ON COMPOSITIONS IN EQUIAFFINE SPACE

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Abstract. *In an equiaffine space* E_{q^N} *using the connection* $\Gamma^{\sigma}_{\alpha\beta}$ *define with projective tensors* $\overset{n}{a}{}^{\beta}_{\alpha}$ *and* $\overset{m}{a}{}^{\beta}_{\alpha}$ *the connections* ${}^1\Gamma^{\sigma}_{\alpha\beta}$, ${}^2\Gamma^{\sigma}_{\alpha\beta}$ *and* ${}^3\Gamma^{\sigma}_{\alpha\beta}$. *For the spaces* ${}^1A_N, {}^2A_N$ *and* ${}^3A_N$, *with coefficient of connection* ${}^1\Gamma^{\sigma}_{\alpha\beta}$, ${}^2\Gamma^{\sigma}_{\alpha\beta}$ *and* ${}^3\Gamma^{\sigma}_{\alpha\beta}$ *respectively, we proved that the affinor of composition and the projective affinors have equal covariant derivatives. It follows that the connection* ${}^3\Gamma^{\sigma}_{\alpha\beta}$ *is equaffine as well, and the connections* $\Gamma^{\sigma}_{\alpha\beta}$ *and* ${}^3\Gamma^{\sigma}_{\alpha\beta}$ *are projective to each other. In the case where* E_{q^N} *and* ${}^3A_N$ *have equal Ricci tensors, we find the fundamental n-vector* ε.

In [4] compositions with structural affinor a^{β}_{α} *are studied. Space containing compositions with symmetric connection and Weyl connection are studied in [6] and [7] respectively.*

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1. Preliminaries

Let in differentiable manifold X_N consider field of affinor a^{β}_{α} satisfying

(1) $$a^{\beta}_{\alpha}\, a^{\sigma}_{\beta} = \delta^{\sigma}_{\alpha}.$$

In X_N the n-vector $\varepsilon_{i_1 \dots i_n}$ defines equiaffine connection $\Gamma^{\sigma}_{\alpha\beta}$ by

(2) $$\Gamma_{\alpha} = \Gamma^{\beta}_{\alpha\beta} = \partial_{\alpha}\, ln\,\varepsilon,$$

where $\varepsilon = \varepsilon_{1,2,\dots,n}$ is the fundamental n-vector of the space $\varepsilon_{i_1 \dots i_n}$ [1, p.150]. Denote by E_{q^N} the space in which $\Gamma^{\sigma}_{\alpha\beta}$ is given. The affinor a^{β}_{α}, for which (1) and

(3) $$a^{\sigma}_{\beta}\, \nabla_{[\alpha} a^{\nu}_{\sigma]} - a^{\sigma}_{\alpha}\, \nabla_{[\beta} a^{\nu}_{\sigma]} = 0$$

hold, defines the composition $X_n \times X_m$, $(m+n=N, m=n+1,\dots,N)$ in E_{q^N} [2], [3].

Through any point of the space of compositions $E_{q^N}(X_n \times X_m)$ there are two positions - $P(X_n)$ and $P(X_m)$ of the base manifolds X_N and X_N. Assume that $E_{q^N}(X_n \times X_m)$ has integrable structure.

The projective tensors $\overset{n}{a}{}_\alpha^\beta$ and $\overset{m}{a}{}_\alpha^\beta$ are defined by [3]:

(4) $$\overset{n}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta), \quad \overset{m}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta),$$

and satisfy

(5) $$\overset{n}{a}{}_\alpha^\beta \overset{n}{a}{}_\beta^\sigma = \overset{n}{a}{}_\alpha^\sigma, \quad a_\alpha^\beta \overset{n}{a}{}_\beta^\sigma = \overset{n}{a}{}_\alpha^\sigma, \quad \overset{m}{a}{}_\alpha^\beta \overset{m}{a}{}_\beta^\sigma = \overset{m}{a}{}_\alpha^\sigma,$$

$$a_\alpha^\beta \overset{m}{a}{}_\beta^\sigma = -\overset{m}{a}{}_\alpha^\sigma, \quad \overset{n}{a}{}_\alpha^\beta + \overset{m}{a}{}_\alpha^\beta = \delta_\alpha^\beta, \quad \overset{n}{a}{}_\alpha^\beta \overset{m}{a}{}_\beta^\sigma = 0.$$

The projective tensors transform the vectors of their positions into themselves, and the vectors of transversal positions into zero vectors.

Any vector $v^\alpha \in E_{q^N}(X_n \times X_m)$ has the following representation

$$v^\alpha = \overset{n}{a}{}_\sigma^\alpha v^\sigma + \overset{m}{a}{}_\sigma^\alpha v^\sigma = \overset{n}{v}{}^\alpha + \overset{m}{v}{}^\alpha,$$

where $\overset{n}{v}{}^\alpha = \overset{n}{a}{}_\sigma^\alpha v^\sigma \in P(X_n)$ and $\overset{m}{v}{}^\alpha = \overset{m}{a}{}_\sigma^\alpha v^\sigma \in P(X_m)$. For any covariant vector, we can find the projections onto $P(X_n)$ and $P(X_m)$ [5].

From (4) and (5), for $\overset{n}{\Gamma}_\alpha = \overset{n}{a}{}_\alpha^\sigma \Gamma_\sigma$ and $\overset{m}{\Gamma}_\alpha = \overset{m}{a}{}_\alpha^\sigma \Gamma_\sigma$, it follows that

(6) $$\Gamma_\alpha = \overset{n}{\Gamma}_\alpha + \overset{m}{\Gamma}_\alpha.$$

In [3] the invariant characteristics of special composition in multi-dimensional spaces are obtained from affinor a_α^β and projective tensors.

The composition $X_n \times X_m$ is Cartesian ($C-C$), if the positions $P(X_n)$ and $P(X_m)$ translate parallely along any line of the space. The composition is Chebichevian ($Ch-Ch$), if the positions $P(X_n)$ and $P(X_m)$ translate parallely along $P(X_m)$ and $P(X_n)$ respectively. The composition is geodesic ($G-G$), if the positions $P(X_n)$ and $P(X_m)$ translate parallely along $P(X_n)$ and $P(X_m)$ respectively. These conditions are characterized by:

(7) $\nabla_\sigma a_\alpha^\beta = 0$ $\qquad C-C$

(8) $\nabla_{[\sigma} a_{\alpha]}^\beta = 0$ $\qquad Ch-Ch$

(9) $a_\beta^\sigma \nabla_\alpha a_\sigma^\nu + a_\alpha^\sigma \nabla_\sigma a_\beta^\nu = 0$ $\qquad G-G$

Composition is $C-Ch$ ($Ch-C$), if the positions $P(X_n)$ and $P(X_m)$ translate parallely along $P(X_n)$ ($P(X_m)$).The composition is $X_n - C$ (C - X_m), if the position $P(X_m)(P(X_n))$ translates parallely along any line of the space. The composition is Ch - X_m (X_n-Ch), if the position $P(X_n)(P(X_m))$ translates parallely along $P(X_m)(P(X_n))$. The composition is G - X_m (X_n-G), if the positions $P(X_n)$ ($P(X_m)$) translate parallely along $P(X_n)(P(X_m))$. These conditions are characterized by:

(10) $\overset{n}{a}{}_\alpha^\sigma \nabla_\sigma \overset{n}{a}{}_\beta^\nu = 0$ ($\overset{m}{a}{}_\alpha^\sigma \nabla_\sigma \overset{m}{a}{}_\beta^\nu = 0$) $\quad C-Ch$ ($Ch-C$)

(11) $\overset{m}{a}{}_\alpha^\sigma \nabla_\sigma \overset{n}{a}{}_\beta^\nu = 0$ ($\overset{n}{a}{}_\alpha^\sigma \nabla_\sigma \overset{m}{a}{}_\beta^\nu = 0$) $\quad X_n - C$ (C- X_m)

(12) $\overset{m}{a}{}_\alpha^\sigma \overset{n}{a}{}_\beta^\nu \nabla_\sigma \overset{n}{a}{}_\nu^\rho = 0$ ($\overset{n}{a}{}_\alpha^\sigma \overset{m}{a}{}_\beta^\nu \nabla_\sigma \overset{m}{a}{}_\nu^\rho = 0$) $\quad Ch$ - X_m (X_n-Ch)

(13) $\overset{n}{a}{}_\alpha^\sigma \overset{n}{a}{}_\beta^\nu \nabla_\sigma \overset{n}{a}{}_\nu^\rho = 0$ ($\overset{m}{a}{}_\alpha^\sigma \overset{m}{a}{}_\beta^\nu \nabla_\sigma \overset{m}{a}{}_\nu^\rho = 0$) $\quad G$ - X_m (X_n-G)

2. EQUIAFFINE SPACES OF COMPOSITIONS

Consider the following connections :

(14) $${}^1\Gamma_{\alpha\beta}^\sigma = \Gamma_{\alpha\beta}^\sigma + \overset{n}{a}{}_\alpha^\sigma \overset{n}{\Gamma}_\beta + \overset{n}{a}{}_\beta^\sigma \overset{n}{\Gamma}_\alpha ,$$

(15) $${}^2\Gamma_{\alpha\beta}^\sigma = \Gamma_{\alpha\beta}^\sigma + \overset{m}{a}{}_\alpha^\sigma \overset{m}{\Gamma}_\beta + \overset{m}{a}{}_\beta^\sigma \overset{m}{\Gamma}_\alpha ,$$

(16) $${}^3\Gamma_{\alpha\beta}^\sigma = \frac{1}{2}({}^1\Gamma_{\alpha\beta}^\sigma + {}^2\Gamma_{\alpha\beta}^\sigma).$$

Connection ${}^3\Gamma_{\alpha\beta}^\sigma$ is called the average connection of ${}^1\Gamma_{\alpha\beta}^\sigma$ and ${}^2\Gamma_{\alpha\beta}^\sigma$ [1, p.164].

Suppose E_{q^N}, ${}^1A_N, {}^2A_N$ and ${}^3A_N$ are spaces with coefficients of connection $\Gamma_{\alpha\beta}^\sigma$, ${}^1\Gamma_{\alpha\beta}^\sigma$, ${}^2\Gamma_{\alpha\beta}^\sigma$ and ${}^3\Gamma_{\alpha\beta}^\sigma$ respectively. Let $\nabla, {}^1\nabla, {}^2\nabla, {}^3\nabla$ be the covariant derivatives in the spaces E_{q^N}, ${}^1A_N, {}^2A_N$ and ${}^3A_N$, respectively. By (1) E_{q^N}, ${}^1A_N, {}^2A_N$ and ${}^3A_N$ are spaces of compositions $X_n \times X_m$ ($m+n=N$) [2], [3].

Theorem 1. The covariant derivatives of the affinor a_α^β of composition in E_{q^N}, ${}^1A_N, {}^2A_N$ and ${}^3A_N$ are equal.

Proof. According to (14) we have

$$^1\nabla_\alpha a^\sigma_\beta - \nabla_\alpha a^\sigma_\beta = a^\nu_\beta(^1\Gamma^{\ \sigma}_{\alpha\nu} - \Gamma^{\ \sigma}_{\alpha\nu}) - a^\sigma_\nu(^1\Gamma^{\ \nu}_{\alpha\beta} - \Gamma^{\ \nu}_{\alpha\beta})$$

$$= a^\nu_\beta(\overset{n}{a}{}^\sigma_\alpha \overset{n}{\Gamma}_\nu + \overset{n}{a}{}^\sigma_\nu \overset{n}{\Gamma}_\alpha) - a^\sigma_\nu(\overset{n}{a}{}^\nu_\alpha \overset{n}{\Gamma}_\beta + \overset{n}{a}{}^\nu_\beta \overset{n}{\Gamma}_\alpha).$$

From (5) and (6) it follows that

$$^1\nabla_\alpha a^\sigma_\beta - \nabla_\alpha a^\sigma_\beta = \overset{n}{a}{}^\sigma_\alpha \overset{n}{\Gamma}_\beta + \overset{n}{a}{}^\sigma_\beta \overset{n}{\Gamma}_\alpha - \overset{n}{a}{}^\sigma_\alpha \overset{n}{\Gamma}_\beta - \overset{n}{a}{}^\sigma_\beta \overset{n}{\Gamma}_\alpha = 0, \text{i.e.}$$

(17) $$^1\nabla_\alpha a^\sigma_\beta = \nabla_\alpha a^\sigma_\beta.$$

Using (15) we obtain

$$^2\nabla_\alpha a^\sigma_\beta - \nabla_\infty a^\sigma_\beta = a^\nu_\beta(^2\Gamma^{\ \sigma}_{\alpha\nu} - \Gamma^{\ \sigma}_{\alpha\nu}) - a^\sigma_\nu(^2\Gamma^{\ \nu}_{\alpha\beta} - \Gamma^{\ \nu}_{\alpha\beta})$$

$$= a^\nu_\beta(\overset{m}{a}{}^\sigma_\alpha \overset{m}{\Gamma}_\nu + \overset{m}{a}{}^\sigma_\nu \overset{m}{\Gamma}_\alpha) - a^\sigma_\nu(\overset{m}{a}{}^\nu_\alpha \overset{m}{\Gamma}_\beta + \overset{m}{a}{}^\nu_\beta \overset{m}{\Gamma}_\alpha),$$

Similarly from (5) and (6) we establish,

$$^2\nabla_\alpha a^\sigma_\beta - \nabla_\alpha a^\sigma_\beta = -\overset{m}{a}{}^\sigma_\alpha \overset{m}{\Gamma}_\beta - \overset{m}{a}{}^\sigma_\beta \overset{m}{\Gamma}_\alpha + \overset{m}{a}{}^\sigma_\alpha \overset{m}{\Gamma}_\beta + \overset{m}{a}{}^\sigma_\beta \overset{m}{\Gamma}_\alpha = 0, \text{ i.e.}$$

(18) $$^2\nabla_\alpha a^\sigma_\beta = \nabla_\alpha a^\sigma_\beta.$$

From (17) and (18) it follows that $\nabla_\alpha a^\sigma_\beta = {}^1\nabla_\alpha a^\sigma_\beta = {}^2\nabla_\alpha a^\sigma_\beta$. From the last equations and according to (16) we have

$$^3\nabla_\alpha a^\sigma_\beta = \frac{1}{2}(^1\nabla_\alpha a^\sigma_\beta + {}^2\nabla_\alpha a^\sigma_\beta) = \nabla_\alpha a^\sigma_\beta, \text{ i.e.}$$

(19) $$\nabla_\alpha a^\sigma_\beta = {}^1\nabla_\alpha a^\sigma_\beta = {}^2\nabla_\alpha a^\sigma_\beta = {}^3\nabla_\alpha a^\sigma_\beta.$$

Corollary 1. If one of the spaces E_{q^N}, ${}^1A_N, {}^2A_N$ or ${}^3A_N$ has integrability of the structure, then the others also have integrability of the structure.

Corollary 1 follows from (1) and (19).

Corollary 2. The projective tensors $\overset{n}{a}{}^\beta_\alpha$ and $\overset{m}{a}{}^\beta_\alpha$ have equal covariant derivative in E_{q^N}, ${}^1A_N, {}^2A_N$ and ${}^3A_N$.

Corollary 2 follows from (4) and (19).

Corollary 3. If the composition $X_n \times X_m$ is some of $C-C$, $Ch-Ch$, $G-G$, $G-Ch$, $Ch-G$, X_n-C, $C-X_m$, X_n-Ch, $Ch-X_m$, X_n-G, or $G-X_m$ in one of the spaces E_{q^N}, ${}^1A_N, {}^2A_N, {}^3A_N$, then it is of the same kind in the rest of these spaces.

Corollary 3 follows from the invariant characteristics, corollaries 1 and 2.

Theorem 2. Connections $\Gamma_{\alpha\beta}^{\sigma}$ and ${}^3\Gamma_{\alpha\beta}^{\sigma}$ are projective between each other.

Proof. From (14) and (15), taking into account (4) we have

$$ {}^1\Gamma_{\alpha\beta}^{\sigma} = \Gamma_{\alpha\beta}^{\sigma} + \frac{1}{2}(\delta_{\alpha}^{\sigma} + a_{\alpha}^{\sigma})\overset{n}{\Gamma}_{\beta} + \frac{1}{2}(\delta_{\beta}^{\sigma} + a_{\beta}^{\sigma})\overset{n}{\Gamma}_{\alpha}\ , $$

$$ {}^2\Gamma_{\alpha\beta}^{\sigma} = \Gamma_{\alpha\beta}^{\sigma} + \frac{1}{2}(\delta_{\alpha}^{\sigma} - a_{\alpha}^{\sigma})\overset{m}{\Gamma}_{\beta} + \frac{1}{2}(\delta_{\beta}^{\sigma} - a_{\beta}^{\sigma})\overset{m}{\Gamma}_{\alpha}\ . $$

From (16) we obtain

$$ {}^3\Gamma_{\alpha\beta}^{\sigma} = \frac{1}{2}({}^1\Gamma_{\alpha\beta}^{\sigma} + {}^2\Gamma_{\alpha\beta}^{\sigma}) = \frac{1}{4}\delta_{\alpha}^{\sigma}(\overset{n}{\Gamma}_{\beta} + \overset{m}{\Gamma}_{\beta}) + \frac{1}{4}\delta_{\beta}^{\sigma}(\overset{n}{\Gamma}_{\alpha} + \overset{m}{\Gamma}_{\alpha})\ , $$

And taking into account (6) we establish

(20) $$ {}^3\Gamma_{\alpha\beta}^{\sigma} = \Gamma_{\alpha\beta}^{\sigma} + \frac{1}{4}(\delta_{\alpha}^{\sigma}\Gamma_{\beta} + \delta_{\beta}^{\sigma}\Gamma_{\alpha}). $$

Thus between $\Gamma_{\alpha\beta}^{\sigma}$ and ${}^3\Gamma_{\alpha\beta}^{\sigma}$ there exists projective correspondence. The vector of the projective transformation is

$$ p_{\alpha} = \frac{1}{N+1}({}^3\Gamma_{\alpha} - \Gamma_{\alpha}) = \frac{1}{N+1}\ \frac{N+1}{4}\ (\delta_{\alpha}^{\beta}\Gamma_{\beta} + \delta_{\beta}^{\beta}\Gamma_{\alpha}) = \frac{1}{4}\Gamma_{\alpha}\ . $$

Theorem 3. The space ${}^3A_N$, with coefficient of connection ${}^3\Gamma_{\alpha\beta}$ is equiaffine.

Proof. For the tensor of affine transformation, from (6) and (16) we have:

(21) $$ T_{\alpha\beta}^{\sigma} = {}^3\Gamma_{\alpha\beta}^{\sigma} - \Gamma_{\alpha\beta}^{\sigma} = \frac{1}{4}(\delta_{\alpha}^{\sigma}\Gamma_{\beta} + \delta_{\beta}^{\sigma}\Gamma_{\alpha}). $$

Denote by $R_{\alpha\beta\sigma.}^{\ \ \ \ \nu}$ and ${}^3R_{\alpha\beta\sigma.}^{\ \ \ \ \nu}$ the tensors of the curvature of E_{q^N} and ${}^3A_N$ respectively. The following equation holds [1, p.133]

$$ {}^3R_{\alpha\beta\sigma.}^{\ \ \ \ \nu} = R_{\alpha\beta\sigma.}^{\ \ \ \ \nu} + 2\nabla_{[\alpha}T_{\beta]\sigma}^{\nu} + 2\,T_{\rho[\alpha}^{\nu}T_{\beta]\sigma}^{\rho}\ . $$

For the Ricci tensors $R_{\beta\sigma}$ and ${}^3R_{\beta\sigma}$ of E_{q^N} and ${}^3A_N$ respectively, after contracting the above equality along the indices α and ν we obtain

$$ {}^3R_{\beta\sigma} = R_{\beta\sigma} + 2\,\nabla_{[\alpha}T_{\beta]\sigma}^{\alpha} + 2\,T_{\rho[\alpha}^{\alpha}T_{\beta]\sigma}^{\rho} $$

$$ = R_{\beta\sigma} + \nabla_{\alpha}T_{\beta\sigma}^{\alpha} - \nabla_{\beta}T_{\alpha\sigma}^{\alpha} + T_{\rho\alpha}^{\alpha}T_{\beta\sigma}^{\alpha} - T_{\rho\beta}^{\alpha}T_{\alpha\sigma}^{\rho}\ . $$

Thus taking into account $\nabla_{\sigma}\Gamma_{\beta} = \nabla_{\beta}\Gamma_{\sigma}$ and (21) we establish

(22) $$ {}^3R_{\beta\sigma} = R_{\beta\sigma} + \frac{N-1}{4}\left(\frac{1}{4}\Gamma_{\beta}\Gamma_{\sigma} - \nabla_{\sigma}\Gamma_{\beta}\right). $$

The tensor $R_{\beta\sigma}$ of the equiaffine space E_{q^N} is symmetric [1, p.150], i.e. the right hand side of (22) is symmetric. Thus the Ricci tensor ${}^3R_{\beta\sigma}$ is symmetric as well. From (9) it follows that the coefficients of connection of ${}^3A_N$ ${}^3\Gamma_{\alpha\beta}$ are symmetric. A space with symmetric connection, having symmetric Ricci tensor, then the space is equiaffine i.e. ${}^3A_N$ is equiaffine.

Example. Given coordinate system u^α ($\alpha = 1, 2, \ldots, n + m = N$) in E_{q^N}, we want to find is the fundamental n-vector of the space E_{q^N}, whenever Ricci tensors of E_{q^N} and ${}^3A_N$ are equal. From (22), for the coefficients of connection we have:

$$\Gamma_\beta \Gamma_\sigma - 4\ \Gamma_{\beta\sigma} = 0\,.$$

From where we obtain

(23) $$\Gamma = -4\ln\left|u_1 + \ldots + u_n + \ldots + u_N\right|.$$

Thus, taking into account (2), for the fundamental n-vector of the space E_{q^N} we establish

$$\varepsilon = (u_1 + \ldots + u_n + \ldots + u_N)^{-4}\,.$$

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