorem 1 hold. Hence the ordinary equation (1), (2) has $L_p(k)$-solution, which induced $L_p(k)$-solution of the partial equation (16) – (19) for any $x \in \Omega$.

References

[1] L. Erbe, H. Freedman, X. Liu, J. Wu: Comparison principle for impulsive parabolic equations with applications to models of single species growth, *J. Austral. Math. Soc. Ser. B* **32**, (1991), 382–400.

[2] J. A. Goldstein: *Semigroups of Linear Operators and Applications*, Oxford University Press, New York, (1985), 235 p.

[3] M. Krasnoselski, P. Zabreiko, E. Pustilnic, P. Sobolevski: *Integrable Operators in Spaces of Integrable Functions*, Moskva, (1966), 499 (In Russian).

[4] A. Pazy: *Semigroups of Linear Operators*, Springer-Verlag, (1983), 279 p.

Atanaska Georgieva
Faculty of Mathematics and Informatics
Plovdiv University
236 Bulgaria Blvd.
4003 Plovdiv, Bulgaria,
e-mail: afi2000@abv.bg