APPLICATION OF AN ACCELERATION SCHEME FOR AN AGE-STRUCTURED DIFFUSION MODEL

Doychin Boyadzhiev

Abstract. *In this paper we propose an optimized algorithm, which is faster compared to previously described finite difference acceleration scheme, namely the Modified Super-Time-Stepping (Modified STS) scheme for age-structured population models with diffusion.*

Keywords: additional numerical acceleration, modified super-time-stepping

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1. Introduction

The Super-Time-Stepping (STS) scheme is proved to be a simple and very effective method which accelerates explicit time stepping schemes for parabolic problems [1]. Even though the method is quite old, it is not known by most of the people working in the computational PDE world. Pelovska in [11] has applied it on equations of age-dependent population diffusion. While the analytical properties of such models have been extensively studied since years (see for instance [3, 4, 7] and the references therein), only several authors have dealt with the numerical study of age and space dependent population models. Kim [6], Kim-Park [5] and Milner [9] deal with nonlinear diffusion models. They propose some mixed numerical algorithms combining finite difference methods along characteristics and finite element methods in the spatial variables. In the case of linear fertility and mortality functions, Lopez and Trigiante [8] have developed a finite difference scheme for an age-dependent model with Dirichlet boundary conditions and linear population flux. Ayati [2] proposes a numerical method for a nonlinear model with nonlinear diffusion which allows the use of variable time steps and independent age and time discretization.

The authors' goal in this paper is to present an improved version of the Modified STS scheme (see [11]) adapted for solving an age-dependent population model with linear spatial diffusion. Let $p(a,t,x)$ be the density of a population having age $a \in [0,a_+]$, where a_+ is the maximum age; $t \in (0,T]$ denotes time, where T is the final time; $x \in (0,1)$ denotes spatial position and $D > 0$ is the coefficient of diffusion. Then, following [4], a mathematical model describing the evolution of the population $p(a,t,x)$ staring at time $t = 0$ with initial distribution

$$p(a,0,x) = p_0(a,x), \quad a \in [0,a_+],\ x \in (0,1) \tag{1.1}$$

is:

$$p_t + p_a + \mu(a)p = Dp_{xx}, \quad a \in [0, a_+],\ t \in (0, T],\ x \in (0, 1), \tag{1.2}$$

where $\mu(a) \geq 0$ is the natural death rate of the species. We add to this model the birth process

$$p(0, t, x) = \int_0^{a_+} \beta(a)p(a, t, x)\ \mathrm{d}a, \quad t \in (0, T],\ x \in (0, 1), \tag{1.3}$$

with $\beta(a) \geq 0$ representing the age specific fertility, and the following Dirichlet conditions on the boundary

$$p(a, t, 0) = p(a, t, 1) = 0, \quad a \in [0, a_+],\ t \in (0, T]. \tag{1.4}$$

Aiming to present a more realistic case where the species are with a finite life span, we assume the maximum age a_+ to be finite ($a \in [0, a_+]$, where $a_+ < +\infty$) and we require that the survival probability

$$\pi(a) = e^{-\int_0^a \mu(\tau)\mathrm{d}\tau} \tag{1.5}$$

vanishes at a_+.

In order to approximate our model we shall use a first order method combined with the trapezoidal rule for the integral terms. In [10] it is shown that this creates problems every time when an evaluation of the mortality function at the right endpoint a_+ of the interval is required, since $\lim\limits_{a \to a_+} \mu(a) = \infty$. Following [10] we take

$$u(a, t, x) = \pi^{-1}(a)p(a, t, x) \tag{1.6}$$

and then substituting with the new variable $u(a, t, x)$ in the equations above, we obtain a reformulation of the discussed model

$$\begin{aligned}
&1)\ u_t + u_a = Du_{xx}, \quad a \in [0, a_+],\ t \in (0, T],\ x \in (0, 1)\\
&2)\ u(0, t, x) = \int_0^{a_+} \beta(a)\pi(a)u(a, t, x)\ \mathrm{d}a, \quad t \in (0, T],\ x \in (0, 1)\\
&3)\ u(a, 0, x) = u_0(a, x), \quad a \in [0, a_+],\ x \in (0, 1)\\
&4)\ u(a, t, 0) = u(a, t, 1) = 0, \quad a \in [0, a_+],\ t \in (0, T]
\end{aligned} \tag{1.7}$$

Using this form of the equations describing our model, we can apply a finite difference scheme, since the qualitative features of the model are preserved but there are no more problems with its numerical treatment (see [10] for details).

2. Optimization of the Modified Super-Time-Stepping scheme

The Super-Time-Stepping algorithm [1] is an acceleration method for explicit schemes for parabolic problems. It relaxes the condition of stability at the end of each time step that is imposed for the normal explicit scheme and

demands stability at the end of each super-step ΔT, consisting of K sub-steps $\tau_1, \tau_2, \ldots, \tau_K$ with different length. These sub-steps can be found by the following explicit formula

$$\tau_k = \tau \left((-1+\nu) \cos\left(\frac{(2k-1)\pi}{2K} \right) + 1 + \nu \right)^{-1}, \quad k = 1, \ldots, K \tag{2.1}$$

where τ is the time step for the explicit scheme (2.5), calculated in such a way that the CFL (stability) condition is satisfied; ν is a number in the interval $\left(0, \frac{\lambda_{\min}}{\lambda_{\max}}\right]$ with $\lambda_{\min}$ and $\lambda_{\max}$ being the smallest and the biggest eigenvalues respectively of the matrix A in (2.6). It implies that we can take larger time steps and consequently the total number of steps is reduced which speeds the computations up, compared with the standard explicit scheme. The inner steps have no approximation properties and can be chosen explicitly in such a way that stability is ensured over the super-step and we obtain a maximum duration of

$$\Delta T = \sum_{k=1}^{K} \tau_k \,. \tag{2.2}$$

Inspired by the fact that along characteristics in the age-time direction the governing equation in (1.7) can be treated as parabolic differential equation (see [11] for details), we proceed as follows introducing some convenient notation. We assume the step size in age identical to the step size in time and we choose $\tau > 0$ to be the age and time discretization parameter, where $\tau = \frac{a_+}{L}$ (L is the number of subintervals in age). We assume T is a multiple of a_+, so that we have $T = L_1 a_+ = L_1 L \tau = N\tau$, where L_1 is an integer and N is the total number of subintervals in time. Let $h = \frac{1}{M}$ be the discretization step in space, where M is the number of subintervals in space. Then for each time level $t^n = n\tau$, $n = 0, \ldots, N$ we have the following grid: $\Gamma = \{(a^j, x_i) : a^j = j\tau,\, j = 0, \ldots, L;\, x_i = ih,\, i = 0, \ldots, M\}$. With this notation, we approximate the directional derivative $\frac{\partial}{\partial t} + \frac{\partial}{\partial a}$, setting

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) u(a^j, t^n, x_i) \approx \frac{\widehat{U}_i^{j+1} - U_i^j}{\tau} \,, \tag{2.3}$$

where the discrete function U_i^j is an approximation of the solution of (1.7) at time level t^n at grid point (a^j, x_i) and $\widehat{U}_i^{j+1}$ – at time level t^{n+1} at grid point (a^{j+1}, x_i).

An approximation of the Laplace operator is given by

$$U_{xx} = \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2} \,. \tag{2.4}$$

Consequently an approximation of problem (1.7) by an Euler explicit scheme (analogous to the one applied to the heat equation in [1]) is given by

$$\widehat{U}_i^{j+1} = \frac{D\tau}{h^2}U_{i-1}^j + \left(1 - \frac{2D\tau}{h^2}\right)U_i^j + \frac{D\tau}{h^2}U_{i+1}^j, \quad i = 1, \dots, M-1;\ j = 0, \dots, L-1 \tag{2.5}$$

$$\widehat{U}_0^{j+1} = \widehat{U}_M^{j+1} = 0,\ j = 0, \dots, L-1$$

or written in a more convenient form

$$\widehat{U}^{j+1} = AU^j,\ j = 0, \dots, L-1 \tag{2.6}$$

where A is an $(M-1) \times (M-1)$ symmetric and three-diagonal matrix.

We couple (2.5) with the trapezoidal rule for the boundary condition (1.7, 2)

$$\widehat{U}_i^0 = \tau \sum_{j=1}^{L-1} \beta_j \pi(a_j)\widehat{U}_i^j + \frac{\tau}{2}\left[\beta_0\pi(a_0)\widehat{U}_i^0 + \beta_L\pi(a_L)\widehat{U}_i^L\right], \quad i = 0, \dots, M \tag{2.7}$$

At the initial time $t = 0$ we take $U_i^j = \dfrac{p_0(a^j, x_i)}{\pi(a^j)}$, $j = 0, \dots, L$, $i = 0, \dots, M$.

This scheme is easy to be implemented, but it is conditionally stable, i.e. it is stable if the time step is very small, namely $\tau \leq \dfrac{2}{\lambda_{\max}}$ ($\lambda_{\max}$ being the biggest eigenvalue of the matrix A in (2.6)). In order to overcome this drawback and to increase the efficiency of the method while keeping the accuracy at the same time, we adapt the STS scheme for parabolic problems (see [1]) to the age-structured model as shown on the graph:

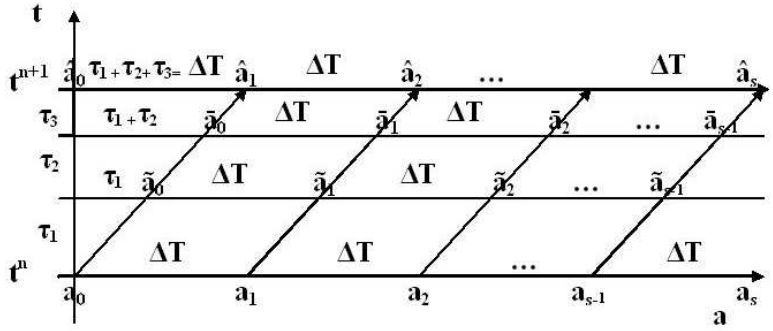

FIGURE 1. One super-time-step with $K = 3$ intermediate steps

The figure above shows how one super-time-step looks like. The vertical and the horizontal axis present the time and the age distributions respectively; τ_k, $k = 1, \dots, K$ are the inner-time-steps (on the graph we have taken $K = 3$). The similarity between Modified STS and STS is the way to move in time,

i.e. the super-time-stepping. However, while moving in time, the same steps in age have to be done. This is the basic difference between STS for parabolic problems and the modification presented in [11]. In the modified scheme the solution at the boundary points is calculated, but not at the intermediate time levels (since it is not needed for the approximation of the solution in the next time levels – see Figure 1). Since there are age nodes at each time level (as shown on Figure 1), the "discrete solution" at the k^{th} inner time level $k = 1, \ldots, K-1$ is calculated as follows

$$\begin{aligned} &\widehat{U}_i^j = \frac{D\tau_k}{h^2}(U_{i-1}^j + U_{i+1}^j) + \left(1 - \frac{2D\tau_k}{h^2}\right) U_i^j, \\ &\qquad\qquad i = 1, \ldots, M-1; j = 0, \ldots, s-1 \\ &\widehat{U}_0^j = \widehat{U}_M^j = 0, \quad j = 0, \ldots, s-1, \end{aligned} \tag{2.8}$$

where U_i^j is the "discrete solution" at the $(k-1)^{\text{st}}$ time level and it is considered as known; $s = \dfrac{a_+ K}{T}$ is the number of age-nodes (see Figure 1), which depends on K, i.e. on the length of one super-step ΔT. The "discrete solution", calculated at these inner steps has no approximation properties and it is not outputted. The approximation only at the end level – K corresponding to t^{n+1} time level is used. It is found by formula (2.5), but with time step τ_K, i.e. $\tau = \tau_K$. As we mentioned before at this level the solution at the boundary point is calculated as well, by formula (2.7) and time step ΔT. This procedure is repeated until the end of the time interval.

In [11] it is proved that in some cases the Modified STS algorithm can speed up the explicit scheme more than K^2 times. Additional acceleration of the Modified STS can be achieved when using some of the properties of its coefficients, namely that the coefficients c_l^k, $k = 1, \ldots, K$, $l = 0, \ldots, k$ of the k^{th} inner level, can be obtained by the coefficients c_l^{k-1} of the previous, $(k-1)^{\text{st}}$ intermediate time level by the following recursive formulas

$$\begin{aligned} &c_0^k = (1 - 2\sigma_k)\, c_0^{k-1} + 2\sigma_k c_1^{k-1}, \\ &c_l^k = \sigma_k \left(c_{l-1}^{k-1} + c_{l+1}^{k-1}\right) + (1 - 2\sigma_k)\, c_l^{k-1}, \quad l = 1, \ldots, k-1, \\ &c_k^k = \sigma_k c_{k-1}^{k-1}, \end{aligned} \tag{2.9}$$

where $\sigma_k = \dfrac{D\tau_k}{h^2}$, $c_l^{k-1} = 0$ for $l \geq k$ and we assume that in the beginning $c_0^0 = 1$, $c_l^0 = 0$, $l \geq 1$.

Using this dependence between c_l^k, $k = 1, \ldots, K$, $l = 0, \ldots, k$ and c_l^{k-1} and the fact, that the inner steps have no approximation properties (for the discrete solution), we can make only steps with length ΔT. Moreover, after one super step we have the following form of the Modified STS scheme

$$(2.10)\quad \widehat{U}_i^{j+1} = c_0^K U_i^j + \begin{cases} \displaystyle\sum_{l=1}^{i} c_l^K (U_{i+l}^j + U_{i-l}^j) + \sum_{l=i+1}^{K} c_l^K (U_{i+l}^j - U_{l-i}^j), \\ \qquad\qquad 1 \le i \le K-1 \\ \displaystyle\sum_{l=1}^{k} c_l^k (U_{i-l}^j + U_{i+l}^j), \quad K \le i \le M-K \\ \displaystyle\sum_{l=1}^{M-i} c_l^K (U_{i+l}^j + U_{i-l}^j) + \sum_{l=M-i+1}^{K} c_l^K (U_{i-l}^j - U_{2M-l-i}^j), \\ \qquad\qquad M-K+1 \le i \le M-1, \end{cases}$$

where $j = 0, \ldots, s-1$ and c_k^K, $k = 0, \ldots, K$ can be obtained explicitly by formula (2.9). In this way we present the solution at the new time level t^{n+1} as a linear combination of $2K+1$ nodes of the previous time level and we reduce the number of the arithmetical operations we do. Comparing the first equation in (2.8) and formulas (2.10) we see that the number of multiplications is reduced from $2K$ for the Modified STS to $K+1$ for the optimized algorithm. This means that we perform almost 2 times less multiplications (that is much time consuming arithmetical operation) for $K > 1$. Thus we save computational time while keeping the accuracy of the Modified STS scheme.

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Doychin Boyadzhiev
Faculty of Mathematics and Informatics
Plovdiv University
236, Bulgaria Blvd.
4003 Plovdiv, Bulgaria
e-mail: dtb@uni-plovdiv.bg