GROUP RINGS – A BRIEF HISTORY AND SOME THEIR GENERALIZATIONS

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Abstract. *We give a brief exposition of the history of the group rings and some their generalizations. Also we indicate some information of certain problems and results.*

Keywords: Group rings, Crossed products, Group graded rings, Partial group rings

2010 Mathematics Subject Classification: 16S34, 16S35, 20C05, 20C07

1. Introduction

Let G be a multiplicative group and let K be an associative ring. Denote by KG the free left K-module with a basis G. Then every element $u \in KG$ has the form

$$u = \sum_{g \in G} \alpha_g g \qquad (\alpha_g \in K),$$

where $Supp(u) = \{g \in G \mid \alpha_g \neq o\}$ is a finite set. The subgroup $\langle Supp(u) \rangle$ of G, generated by $Supp(u)$, is said to be support subgroup of $u \in KG$.

Note that the definition of KG implies that the element u and the element

$$v = \sum_{g \in G} \beta_g g \qquad (\beta_g \in K)$$

are equal if and only if $\alpha_g = \beta_g$ for all $g \in G$. Moreover,

$$u + v = \sum_{g \in G} (\alpha_g + \beta_g) g$$

and

$$\alpha u = \sum_{g \in G} (\alpha \alpha_g) g \qquad (\alpha \in K).$$

For arbitrary elements $u, v \in KG$ can be defined a product

$$uv = \sum_{g,h \in G} \alpha_g \beta_h gh = \sum_{f \in G} \gamma_f f,$$

where

$$\gamma_f = \sum_{gh = f} \alpha_g \beta_h .$$

Since $(\alpha g)(\beta h) = (\alpha\beta) gh$ for all $\alpha, \beta \in K$ and $g, h \in G$, the product uv can be defined by this rule and the distributive law.

It is easily verified that with the above operations KG is a ring, which is called the group ring of the group G over the ring K. If K is commutative, then

KG is algebra over K, and in this case KG is frequently called a group algebra of G over K.

Another way of looking at KG is as set of all functions $u : G \to K$ with almost all values $u(g)$ equal to zero with pointwize addition $(u+v)(g) = u(g) + v(g)$ and convolution

$$(uv)(g) = \sum_{hf=g} u(h)v(f) .$$

If G is a semigroup, then KG is a semigroup ring. In fact, there exist natural and useful extensions of this concept.

2. A brief history

The concept of the group rings is relatively old. This theory is a union of two theories – Theory of group and Theory of rings. Thus the history of the group rings is an history of groups and rings.

It is well known that the attention on the permutations was focused in 1730 by the work on the algebraic equations of J. L. Lagrange (1736 - 1813), following in 1799 by P. Ruffini (1765 - 1822) and N. H. Abel (1802 - 1829) in 1826. In his classical work in 1830 E. Galois (1811 - 1832) was the first to consider groups and subgroups of permutations, using the term group in its modern sense – though restricted to permutations – and introducing such concepts as those of normal subgroup, solvable group etc. Implicitly, examples of groups are studied also by L. Euler (1707 - 1783) and K. F. Gauss (1777 – 1855).

A. L. Cauchy (1789 - 1857) was a pioneer in understanding the relevance of permutation groups as an independent subject. He wrote a series of interesting papers about them, in period 1844 -1846. Influenced by Cauchy's work, in 1854 Arthur Cayley (1821 - 1895) recognized that the notion of a group could be formulated in a more abstract setting and gave the first definition of an abstract group in [11]. The paper [11] is considered by several authors as the beginning of abstract group theory. It contains a number of important features:

- Gives an abstract definition of a group in multiplicative notion;
- Introduces the table of an operation;
- Shows that there exist two nonisomorphic groups of order four, giving explicit examples;
- Shows that there exist two nonisomorphic groups of order six, one at them being commutative and other is isomorphic to S_3;
- Shows that the order of every element is a divisor of the order of the group.

The study of rings originated from the theory of polynomials. In the 16th century were introduced the complex numbers as a result of the work of Italian mathematicians while studying equations of the third degree. A long controversy regarding their existence and meaning was raised, and they gradually gained acceptance after a geometrical interpretation was given by Caspar Wessel (1745 - 1818), Jean-Robert Argan (1768 - 1822) and K. F. Gauss (1777 – 1855). However, though better understood, a need for an algebraic system in which the square of a "quantity" would actually be equal to -1 was still felt.

In 1837 W. R. Hamilton (1805 - 1865) gave the first formal theory of complex numbers, defining them as ordered pairs of real numbers, just as is done nowadays, thus ending almost three hundred years of discussions regarding their legitimacy.

Moreover, he came to consider elements of the form

(1) $$\alpha = a + bi + cj + dk \qquad (a, b, c, d \in R),$$

which he called quaternions. It was obvious to him that such elements should be added componentwize. The main difficulty was to define the product of two elements in a reasonable way. Since this product should have the usual properties of a multiplication, such as the distributive law, it is would actually be enough to decide how to multiply the symbols i, j, k among themselves. Hamilton also assumed implicitly that the product should be commutative. Finally, in October 1843 he discovered the fundamental laws of the product of quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

So Hamilton [24] discovers the first noncommutative algebraic structure in the history of mathematics.

The quaternions are studied also by Gauss, but he does not publish its results.

Although the theory of determinants began in 18th century with the works of G. W. Leibnitz (1646 - 1716) in 1693, C. Maclaurin (1698 - 1746) in 1729 and G. Cramer (1704 - 1752) in 1750 in connection with the resolution of linear systems of equations, it precedes the explicit formulation of the notion matrix with more of 100 years. The notion of matrix first is defined by J. J. Silvester (1814 - 1897) in 1848 and Cayley in 1855 as a convenient notation to express linear systems and quadratic forms. In a subsequent paper in 1858 Cayley defines the addition, the multiplication by scalars and the product of matrices, studying the properties of these operations without explicit mention of connection with hypercomplex system. After, Sylvester in 1884 and E. Study (1862 – 1930) in 1889 observe that the total $n \times n$ matrix algebra can be viewed as an n^2-dimensional vector space.

In 1845 Cayley introduced a new set of numbers, the octionions of the form

$$\alpha = a_0 + a_1 e_1 + a_2 e_2 + ... + a_7 e_7 \qquad (a_k \in R),$$

where the symbols e_i $(1 \le i \le 7)$ are the basic units. The octonions are known as Cayley numbers. Hamilton himself realized that it was possible to extend this construction even further and he first defined biquaternions, which are elements of the form (1), where a, b, c, d are complex numbers. Soon afterwards he introduced the hypercomplex systems. These are elements of the form

$$\alpha = a_0 + a_1 e_1 + a_2 e_2 + ... + a_n e_n ,$$

where again the sum is defined componentwise and the multiplication is determined by the distributive law and the formulas

$$e_i e_j = \sum_{k=1}^{n} \alpha_k (i, j) e_k \ .$$

Thus, in fact he introduces the contemporary notions for algebras.

These facts were the first steps in the development of the ring theory. In 1871 Benjamin Pirce (1809 - 1880) gave a classification of algebras known at the time and determined 162 algebras of a dimension $n \le 6$. As tools of his methods of classification, B. Pirce introduced some very important ideas in ring theory, such as idempotent and nilpotent elements, and the use of idempotents to obtain a decomposition of a given algebra. Impressed by these results both Theodor Molien (1861 – 1941) and E. J. Cartan (1869 – 1951) obtained, independently, important results regarding the structure theory of finite-dimensional real or complex algebras, introducing in this context the notions of simple and semisimple algebras and characterizing the simple algebras as full matrix algebras.

The concept of a ring first arose from attempts to prove Fermat's last theorem, starting with Richard Dedekind (1831 - 1916) in the 1880s. The term ring was coined by David Hilbert (1862 - 1943) in 1892 for a specific ring. The first axiomatic definition of a ring was given by Adolf Fraenkel (1891 - 1965) in 1914. In 1921 Emmy Noether (1882 - 1935) gave the first axiomatic foundation of the theory of commutative rings.

The concept of group rings appears implicitly in the paper [11] of A. Cayley in 1854, which is considered as the first work in the abstract theory of groups. But the concept of group rings was explicitly introduced in 1892 by T. Molien [46].

In his paper Cayley actually gave the formal construction of the group ring CS_3 and, in essentially, this way we do today. However, not even the basic notions of the theory of rings and algebras, were formulated at this time and this concept remained in noticed. T. Molien (Fedor Eduardovich Molin) obtained important results regarding the structure of finite-dimensional real and complex algebras, introducing the notions of simple and semisimple algebras and characterizing the simple algebras as complete matrix algebras. In a subsequent paper [47] in 1897 Molien obtained important results relating the representability of a given discrete group in the form of a homogeneous linear substitution group. In this way, he discovered some of the basic results in theory of complex representations of finite groups, including the orthogonality relations for group characters.

The connection between group representation theory and the structure theory of algebras – which is obtained through group algebras – was widely recognized after the most influential papers of Emmy Noether and Richard Brauer (1901 - 1977), giving the subject a new impulse. A classic and encyclopedic book on the theory of representation of finite groups and associative algebras is the book of Ceartis and Reiner [14].

Let us remember that Graham Higman (1917-2008) investigated and raised important questions about units of integral group rings in [26]. Those questions originated from his investigations of the Whitehead torsion in topology. An important role in this direction has and the paper [2] of S. D. Berman.

Later, the subject gained importance of its own after the inclusion of questions on group rings in Kaplansky's famous list of problems [28, 29]. Other important facts to stimulate the area where the paper by I. Connell [12] considering ring-theoretical questions about group rings, it follows the inclusion of chapters on group rings in the books on ring theory by Lambek [35] and Ribenboim [57], as well as the publication of the first book entirely devoted to the subject, by Donald S. Passman [51].

Several new books on the subject have been published in recent years by A. A. Bovdi [5, 6, 7], G. Karpilovsky [30, 32], I. B. S. Passi [50], C. Polcino-Milies and S. K. Sehgal [55], K. W. Roggekamp and M. J. Taylor [58], S. K. Sehgal [60, 61]. Passman's "The algebraic structure of group rings" [52] is truly classic and encyclopedic. For more detail on the history of group rings and some other references see [54, 62, 63].

3. Some generalizations of group rings

From here on for the sake of brevity, we shall use the terminology of [1, 34].

Let KG be an arbitrary group ring of a group G over a ring K and let H be a normal subgroup of G with a fixed transversal $T(G/H)=\{g_i \mid i\in I\}$. Then

$$G=\bigcup_{i\in I} Hg_i$$

and every element $u\in KG$ can be represented in the form

$$u=\sum_{i\in I} a_i g_i \qquad \left(g_i\in T(G/H),\ a_i\in KH \right).$$

If

$$v=\sum_{i\in I} b_i g_i \qquad \left(g_i\in T(G/H),\ b_i\in KH \right)$$

is an other element of KG, then $u=v$ if and only if $a_i=b_i$ for all $i\in I$. Thus KG is a left free KH-module with a basis $T(G/H)$. Since H is a normal subgroup of G and

$$(a_i g_i)(b_j g_j)=a_i(g_i b_j g_i^{-1})g_i g_j=a_i b_j^{\sigma(g_i)}\rho(g_i,g_j)g_k,$$

where

$$g_i b_j g_i^{-1}=b_j^{\sigma(g_i)}\in KH, \quad g_i g_j=\rho(g_i,g_j)g_k\in Hg_k, \quad k\in I,$$

we conclude that the product uv again is an element of the KH-module KG. Observe that the map $\sigma(g_i):KH\to KH$ is an automorphism of KH and $\rho(g_i,g_j)$ is an invertible element of KH for all $i,j\in I$. So we receive a new algebraic structure, which can be defined by the following way.

Let G be a group and K a ring with identity. Suppose that are given a function $\rho:G\times G\to K^*$ and a mapping $\sigma:G\to AutK$, where K^* is the multiplicative group of K and $AutK$ is the group of the automorphisms of K. Let $K*G$ be a free left K-module with basis $\overline{G}=\{\ \overline{g}\mid g\in G\ \}$, where every element $\overline{g}\in G$ is a symbol, corresponding to $g\in G$. Thus each element $u\in K*G$ is a finite sum of the form

$$u = \sum_{g\in G} \alpha_g \overline{g} \qquad (\alpha_g \in K).$$

Moreover, the equality and the addition in $K * G$ are defined componentwize. Assume that

$$\overline{g}\overline{h} = \rho(g,h)\overline{gh}, \qquad \overline{g}\alpha = \alpha^{\sigma(g)}\overline{g} \qquad (g,h \in G,\ \alpha \in K),$$

where $\rho(g,h) \in K^*$ is an invertible element of K and $\alpha^{\sigma(g)}$ is the image of $\alpha \in K$ under the action of the automorphism $\sigma(g) \in AutK$. Then these conditions induce a multiplication

$$\left(\sum \alpha_g \overline{g}\right)\left(\sum \beta_h \overline{h}\right) = \sum \alpha_g \beta_h^{\sigma(g)} \rho(g,h)\overline{gh} = \sum \gamma_f \overline{f} \in K * G,$$

where

$$\gamma_f = \sum_{gh=f} \alpha_g \beta_h^{\sigma(g)} \rho(g,h) \in K.$$

When the basis $\overline{G} = \left\{ \overline{g} \mid g \in G \right\}$ satisfies the conditions

(2) $$\overline{f}(\overline{g}\overline{h}) = (\overline{f}\overline{g})\overline{h}, \quad \overline{g}(\overline{h}\alpha) = (\overline{g}\overline{h})\alpha, \quad \alpha \in K,$$

with these so defined two operations $K * G$ is an associative ring, called crossed product of a group G over a ring K with system of factors ρ and mapping σ. We us also the designations (G, K, ρ, σ) and $K_\rho^\sigma G$.

It is easy to verify that the conditions (2) are equivalent with the conditions

$$\rho(f,g)\rho(fg,h) = \rho(g,h)^{\sigma(f)} \rho(f,gh),\ \alpha^{\sigma(gh)} = \rho(g,h)^{-1} \alpha^{\sigma(h)\sigma(g)} \rho(g,h).$$

As a special case, if $\rho(g,h) = 1$ for all $g, h \in G$, then we get the skew group ring $(G, K, 1, \sigma) = K^\sigma G$. If $\sigma(g) = 1$ for all $g \in G$, then we have the twisted group ring $(G, K, \rho, 1) = K_\rho G$. In addition, if we have $\rho(g,h) = 1$ and $\sigma(g) = 1$ for all $g, h \in G$, then we obtain the group ring KG.

The preceding facts show that every group ring KG can be regarded as a crossed product $KG = KH * G/H$, where $H \lhd G$, i.e. H is a normal subgroup of G. So the methods of the crossed products can be used in the theory of group rings. Moreover, if $H \lhd G$, then in [10] is proved that $K * G$ is a suitable crossed product of the quotient group G/H over the subring $K * H$, i.e. $K * G = (K * H) * G/H$. Moreover, if J is a G-invariant ideal of K, i.e. $J^{\sigma(g)} = J$ for all $g \in G$, then $J * G = J(K * G)$ is an ideal of $K * G$ and $(K * G)/(J * G) \cong (K/J) * G$, where $(K/J) * G$ is a crossed product of G over the quotient ring K/J. At first, these methods are used in [10] and [36].

Classically, crossed products of arbitrary finite groups over fields were introduced by E. Noether [49, 64] in 1929 in her lectures in Gottingen. Earlier, the special case of cyclic algebras was defined by Dickson [16, 17] in 1906 and the first significant result about them was proved by Wedderburn [65] in 1914. After, in the early 1940s N. Jacobson introduces crossed products of finite groups over division rings. Although he was working over a division rings, all the essential ingredients for the general case appear in his book [27]. However, the crossed products of general groups and rings are introduced 20 years later by A. A. Bovdi [3, 4]. Namely A. Bovdi at first regards the crossed products as a generalization of the group rings. Now the main results on this area are collected in [31, 53].

Other more large generalization of the group rings is the notion group graded ring.

Let G be an arbitrary multiplicative group. A ring R is said to be G-graded if there is a family $\left\{ R_g \mid g \in G \right\}$ of additive subgroup of R such that the additive group of R is a direct sum

$$R = \sum_{g \in G} \oplus R_g$$

and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Moreover, if $R_g R_h = R_{gh}$ holds for all $g, h \in G$, then R is said to be strongly G-graded ring. Here the product $R_g R_h$ is the usual module, the product consisting of all finite sums of ring products $r_g r_h$ of elements $r_g \in R_g$ and $r_h \in R_h$.

Group-graded rings were introduced by E. C. Dade [15] in 1980 as a formal way to deal with finite group representation problems. In addition, group-graded rings occur naturally certain Galois theory situations and, of course, they are related to crossed products.

Other generalizations of the crossed products are as called Hopf algebras, C^*-algebras, smash products, partial skew group rings, partial crossed products etc. As an example we shall define the notion partial skew group ring.

Let G be a multiplicative group and K be an associative ring with identity. A partial action σ of G on K is a collection of ideals $D_g \lhd K$ $(g \in G)$ and ring isomorphisms $\sigma(g): D_{g^{-1}} \to D_g$ such that

(i) $D_1 = K$ and $\sigma(1)$ is the identity map of K;

(ii) $(D_{g^{-1}} \cap D_h)^{\sigma(g)} = D_g \cap D_{gh}$;

(iii) $a^{\sigma(h)\sigma(g)} = a^{\sigma(gh)}$ for all $a \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

Then the partial skew group ring of K and G is defined to be the projective left K-module

$$K^{\sigma}_{par} G = \sum_{g \in G} \oplus D_g \overline{g},$$

where the multiplication is defined by the conditions

$$(a_g \overline{g})(b_h \overline{h}) = (a_g{}^{\sigma(g)^{-1}} b_h)^{\sigma(g)} \overline{gh}.$$

It is easy to see, using the conditions (i) – (iii), that $(a_g\overline{g})(b_h\overline{h}) \in D_{gh}\overline{gh}$ and thus the multiplication in $K_{par}^{\sigma}G$ is well defined. So $K_{par}^{\sigma}G$ is a ring, which is not always associative.

The notion of a partial action was been introduced in 1994 by R. Exel [21] in the study of C^*-algebras. More details for partial group rings and partial crossed products can be seen in [13, 20, 21, 22].

4. Main problems and some results

As was market above, the theory of group rings is a product of the theory of group and the theory of rings. Thus this theory is a meeting point of various algebraic theories. It is worthwhile to mention that group rings are important in other branches of mathematics, such as homological algebra, algebraic topology and algebraic K-theory, and that during the last decades significant applications have been obtained in the theory of error correcting codes which are used in digital transmissions, allowing the creation of new codes which are simultaneously efficient and reliable. To get an idea about the importance of group rings in algebraic research, it is enough to observe that several great contemporary algebraists have worked at some point of their lives in the area, contributing fundamentally to its development. Among them we can mention S. A. Amitsur, H. Bass, E. Formanek, N. D. Gupta, I. N. Herstein, G, Higman, A. V. Jategaonkar, I. Kaplansky, W. May, D. S. Passman, K. W. Roggenkamp, W. Rudin, S. K. Sehgal, H. J. Zassenhaus etc. Also in the ex-Soviet Union there was great interest in the area of group rings which was represented by such known scientists as S. D. Berman, A. A. Bovdi, A. E. Zalesskii and A. V. Mikhalev. It is worth noting that S. D. Berman was one of the great specialists in the representation theory of groups.

Conditionally, the problems in the theory of group rings and its generalizations we divide in tree parts: Ring-theoretic properties, Group-theoretic properties and Problem of the isomorphism.

In view of the intimate connection with representation theory, it is natural the question when the group ring KG determines the group G. More formally, is it true that the isomorphism $KG \cong KH$ implies that $G \cong H$? A special interest is the case when $K = Z$, the ring of the integers. The first positive results on this conjecture were obtained in 1940 by G. Higman [25, 26] for finite abelian groups and for the Hamiltonian 2-groups. Until now the problem has not been solved completely, but several deep results have been obtained for various classes of groups as: finite metabelien groups, symmetric and alternating groups, finite groups which are multiplicative groups of some rings, finite nilpotent groups etc. In the early seventies, H. J. Zassenhaus formulated various conjectures about units and the isomorphism of integral group rings ZG (see [6, 7, 61, 62].

Isomorphic crossed products of some torsion free groups over prime rings are investigated only in [4, 41, 45]. The self-injective crossed products are studied in [39].

The multiplicative group $U(KG)$ of commutative group rings KG is studied by G. Higman, S. Berman, W. May, T. Mollov, N. Nachev (see [7, 30]). The group $U(KG)$ for noncommutative ring KG is studied by several authors (see [61]). The main problem here is to find necessary and sufficient conditions under which $U(KG)$ possess certain group-theoretic properties, for example to be solvable, nilpotent, torsion, locally finite, FC-group etc. Other problems are connected with free subgroups of $U(KG)$, dimension subgroups, group identities in $U(KG)$, Lie n-Engel subgroups, atomorphisms of KG etc. These results are exposed in [6, 7, 61]. Some results for the multiplicative group of crossed products $K*G$ are received in [45].

The first completely solved ring-theoretic problems are published by Ian G. Connell. In [12] Connell was proved that the group ring KG satisfies the descending chain condition (DCC) on the left ideals if and only if K satisfies this condition and G is a finite group. Moreover, KG is without nilpotent ideals if and only if K is without nilpotent ideals and the order of every finite normal subgroup of G is not zero divisor in K (see [35, 57]). After that, in [37, 66] the authors independently prove that KG satisfies the DCC on the principal left ideals if and only if K satisfies this condition and G is a finite group

In the area of the ring-theoretic properties the main problems are connected with idempotents, zero divisors, ideals, radicals, polynomials identities, Lie identities, rings of quotients, regularity, chain conditions, modules over group rings etc. Briefly we may say that all problems in the theory of rings and the theory of groups are problems in the theory of group rings and they generalizations. Some of these problems are as Fermat's Last Theorem, whose enunciation are easily comprehended, but they solution is very difficult. For example, by the early 1960s there had emerged two major and easily comprehended problems, namely:

1) If K has no nontrivial zero divisors it is true that KG also has no proper zero divisors for all torsion-free groups G?
2) It is true that the rational group ring QG is semisimple for all groups G?

More systematic information on the problems in the theory of group rings and the crossed products may be discovered in [63] and [48], respectively.

Finally, as examples, for an illustration of the theory, we shall indicate only some results, which are obtained in University of Plodiv.

First, we shall recall some definitions.

Let $K*G = K_\rho^\sigma G$ be any crossed product of G over K. Put

$$G_{\ker} = \{g \in G \mid \sigma(g) \in I(K)\}, \quad G_{inn} = \{g \in G \mid \sigma(g) \in X(K)\},$$

where $I(K)$ is the group of the inner automorphisms of K and $X(K)$ is the group of the X-inner automorphisms of K (see [51]). It is known that $G_{\ker}$ and G_{inn} are normal subgroups of G. Moreover, a subring S of the ring K is said to be G-invariant if $s^{\sigma(g)} \in S$ for all $s \in S$ and $g \in G$. It is easy to verify that

$$G(S) = \{g \in G | s^{\sigma(g)} = s \ \ for \ all \ \ s \in S\}$$

is a normal subgroup of G.

In [59] Rudin and Schneider formulated the hypothesis that every central idempotent $e = e^2$ of an arbitrary group ring KG has a finite support subgroup $\langle Supp(e) \rangle$. First this hypothesis is proved in [8, 9, 10]. For G-graded rings the notion support subgroup is defined as for the group rings.

So in [40] are proved the following theorems.

Theorem 1 [40]. If R is a semiprime strongly G-graded ring, then the support subgroup of every central algebraic element of R is a finite normal subgroup of G.

Theorem 2 [40]. If R is a strongly G-graded ring, then the support subgroup of every central idempotent of R is a finite normal subgroup of G.

Theorem 3 [40]. All idempotents of a strongly G-graded ring have finite support subgroups if and only if they are central, or G is a locally finite group.

From these theorems the main results of [8, 9, 10] follow.

There exists an hypothesis [8] that the group ring KG over commutative domain K contains non trivial idempotents if and only if the order of some element $g \in G$ is invertible in K. For finite groups this problem is solved by Colleman (see [60]) and for locally nilpotent groups it is solved in [8]. E. Formanek [23] shoved that the hypothesis has positive solution and for Noetherian groups. For twisted group rings of finite groups over commutative domain the problem also has a positive solution [33]. But the general case is open.

The finite support subgroups of the central idempotents are used for a characterization of the biregular group rings. Recall that the ring K is said to be biregular [1], if every principal two-sided ideal of K is generated by central idempotent. Therefore, every simple associative ring K is biregular.

In [38] is proved the following

Theorem 4 [38]. If K is a commutative ring, then the group ring KG is biregular if and only if G and K satisfy the following condition:

(*) G is locally finite, K is biregular and the order of every element $g \in G$ is an invertible element of K.

For crossed products the condition (*) is not necessary. So in [43] are funded conditions under which $K * G$ is a simple ring, where G can be an arbitrary group. Obviously, such crossed products are biregular. Namely, we have

Theorem 5 [43]. If K has no G-invariant ideals, then $K*G$ is simple if and only if $K*G_{\ker}$ has no G-invariant ideals.

Nevertheless, we have the following

Theorem 6 [38]. Assume that K satisfies the ACC, or DCC on the principal two-sided ideals. If G and K satisfy the condition (*), then every crossed product $K*G$ is biregular.

Denote by $P(K)$, $L(K)$, $U(K)$, $J(K)$ and $B(K)$, respectively, the lower nil radical, the locally nilpotent radical, the upper nil radical, the radical of Jacobson and the radical of Braun-McCoy of an associative ring K [1]. Then

$$P(K) \subseteq L(K) \subseteq U(K) \subseteq J(K) \subseteq B(K).$$

In [19] are proved the following main theorems.

Theorem 7 [19]. If $K\rho G$ is a twisted group ring of the group G over the ring K and the order of every torsion element of G is not a zero divisor in K, then

$$P(K)_{\rho}G \subseteq U(K\rho G) \subseteq U(K)\rho G.$$

Theorem 8 [19]. Let $K*G$ be a crossed product of a group G over a prime ring K of characteristic $p \geq 0$. If the subgroup G_{inn} has no p-elements when $p > 0$ and $U(K) = 0$, then $U(K*G) = 0$.

We shall say that the normal subgroup H of the group G controls the ideal I of $K*G$ if $I = (I \cap K*H)(K*G)$, i. e. the intersection $I \cap K*H$ generates I as an ideal of $K*G$. The main result of [42] is the following

Theorem 9 [42]. Let $K*G$ be any crossed product over the F-algebra K. If F is a G-invariant subfield of K, then $G(F)$ controls all ideals of $K*G$.

Hence we obtain

Theorem 10 [42]. Let $K*G$ be a crossed product over the F-algebra K and let F be a G-invariant subfield of K with $charF = p \geq 0$.

(i) If H is a normal subgroup of the group $G(F)$ such that $G(F)/H$ is a solvable group and all factors of its commutator series have no p-elements when $p > 0$, then $J(K*G) \subseteq J(K*H)(K*G)$;

(ii) If $G(F)$ is a locally nilpotent group without p-elements in the case $p > 0$, then $J(K*G) \subseteq J(K)(K*G)$.

A series of other results for the radicals $J(K*G)$ and $B(K*G)$ are obtained in [18, 44, 45].

In [33] K. Kolikov studies the ring-theoretical properties of crossed products as idempotents, algebraic elements, subrings, ideals, quotient rings, DCC on the left ideals and Lie nilpotent crossed products. He proves that all Lie nilpotent crossed products are twisted group rings. In particular, if the characteristic of the field K is not divisor of the order of every torsion element of G with finite many conjugate elements, then $K*G$ is Lie nilpotent if and only if $\sigma=1$, G is abelian and the factor set ρ is symmetric.

The notions for generalized regular near-rings and associative rings are studied in [56]. For example, the ring K is called ξN-ring if for every element $a\in K$ there exists an element $x\in K$, such that a^2x-a is a central nilpotent element of K. Such classes of rings are introduced in 1957 by Y. Ytumi. Rakhnev proves that the group ring KG over a commutative ring K with finite characteristic is ξN-ring if and only if G is a torsion abelian group. In [56] Rakhnev has obtained also several interesting results for the generalized regular near-rings and associative rings. Thus Rakhnev is the first Bulgarian author, which studies the near-rings.

Nowadays research in the theory of group rings and its generalizations is rather intense. Consulting the Zentralblatt MATH, one discovers that practically each month several articles are published in this area. Moreover, group rings occupy an important place in various international conferences both in group theory and ring theory. For more information see http://mat.polsl.pl/groups/ and http://www.math.wisc.edu/~passman/program.pdf.

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