Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 1 (2010) ISSN 1311-0454

ON A HYPERSINGULAR EQUATION OF A PROBLEM, FOR A CRACK IN ELASTIC MEDIA

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Abstract

We give a procedure to reduce a hypersingular integral equation, arising in $2d$ diffraction problems on cracks in elastic media, to a Fredholm integral equation of the second kind, to which it is easier and more effectively to apply numerical methods than to the initial hypersingular equation.

Mathematics Subject Classification 2010: 45DB05, 45E05, 78A45
Key Words and Phrases: hypersingular integrals, diffraction, cracks, Fredholm integral equations, singular integral equations

1. Introduction

A static $2d$ problem for a crack in elastic media ([6]) in a scalar approximation is described by the following hypersingular integral equation of the potential theory

$$\int\limits_{\ell} \frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}} \ln |\mathbf{y}-\mathbf{x}| u(\mathbf{y}) d\ell_{\mathbf{y}} = f(\mathbf{x}), \quad \mathbf{x} \in \ell, \tag{1}$$

where $\mathbf{n_x}, \mathbf{n_y}$ mean normal vectors at the points $\mathbf{x}, \mathbf{y}$ of the curve ℓ, $f(\mathbf{x}) \in C^\infty(\ell)$ and the crack contour ℓ is assumed to be a smooth curve. It may be open or closed. For definiteness we suppose that it is open. Let (a, c) and (b, d) be its end-points. In applications there exist various approximative approaches applied directly to such equations, see for instance [3]. The goal of this note is to give a direct closed form reduction of equation (1) to a Fredholm integral equation of the 2nd kind on an interval of the real axis, via the well known procedure of determining bounded solutions of singular integral equations with Cauchy kernel. An application of this or other numerical method to the obtained Fredholm integral equation is already a well studied matter.

We do not dwell on details related to the proper interpretation of the divergent hypersingular integrals, this follows the standard ideas based either on the Hadamard finite part or treating the hypersingular integral as a result of application of a differential operator to the corresponding integral operator, both ways coinciding in the case under the consideration, we refer to [5] for details.

2. Notation and preliminaries

In the sequel we denote

$$\mathbf{x} = (x, y), \quad \mathbf{y} = (s, t), \quad \mathbf{r} = \mathbf{y} - \mathbf{x} = (s - x, t - y), \quad r = |\mathbf{r}|.$$

We have

$$\frac{\partial(\ln r)}{\partial n_{\mathbf{y}}} = \frac{1}{r} \cos(\mathbf{r}^\wedge \mathbf{n}_{\mathbf{y}}) = \frac{(\mathbf{r} \cdot \mathbf{n}_{\mathbf{y}})}{r^2},$$

and then after easy calculations we obtain

$$\frac{\partial^2(\ln r)}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}} = -\frac{(\mathbf{n}_x \cdot \mathbf{n}_y) + 2\cos(\mathbf{r}^\wedge \mathbf{n}_y)\cos(\mathbf{r}^\wedge \mathbf{n}_x)}{r^2}. \tag{2}$$

Therefore, the initial equation (1) takes the form

$$\int\limits_{\ell} \frac{(\mathbf{n}_x \cdot \mathbf{n}_y) + 2\cos(\mathbf{r}^\wedge \mathbf{n}_y)\cos(\mathbf{r}^\wedge \mathbf{n}_x)}{r^2} u(\mathbf{y}) d\ell_{\mathbf{y}} = -f(\mathbf{x}). \tag{3}$$

3. Reduction of (1) to a Fredholm integral equation

Let $y = \varphi(x)$ be the equation of the crack curve ℓ in cartesian coordinates.

3.1. Reduction of (1) to a Fredholm integral equation

Since $\mathbf{n_y} = \frac{1}{\sqrt{1+\varphi'^2(s)}}(\varphi'(s), -1)$, we have

$$\mathbf{r} \cdot \mathbf{n_y} = \frac{(s-x)\varphi'(s) - (t-y)}{\sqrt{1+\varphi'^2(s)}},$$

and after easy calculations we obtain

$$\cos(\mathbf{r}^\wedge \mathbf{n_y}) = \frac{\mathbf{r} \cdot \mathbf{n_y}}{r} = \frac{sign(s-x)}{\sqrt{1+\varphi'^2(s)}} \cdot \frac{\varphi'(s) - A(x,s)}{\sqrt{1+A^2(x,s)}},$$

where

$$A(x,s) = \frac{\varphi(x) - \varphi(s)}{x-s}.$$

Similar calculations give

$$\cos(\mathbf{r}^\wedge \mathbf{n_x}) = \frac{sign(s-x)}{\sqrt{1+\varphi'^2(x)}} \cdot \frac{\varphi'(x) - A(x,s)}{\sqrt{1+A^2(x,s)}}.$$

Therefore,

$$\cos(\mathbf{r}^\wedge \mathbf{n_x})\cos(\mathbf{r}^\wedge \mathbf{n_y}) = \frac{[\varphi'(x) - A(x,s)][\varphi'(s) - A(x,s)]}{[1+A^2(x,s)]\sqrt{1+\varphi'^2(x)}\sqrt{1+\varphi'^2(s)}}. \tag{4}$$

Note that from (4) it is seen that $|\cos(\mathbf{r}^\wedge \mathbf{n_x})\cos(\mathbf{r}^\wedge \mathbf{n_y})| \le Cr^2 \le C_1(x-s)^2$ so that the kernel

$$\frac{(\mathbf{n}_x \cdot \mathbf{n}_y) + 2\cos(\mathbf{r}^\wedge \mathbf{n}_y)\cos(\mathbf{r}^\wedge \mathbf{n}_x)}{r^2}$$

$$= \frac{(\mathbf{n}_x \cdot \mathbf{n}_y)}{r^2} + 2\frac{\cos(\mathbf{r}^\wedge \mathbf{n}_y)\cos(\mathbf{r}^\wedge \mathbf{n}_x)}{r^2}$$

has singularity only in the first term. We substitute into (3), take into account that $r = |x-s|\sqrt{1+A^2(x,s)}$ and obtain

$$\int_a^b \frac{B(x,s)}{(x-s)^2} u(s)ds = -f(x), \tag{5}$$

where

$$B(x,s)=\frac{(1+\varphi'(x)\varphi'(s))(1+A^2(x,s))+2[\varphi'(x)-A(x,s)][\varphi'(s)-A(x,s)]}{(1+A^2(x,s))^2\sqrt{1+\varphi'^2(x)}\sqrt{1+\varphi'^2(s)}}$$

and for brevity we write

$$u(s)=u[s,\varphi(s)] \quad \text{and} \quad f(s)=f[s,\varphi(s)].$$

Note that $B(x,x)=\frac{1}{1+\varphi'^2(x)}$.

3.2. Reduction to a singular integral equation

We rewrite (5) as

$$\int\limits_a^b \left[\frac{1}{(x-s)^2}+C(x,s)\right]u(s)ds=g(x), \tag{6}$$

where $g(x)=-f(x)(1+\varphi'^2(x))$ and

$$C(x,s)=\frac{B_1(x,s)-1}{(x-s)^2}$$

is a regular kernel with

$$B_1(x,s)=\frac{B(x,s)}{B(x,x)}$$

$$=\frac{(1+\varphi'(x)\varphi'(s))(1+A^2(x,s))+2[\varphi'(x)-A(x,s)][\varphi'(s)-A(x,s)]}{(1+A^2(x,s))^2} \times\frac{\sqrt{1+\varphi'^2(x)}}{\sqrt{1+\varphi'^2(s)}}\,,$$

and $B_1(x,x)=1$. Expanding $B_1(x,s)$ by Taylor formula with respect to the variable s, it is not difficult to show that $|B_1(x,s)-1|\le C(x-s)^2$ in the case where $B(x,s)$ is differentiable up to order 3. Then $C(x,s)$ is a bounded function.

Equation (6) is nothing else but

$$\frac{d}{dx}\int\limits_a^b \left[\frac{1}{s-x}+D(x,s)\right]u(s)ds=g(x), \tag{7}$$

where $D(x,s)$ is a primitive of the kernel $C(x,s)$ with respect to the variable x. We may take it, for instance, in the form

$$D(x,s) = \int_a^x C(\mathbf{x}i, s)\, d\mathbf{x}i.$$

From (6) we have

$$\int_a^b \frac{u(s)}{s-x}\, ds + Ku(x) = G(x) + C, \tag{8}$$

where $G(x) = \int_a^x g(t)\, dt$ and

$$Ku(x) = \int_a^b D(x,t)u(t)\, dt$$

is an integral operator with a "good" kernel $D(x,t)$. The constant C will be determined from the condition that we look for bounded solutions $u(t)$.

3.3. Reduction to the Fredholm equation

Let

$$Su(x) = \frac{1}{\pi}\int_a^b \frac{u(t)}{t-x}\, dt$$

be the well known singular integral operator. It is well known ([1], [4]), that the equation

$$Su(x) = f(x)$$

has a unique solution in the class of bounded functions if and only if the right-hand side $f(x)$ satisfies the condition

$$\int_a^b \frac{f(t)\, dt}{\sqrt{(t-a)(b-t)}} = 0$$

and under this condition the unique solution is

$$u(x) = -\frac{\sqrt{(x-a)(b-x)}}{\pi}\int_a^b \frac{f(t)\, dt}{\sqrt{(t-a)(b-t)}(t-x)} =: S^{-1}f(x). \tag{9}$$

Therefore, inverting the singular operator in (8), we arrive at

$$u(x) + S^{-1}Ku(x) = S^{-1}G(x) + S^{-1}(C) \tag{10}$$

under the condition that

$$\int_a^b \frac{G(t) + C - Ku(t)}{\sqrt{(t-a)(b-t)}}\,dt = 0.$$

The latter condition provides the value of C (expressed in terms of the solution $u(t)$):

$$C \cdot B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_a^b \frac{Ku(t) - G(t)}{\sqrt{(t-a)(b-t)}}\,dt,$$

from where

$$C = \frac{1}{\pi}\int_a^b u(s)ds \int_a^b \frac{D(t,s)}{\sqrt{(t-a)(b-t)}}\,dt - \frac{1}{\pi}\int_a^b g(s)\,ds \int_s^b \frac{dt}{\sqrt{(t-a)(b-t)}}. \tag{11}$$

The integral $\int_s^b \frac{dt}{\sqrt{(t-a)(b-t)}}$ is calculated in terms of the Gauss hypergeometric function $F(a,b;c;z)$:

$$\int_s^b \frac{dt}{\sqrt{(t-a)(b-t)}} = \sqrt{\frac{b-s}{s-a}} \int_0^1 \frac{d\xi}{\sqrt{(1-\xi)\left(1+\xi\frac{b-s}{s-a}\right)}}$$
$$= 2zF\left(\frac{1}{2}, 1, \frac{3}{2}, -z^2\right),$$

where $z = \sqrt{\frac{b-s}{s-a}}$. Making use of formula (9.121.27) from [2], we obtain

$$\int_s^b \frac{dt}{\sqrt{(t-a)(b-t)}} = 2\,arctg\sqrt{\frac{b-s}{s-a}}.$$

We denote

$$E(s) = \frac{1}{\pi}\int_a^b \frac{D(t,s)}{\sqrt{(t-a)(b-t)}}\,dt$$

for brevity. Then (11) takes the form

$$C = \int_a^b E(s)u(s)ds - \frac{2}{\pi}\int_a^b arctg\sqrt{\frac{b-s}{s-a}}\ \ g(s)\,ds.$$

Remark. We calculated the value of the constant C, but in reality we do not need it, because $S^{-1}(C) \equiv 0$ for an arbitrary constant, since $S^{-1}(1) \equiv 0$, which follows from the known formulas. So we may just forget about $S^{-1}(C)$ in (10):

$$u(x) + S^{-1}Ku(x) = S^{-1}G. \tag{12}$$

However, we must understand that the solution $u(x)$ of the last equation will satisfy the "initial" equation (8) not with $C = 0$, but just with the value C determined above.

Equation (12) is Fredholm integral equation. It has the form

$$u(x) - \int_a^b \mathcal{K}(x,t)u(t)\,dt = F(x), \tag{13}$$

where

$$\mathcal{K}(x,t) = \frac{\sqrt{(x-a)(b-x)}}{\pi}\int_a^b \frac{ds}{\sqrt{(s-a)(b-s)}(s-x)}\int_a^s C(\xi,t)\,d\xi$$

and

$$F(x) = \int_a^b L(x,t)f(t)dt,$$

where

$$L(x,t) = \frac{\sqrt{(x-a)(b-x)}}{\pi}(1+\varphi'^2(t))\int_t^b \frac{ds}{\sqrt{(s-a)(b-s)}(s-x)}\,.$$

Acknowledgements. Supported by Russian Federal Targeted Programme "Scientific and Research-Educational Personnel of Innovative Russia" for 2009-2013, Project N 02.740.11.5024.

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Received: November 2, 2009