Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 1 (2010) ISSN 1311-0454

ON Y. NIEVERGELT'S INVERSION FORMULA FOR THE RADON TRANSFORM

E. Ournycheva *, **B. Rubin** **

Abstract

In 1986 Y. Nievergelt suggested a simple formula which allows to reconstruct a continuous compactly supported function on the 2-plane from its Radon transform. This formula falls into the scope of the classical convolution-backprojection method. We show that elementary tools of fractional calculus can be used to obtain more general inversion formulas for the k-plane Radon transform of continuous and L^p functions on $\mathbb{R}^n$ for all $1 \le k < n$. Further generalizations and open problems are discussed.

Mathematics Subject Classification 2010: 42C40, 44A12
Key Words and Phrases: The k-plane Radon transform, Nievergelt's inversion formula, the convolution-backprojection method.

1. Introduction

Inversion formulas for Radon transforms of different kinds are of great importance in mathematics and its applications; see, e.g., [2, 4, 7, 10, 11, 18, 20, 21, 24, 29], and references therein. Since many of them are pretty involved or applicable under essential restrictions, every more or less elementary inversion algorithm deserves special consideration. In 1986, Yves Nievergelt came up with the paper [12], entitled "Elementary inversion of Radon's transform". His result can be stated as follows.

THEOREM 1. *Let*

$$G_a(t) = \begin{cases} 1/(\pi a^2) & \text{if } |t| \le a, \\ \dfrac{1}{\pi a^2}\left(1 - \dfrac{1}{\sqrt{1-a^2/t^2}}\right) & \text{if } |t| > a; \end{cases} \tag{1}$$

$a > 0$. *Any continuous compactly supported function on the 2-plane can be reconstructed from the Radon transform over lines in this plane by the formula*

$$f(x,y) = \lim_{a\to 0} \frac{1}{\pi}\int_0^\pi \int_{-\infty}^{\infty} (Rf)(t - x\cos\alpha - y\sin\alpha, \alpha)\, G_a(t)\, dt\, d\alpha, \tag{2}$$

where the double integral on the right-hand side equals the average of f over the disc of radius a centered at (x,y).

Formulas (1) and (2) indeed look elementary. The following questions arise:

1. What is the basic idea of the Nievergelt's method from the point of view of modern developments?

2. Whether this method is applicable to more general k-plane Radon transforms on $\mathbb{R}^n$ for all $1 \le k < n$ and arbitrary continuous or L^p functions, satisfying minimal assumptions at infinity?

In the present article we apply elementary facts from fractional calculus [25] to answer these questions and indicate possible generalizations.

It is worth noting that Y. Nievergelt's inversion formula is based on the relation

$$W_a * f = R^*(w_a * Rf), \tag{3}$$

where $*$ is the convolution operator, R^* is the backprojection, w_a is a delta-sequence, and $W_a = R^*(w_a)$. As it was explained in [11], the relation (3) is a basis of various inversion algorithms, cf. [8, 9]. Our interest in generalization of (2) is purely theoretical and no claim about practical importance of the results is made. However, it is felt that the elementary use of operators of fractional integration might appeal to the applied mathematician as well.

Notation and main results. Let $\mathcal{G}_{n,k}$ and $G_{n,k}$ be the *affine* Grassmann manifold of all non-oriented k-planes τ in $\mathbb{R}^n$, and the ordinary Grassmann manifold of k-dimensional subspaces ζ of $\mathbb{R}^n$, respectively. Given $\zeta \in G_{n,k}$, each vector $x \in \mathbb{R}^n$ can be written as $x = (x', x'') = x' + x''$ where

$x' \in \zeta$ and $x'' \in \zeta^\perp$, $\zeta^\perp$ being the orthogonal complement to ζ in $\mathbb{R}^n$. Each k-plane τ is parameterized by the pair (ζ, x'') where $\zeta \in G_{n,k}$ and $x'' \in \zeta^\perp$. The manifold $\mathcal{G}_{n,k}$ will be endowed with the product measure $d\tau = d\zeta dx''$, where $d\zeta$ is the $SO(n)$-invariant measure on $G_{n,k}$ of total mass 1, and dx'' denotes the usual volume element on $\zeta^\perp$. We write $C_0 = C_0(\mathbb{R}^n)$ for the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ denotes the area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

The k-plane transform of a function f on $\mathbb{R}^n$ is a function $\hat{f}$ on $\mathcal{G}_{n,k}$ defined by

$$\hat{f}(\tau) = \int_\zeta f(x' + x'')\, dx', \qquad \tau = (\zeta, x'') \in \mathcal{G}_{n,k}. \tag{4}$$

This expression is finite for all $\tau \in \mathcal{G}_{n,k}$ if f is continuous and decays like $O(|x|^{-\lambda})$ with $\lambda > k$. Moreover [24, 27, 29], if $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/k$, then $\hat{f}(\tau)$ is finite for almost all planes $\tau \in \mathcal{G}_{n,k}$. The above-mentioned bounds for λ and p are best possible.

Following [23], we define the wavelet-like transform

$$(W_a^* \varphi)(x) = \frac{1}{a^n} \int_{\mathcal{G}_{n,k}} \varphi(\tau)\, w\Big(\frac{|x-\tau|}{a}\Big)\, d\tau, \qquad a > 0, \tag{5}$$

where $|x - \tau|$ denotes the Euclidean distance between the point $x \in \mathbb{R}^n$ and the k-plane τ.

Theorem 2. *[23, Th. 3.1] Let $f \in L^p$, $1 \le p < n/k$, and let $\psi(|\cdot|)$ be a radial function on $\mathbb{R}^n$, which has an integrable decreasing radial majorant. If w is a solution of the Abel type integral equation*

$$cr^{2-n} \int_0^r s^{n-k-1} w(s)(r^2 - s^2)^{k/2-1}\, ds = \psi(r), \qquad c = \frac{\sigma_{k-1}\, \sigma_{n-k-1}}{\sigma_{n-1}}, \tag{6}$$

then

$$(W_a^* \hat{f})(x) = \int_{\mathbb{R}^n} f(x - ay)\, \psi(|y|)\, dy, \tag{7}$$

and therefore,

$$\lim_{a \to 0} (W_a^* \hat{f})(x) = \lambda f(x), \qquad \lambda = \int_{\mathbb{R}^n} \psi(|x|)\, dx. \tag{8}$$

The limit in (8) is understood in the L^p-norm and in the almost everywhere sense. If $f \in C_0 \cap L^p$ for some $1 \le p < n/k$, then (8) holds uniformly on $\mathbb{R}^n$.

This theorem is a core of the convolution-backprojection method for the k-plane Radon transform, and the most difficult task is to choose relatively simple functions w and ψ satisfying (6). The crux is that the left-hand side of (6) has, in general, a bad behavior when $r \to \infty$. Hence, *to achieve integrability of ψ, the solution w must be sign-changing.*

Our first observation is that the essence of Y. Nievergelt's Theorem 1 can be presented in the language of Theorem 2 as follows.

THEOREM 3. *Let $k = 1$, $n = 2$,*

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r > 1. \end{cases} \tag{9}$$

Then (6) has a solution

$$w(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 1 - \dfrac{r}{\sqrt{r^2-1}} & \text{if } r > 1, \end{cases} \tag{10}$$

such that for every compactly supported continuous function f on $\mathbb{R}^2$,

$$(W_a^* \hat{f})(x) = \int_{|y|<1} f(x - ay)\, dy, \tag{11}$$

where W_a^* is the wavelet-like transform (5) generated by w.

For the convenience of presentation, we will keep to the following convention.

DEFINITION 4. The convolution-backprojection algorithm in Theorem 2 will be called *the Nievergelt's method* if ψ is chosen according to (9).

Of course, Theorem 2 deals with essentially more general classes of functions than Theorem 1, however, the main focus of our article is different: *we want to find auxiliary functions ψ and w, having possibly simple analytic expression.*

THEOREM A.

(i) *The Nievergelt's method is applicable to the X-ray transform (the case $k = 1$) in any dimension. Namely, if ψ is chosen according to (9), then (6) has a solution*

$$w(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ -\dfrac{\Gamma\left(\frac{n-1}{2}\right) r^{3-n}}{2\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \displaystyle\int_0^1 v^{n/2-1} (r^2 - v)^{-3/2}\, dv & \text{if } r > 1, \end{cases} \tag{12}$$

and inversion formula (8) holds.

(ii) *If $k > 1$, then Nievergelt's method is inapplicable.*

An integral in (12) can be expressed through the hypergeometric function and explicitly evaluated in some particular cases; see Section 2.2. For instance, if $n = 2$, then (12) is the Nievergelt's function (10).

To include all $1 \le k < n$, we modify the Nievergelt's method by choosing $\psi(r)$ in a different way as follows.

THEOREM B. *Let*

$$\psi(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1, \\ \dfrac{(r^2-1)^\ell}{r^{n+2\ell+1}} & \text{if } r > 1; \end{cases} \tag{13}$$

$\ell \ge 0$. *Then the corresponding function $\psi(|\cdot|)$ has a decreasing radial majorant in $L^1(\mathbb{R}^n)$ and (6) has the following solution:*

(i) *In the case $k = 2\ell$; $\ell = 1, 2, \dots$:*

$$w(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1, \\ c_\ell \, r^{2+2\ell-n} \left(\dfrac{1}{2r}\dfrac{d}{dr}\right)^\ell \left[\dfrac{(r^2-1)^\ell}{r^{2\ell+3}}\right] & \text{if } r > 1, \end{cases} \tag{14}$$

$$c_\ell = \frac{\Gamma(n/2-\ell)}{\Gamma(n/2)}.$$

(ii) *In the case $k = 2\ell+1$; $\ell = 0, 1, 2, \dots$:*

$$w(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1, \\ \tilde{c}_\ell \, r^{3+2\ell-n} \left(\dfrac{1}{2r}\dfrac{d}{dr}\right)^{\ell+1} \left[\dfrac{(r^2-1)^{\ell+1/2}}{r^{2\ell+2}}\right] & \text{if } r > 1, \end{cases} \tag{15}$$

$$\tilde{c}_\ell = \frac{\Gamma((n-1)/2-\ell)\, \ell!}{\Gamma(n/2)\, \Gamma(\ell+3/2)}.$$

In both cases the inversion result in Theorem 2 is valid.

Theorems **A** and **B** are proved in Sections 2 and 3, respectively.

Possible generalizations.

1°. The convolution-backprojection method is well-developed in the general context of totally geodesic Radon transforms on spaces of constant curvature. Apart of $\mathbb{R}^n$, the latter include the n-dimensional unit sphere S^n and the hyperbolic space $\mathbb{H}^n$; see [1, 22, 23]. As above, the key role in this theory belongs to a certain Abel type integral equation and the relevant sign-changing solution w. Moreover, passage to the limit in (8) as $a \to 0$, can be replaced by integration in a from 0 to ∞ against the dilation-invariant measure da/a. This leads to inversion formulas, which resemble the classical Calderón's identity for continuous wavelet transforms [3]. The corresponding wavelet function is determined as a solution of a similar Abel type integral equation; see [1, 22, 23] for details. In all these cases analogues of Theorems **A** and **B** can be obtained. We leave this exercise to the interested reader.

2°. Unlike the classical k-plane transforms on $\mathbb{R}^n$, the corresponding transforms on matrix spaces [5, 15, 16, 17] are much less investigated. To the best of our knowledge, no pointwise inversion formulas (i.e., those, that do not contain operations in the sense of distributions) are available for these transforms if the latter are applied to arbitrary continuous or L^p functions ($p \neq 2$). One of the reasons of our interest in Nievergelt's idea is that it might be applicable to the matrix case. Moreover, as in 1°, it may pave the way to implementation of wavelet-like transforms in the corresponding reconstruction formulas. We plan to study these questions in our forthcoming publication.

2. The case $k = 1$

2.1. Proof of Theorem A

We will be dealing with Riemann-Liouville fractional integrals

$$(I^\alpha_{a+} g)(u) = \frac{1}{\Gamma(\alpha)} \int_a^u (u-v)^{\alpha-1} g(v)\, dv, \qquad \alpha > 0. \tag{16}$$

Changing variables, we transform the basic integral equation

$$cr^{2-n} \int_0^r s^{n-k-1} w(s) (r^2 - s^2)^{k/2-1}\, ds = \psi(r), \quad c = \frac{\sigma_{k-1}\, \sigma_{n-k-1}}{\sigma_{n-1}}, \tag{17}$$

(cf. (6)) to the form

$$I^{k/2}_{0+}\tilde{w} = \tilde{\psi}, \tag{18}$$

where

$$\tilde{w}(u) = u^{(n-k)/2-1} w(\sqrt{u}), \qquad \tilde{\psi}(u) = \frac{2u^{n/2-1}}{c\,\Gamma(k/2)}\,\psi(\sqrt{u}). \tag{19}$$

Suppose that ψ is defined by (9). If $0 < u \le 1$, then, by homogeneity,

$$\tilde{w}(u) = c' u^{(n-k)/2-1}, \qquad c' = \frac{2\Gamma(n/2)}{c\,\Gamma(k/2)\,\Gamma((n-k)/2)} = 1, \tag{20}$$

c being the constant from (17). Hence, for $u > 1$, we necessarily have

$$(I^{k/2}_{1+}\tilde{w})(u) = -\frac{1}{\Gamma(k/2)}\int_0^1 (u-v)^{k/2-1} v^{(n-k)/2-1}\,dv. \tag{21}$$

If $k \ge 2$, this equation has no solution $\tilde{w} \in L^1_{loc}(1,\infty)$, because, otherwise, we get

$$\lim_{x\to 1^+} (l.h.s) = 0, \qquad \lim_{x\to 1^+} (r.h.s) = const \ne 0.$$

This proves the second statement in Theorem **A**.

Consider the case $k = 1$. If $0 < u \le 1$, then $\tilde{w}(u) = u^{(n-1)/2-1}$. If $u > 1$, then, setting $w_0(u) = u^{(n-1)/2-1}$, from (21) we get

$$\begin{aligned}(I^{1/2}_{1+}[\tilde{w} - w_0])(u) &= -\frac{1}{\sqrt{\pi}}\int_0^u (u-v)^{-1/2} v^{(n-1)/2-1}\,dv \\ &= -c_n\,u^{n/2-1}, \qquad c_n = \frac{\Gamma((n-1)/2)}{\Gamma(n/2)}.\end{aligned}$$

This gives

$$\tilde{w}(u) = w_0(u) - \frac{c_n}{\sqrt{\pi}}\frac{d}{du}\int_1^u (u-v)^{-1/2} v^{n/2-1}\,dv$$

or (set $\int_1^u = \int_0^u - \int_0^1$)

$$\tilde{w}(u) = -\frac{c_n}{2\sqrt{\pi}}\int_0^1 (u-v)^{-3/2} v^{n/2-1}\,dv.$$

Thus, if $I^{1/2}_{0+}\tilde w=\tilde\psi$, then, necessarily,

$$\tilde w(u)=\begin{cases} u^{(n-1)/2-1} & \text{if } 0<u\le 1,\\[2ex] -\dfrac{c_n}{2\sqrt{\pi}}\displaystyle\int_0^1 (u-v)^{-3/2} v^{n/2-1}\,dv & \text{if } u>1.\end{cases} \tag{22}$$

One can readily see that function (22) is locally integrable on $(0,\infty)$. Let us prove that it satisfies $(I^{1/2}_{0+}\tilde w)(u)=\tilde\psi(u)$ for all $u>0$. It suffices to show that $(I^{1/2}_{0+}\tilde w)(u)\equiv 0$ when $u>1$. We have $I^{1/2}_{0+}\tilde w=I_1-I_2$, where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{\pi}}\int_0^1 (u-v)^{-1/2} v^{(n-1)/2-1}\,dv,\\ I_2 &= \frac{c_n}{2\pi}\int_1^u (u-v)^{-1/2}\,dv\int_0^1 (v-s)^{-3/2} s^{n/2-1}\,ds. \end{aligned}$$

Both integrals can be expressed in terms of hypergeometric functions. For I_1, owing to 3.197 (3) and 9.131 (1) from [6], we obtain

$$\begin{aligned} I_1 &= \frac{u^{-1/2}}{\sqrt{\pi}}\,B\left(\frac{n-1}{2},1\right)\,F\left(\frac12,\frac{n-1}{2};\frac{n+1}{2};\frac1u\right)\\ &= \frac{\Gamma((n-1)/2)}{\sqrt{\pi}\,\Gamma((n+1)/2)}\,(u-1)^{-1/2}\,F\left(1,\frac12;\frac{n+1}{2};\frac{1}{1-u}\right). \end{aligned} \tag{23}$$

For I_2, changing the order of integration and using [6, 3.238 (3)], we have

$$I_2=\frac{c_n\,(u-1)^{1/2}}{\pi}\int_0^1 \frac{s^{n/2-1}\,(1-s)^{-1/2}}{u-s}\,ds.$$

By [6, 3.228 (3)] this expression coincides with (23).

To complete the proof, we recall that $w(r)=r^{3-n}\tilde w(r^2)$, which gives

$$w(r)=\begin{cases} 1 & \text{if } 0\le r\le 1,\\[2ex] -\dfrac{c_n\,r^{3-n}}{2\sqrt{\pi}}\displaystyle\int_0^1 (r^2-v)^{-3/2} v^{n/2-1}\,dv & \text{if } r>1.\end{cases}$$

This coincides with (12).

2.2. Examples

Let us give some examples of functions $w(r)$ defined by (12) in the case $r > 1$. By [6, 3.197(3)],

$$w(r) = -\frac{\Gamma((n-1)/2)\, r^{-n}}{2\sqrt{\pi}\,\Gamma(n/2+1)}\, F\left(\frac{3}{2}, \frac{n}{2}; \frac{n}{2}+1; \frac{1}{r^2}\right).$$

Keeping in mind that $F(a, b; c; z) = F(b, a; c; z)$, and using formulas 156, 203, and 211 from [19, 7.3.2], we obtain:

For $n = 2$:

$$w(r) = -\frac{1}{2r^2}\, F\left(\frac{3}{2}, 1; 2; \frac{1}{r^2}\right) = 1 - \frac{r}{\sqrt{r^2-1}}. \tag{24}$$

For $n = 3$:

$$w(r) = -\frac{1}{2r^3\sqrt{\pi}\,\Gamma(5/2)}\, F\left(\frac{3}{2}, \frac{3}{2}; \frac{5}{2}; \frac{1}{r^2}\right) = \frac{2}{\pi}\left(\arcsin\frac{1}{r} - \frac{1}{\sqrt{r^2-1}}\right). \tag{25}$$

For $n = 4$:

$$w(r) = -\frac{1}{8r^4}\, F\left(\frac{3}{2}, 2; 3; \frac{1}{r^2}\right) = 1 - \frac{2r^2-1}{2r\sqrt{r^2-1}}. \tag{26}$$

3. The general case

As in Section 2.1, our main concern is integral equation (17), which is equivalent to

$$I^{k/2}_{0+}\tilde{w} = \tilde{\psi}, \tag{27}$$

$$\tilde{w}(u) = u^{(n-k)/2-1} w(\sqrt{u}), \qquad \tilde{\psi}(u) = \frac{2u^{n/2-1}}{c\,\Gamma(k/2)}\, \psi(\sqrt{u}). \tag{28}$$

We want to find relatively simple functions w and ψ, which are admissible in the basic Theorem 2 and such that the corresponding functions $\tilde{w}$ and $\tilde{\psi}$ obey (27). It is convenient to consider the cases of k even and k odd separately.

3.1. The case of k even

Let $k = 2\ell$; $\ell = 1, 2, \ldots$. We choose

$$\psi(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1, \\[2ex] \dfrac{(r^2-1)^\ell}{r^{n+2\ell+1}} & \text{if } r > 1. \end{cases} \tag{29}$$

The corresponding function $x \to \psi(|x|)$ obviously has a decreasing radial majorant in $L^1(\mathbb{R}^n)$. By (28), equation (27) is equivalent to

$$(I^{\ell}_{1+}\tilde{w})(u) = \frac{c_\ell\,(u-1)^\ell}{u^{\ell+3/2}}, \qquad c_\ell = \frac{\Gamma(n/2-\ell)}{\Gamma(n/2)}, \quad u>1. \tag{30}$$

The ℓth derivative

$$\tilde{w}(u) = \left[\frac{c_\ell\,(u-1)^\ell}{u^{\ell+3/2}}\right]^{(\ell)} = c_\ell \sum_{j=0}^{\ell} \binom{\ell}{j} [(u-1)^\ell]^{(j)}\,[u^{-\ell-3/2}]^{(\ell-j)} \tag{31}$$

is integrable on $(1,\infty)$. The fact that (31) satisfies (30) can be easily checked using integration by parts. Thus, the pair of functions $w(r)$ and $\psi(r)$, defined by

$$w(r) = \begin{cases} 0 & \text{if } \ 0 \le r \le 1, \\ c_\ell\, r^{2+2\ell-n} \left(\dfrac{1}{2r}\dfrac{d}{dr}\right)^\ell \left[\dfrac{(r^2-1)^\ell}{r^{2\ell+3}}\right] & \text{if } \ r>1, \end{cases} \tag{32}$$

and (29), falls into the scope of Theorem 2 and the “even part” of Theorem **B** is proved.

3.2. The case of k odd

Let $k = 2\ell+1$; $\ell = 0,1,2,\ldots$. We define $\psi(r)$ by (29), as above. Then, instead of (30), we have

$$(I^{\ell+1/2}_{1+}\tilde{w})(u) = \frac{c'_\ell\,(u-1)^\ell}{u^{\ell+3/2}}, \qquad c'_\ell = \frac{\Gamma((n-1)/2-\ell)}{\Gamma(n/2)}, \quad u>1. \tag{33}$$

This gives $\tilde{w}(u) = g^{(\ell+1)}(u)$, where

$$g(u) = \frac{c'_\ell}{\sqrt{\pi}} \int_1^u \frac{(u-v)^{-1/2}(v-1)^\ell}{v^{\ell+3/2}}\, dv = \tilde{c}_\ell\, \frac{(u-1)^{\ell+1/2}}{u^{\ell+1}},$$

$$\tilde{c}_\ell = \frac{\Gamma((n-1)/2-\ell)\,\ell!}{\Gamma(n/2)\,\Gamma(\ell+3/2)},$$

(use [6, 3.238(3)]). Let us show that $\tilde{w}(u) \equiv g^{(\ell+1)}(u)$ satisfies (33). Integrating by parts, we have $I^{\ell+1/2}_{1+}\tilde{w} = I^{1/2}_{1+} g'$, where

$$g'(u) = \frac{dg(u)}{du} = \tilde{c}_\ell \left[\frac{(\ell+1/2)\,(u-1)^{\ell-1/2}}{u^{\ell+1}} - \frac{(\ell+1)\,(u-1)^{\ell+1/2}}{u^{\ell+2}}\right].$$

This gives $I_{1+}^{\ell+1/2}\tilde{w} = \tilde{c}_\ell\,(I_1 - I_2)$ where (use [6, 3.238(3)] again)

$$\begin{aligned} I_1 &= \frac{\ell+1/2}{\sqrt{\pi}} \int_1^u \frac{(u-v)^{-1/2}(v-1)^{\ell-1/2}}{v^{\ell+1}}\,dv = \frac{\Gamma(\ell+3/2)}{\ell!}\,\frac{(u-1)^\ell}{u^{\ell+1/2}},\\ I_2 &= \frac{\ell+1}{\sqrt{\pi}} \int_1^u \frac{(u-v)^{-1/2}(v-1)^{\ell+1/2}}{v^{\ell+2}}\,dv = \frac{\Gamma(\ell+3/2)}{\ell!}\,\frac{(u-1)^{\ell+1}}{u^{\ell+3/2}}. \end{aligned}$$

Hence,

$$I_1 - I_2 = \frac{\Gamma(\ell+3/2)}{\ell!}\,\frac{(u-1)^\ell}{u^{\ell+3/2}}$$

and therefore,

$$(I_{1+}^{\ell+1/2}\tilde{w})(u) = \frac{\tilde{c}_\ell\,\Gamma(\ell+3/2)}{\ell!}\,\frac{(u-1)^\ell}{u^{\ell+3/2}} = c'_\ell\,\frac{(u-1)^\ell}{u^{\ell+3/2}},$$

as desired. Thus, functions $w(r)$ and $\psi(r)$, defined by

$$w(r) = \begin{cases} 0 & \text{if } \ 0 \le r \le 1,\\ \tilde{c}_\ell\, r^{3+2\ell-n} \left(\dfrac{1}{2r}\,\dfrac{d}{dr}\right)^{\ell+1} \left[\dfrac{(r^2-1)^{\ell+1/2}}{r^{2\ell+2}}\right] & \text{if } \ r > 1, \end{cases} \tag{34}$$

and (29), obey Theorem 2. This completes the proof of Theorem **B**.

Acknowledgements. We are grateful to Professor Yves Nievergelt, who read our article and paid attention to close papers [13, 14, 26]. Our approach differs from that in these works.

The second author (B. Rubin) was partially supported by the NSF grants PFUND-137 (Louisiana Board of Regents) and DMS-0556157.

References

[1] C.A. Berenstein and B. Rubin, Totally geodesic Radon transform of L^p-functions on real hyperbolic space. *Fourier Analysis and Convexity, Series: Applied and Numerical Harmonic Analysis*, Ed. L. Brandolini, L. Colzani, A. Iosevich and G. Travaglini, No VIII (2004), 280.

[2] L. Ehrenpreis, *The Universality of the Radon Transform.* Oxford University Press, Oxford (2003).

[3] M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley theory and the study of function spaces. *CBMS Regional Conference Series in Mathematics*, vol. 79, AMS, Providence, R.I. (1991).

[4] I.M. Gelfand, S.G. Gindikin and M.I. Graev, *Selected Topics in Integral Geometry.* Translations of Mathematical Monographs, v. 220, AMS, Providence, RI (2003).

[5] F.B. Gonzalez and T. Kakehi, Invariant differential operators and the range of the matrix Radon transform. *J. Funct. Anal.* **241** (2006), 232-267.

[6] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products.* Academic Press, NY (1980).

[7] S. Helgason, *The Radon Transform.* Birkhäuser, Boston, Second Ed. (1999).

[8] A.V. Lakshminarayanan and G.N. Ramachandran, Three-dimensional reconstruction from radiographs and electron micrographs: Application of convolutions instead of Fourier transforms. *Proc. Nat. Acad. Sci. U.S.A.* **68** (1971) 2236-2240.

[9] B.F. Logan and L. Shepp, The Fourier Reconstruction of a Head Section. *IEEE Trans. on Nuclear Science* **21**, (1974), 21-43.

[10] A. Markoe, *Analytic Tomography.* Encyclopedia of Mathematics and its Applications, vol. 106, Cambridge University Press, Cambridge (2006).

[11] F. Natterer, *The Mathematics of Computerized Tomography.* SIAM, Philadelphia (2001).

[12] Y. Nievergelt, Elementary inversion of Radon's transform. *SIAM Rev.* **28**, No 1 (1986), 79-84.

[13] Y. Nievergelt, Exact reconstruction filters to invert Radon transforms with finite elements. *J. Math. Anal. Appl.* **120**, No 1 (1986), 288-316.

[14] Y. Nievergelt, Elementary inversion of the exponential X-ray transform. *IEEE Trans. Nuclear Sci.* **38**, No 2, Part II (1991), 873-876.

[15] G. Ólafsson, E. Ournycheva and B. Rubin, Higher rank wavelet transforms, ridgelet transforms, and Radon transforms on the space of matrices. *Appl. Comput. Harmon. Anal.* **21** (2006), 182-203.

[16] E. Ournycheva and B. Rubin, Method of mean value operators for Radon transforms in the space of matrices. *Intern. J. Math.* **19** (2008), 245-283.

[17] E. Ournycheva and B. Rubin, Semyanistyi's integrals and Radon transforms on matrix spaces. *The J. of Fourier Analysis and Appl.* **14**, No 1 (2008), 60-88.

[18] V. Palamodov, *Reconstructive Integral Geometry.* Monographs in Mathematics, vol. 98, Birkhäuser Verlag, Basel (2004).

[19] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integrals and Series, Vol. 3: More Special Functions.* Gordon and Breach, Amsterdam (1990).

[20] E.T. Quinto, M. Cheney and P. Kuchment (Ed-s), *Tomography, Impedance Imaging, and Integral Geometry.* Lectures in Appl. Math., 30, AMS, Providence, RI (1994).

[21] A.G. Ramm and A.I. Katsevich, *The Radon Transform and Local Tomography.* CRC Press, Boca Raton (1996).

[22] B. Rubin, Inversion formulas for the spherical Radon transform and the generalized cosine transform. *Advances in Appl. Math.* **29** (2002), 471-497.

[23] B. Rubin, Convolution-backprojection method for the k-plane transform, and Calderón's identity for ridgelet transforms. *Appl. Comput. Harmon. Anal.* **16** (2004), 231-242.

[24] B. Rubin, Reconstruction of functions from their integrals over k-planes. *Israel J. of Math.* **141** (2004), 93117.

[25] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach Sc. Publ., New York (1993).

[26] D. Schuster, Der Faltungsalgorithmus für dreidimensionale Radon-Probleme. *Numer. Funct. Anal. Optim.*, **10** (1989), 1203-1212.

[27] D.C. Solmon, A note on k-plane integral transforms. *Journal of Math. Anal. and Appl.* **71** (1979), 351-358.

[28] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, Princeton, NJ (1970).

[29] R.S. Strichartz, L^p-estimates for Radon transforms in Euclidean and non-euclidean spaces. *Duke Math. J.* **48** (1981), 699-727.

* *University of Pittsburgh at Bradford*
300, Campus Drive
16701, Bradford, PA, USA
e-mail: elo10@pitt.edu

** *Department of Mathematics*
Louisiana State University
Baton Rouge, LA, 70803, USA
e-mail: borisr@math.lsu.edu

Received: December 9, 2009