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INTEGRAL TRANSFORMS METHOD TO SOLVE A TIME-SPACE FRACTIONAL DIFFUSION EQUATION

Yanka Nikolova [1], **Lyubomir Boyadjiev** [2]

Abstract

The method of integral transforms based on using a fractional generalization of the Fourier transform and the classical Laplace transform is applied for solving Cauchy-type problem for the time-space fractional diffusion equation expressed in terms of the Caputo time-fractional derivative and a generalized Riemann-Liouville space-fractional derivative.

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1. Introduction

It was Namias [15] who introduced officially in 1980 the Fractional Fourier transform (FRFT) as a way to solve certain classes of ordinary and partial differential equations appearing in quantum mechanics. Most probably Namias was unaware of Wiener's paper [25] published in 1929, where the FRFT in the form of fractional powers of Fourier operator has been already introduced. Namias' results were later improved by McBride and Kerr [11], who also developed an operational calculus for the FRFT. The FRFT became much popularity after 1995 because of its numerous applications in quantum mechanics, chemistry, optics, dynamical systems, stochastic processes and signal processing.

Because of the importance of the FRFT, a number of its generalizations have been introduced [1]. In [26], Zayed extended the FRFT to larger classes of functions and generalized functions, whereas in [3] and [20] discrete versions of the FRFT were introduced. In [27] a systematic and unified approach to fractional integral transforms was presented and a new class of fractional integral transforms that includes the FRFT, the fractional Hankel transforms and the fractional integration and differentiation operators as special cases was studied.

A detailed survey on the mathematical background, properties and applications related to the FRFT is provided in [19]. The authors of the book cited present practically independent different definitions of the FRFT and show that every particular modeling process or mathematical problem requires the most suitable definition of the FRFT among the existing to be taken.

The modeling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [8], [21]. Two types of partial differential equations of fractional order deserve special attention. The first type is a generalization of the fractional partial differential equation suggested by Oldham and Spanier as a replacement of Fick's law [18]. The fractional-order diffusion equation suggested by Metzler, Glöckle and Nonnenmacher [12] is an example of the second type of fractional diffusion equation. Another example of the second type is the fractional diffusion equation deduced by Nigmatullin [16], [17] also known as the fractional diffusion-wave equation.

A space-time fractional diffusion equation, obtained from the standard diffusion equation by replacing the second order space-derivative by a fractional Riesz derivative and the first order time-derivative by a Caputo fractional derivative, has been treated by Saichev and Zaslavsky [22], Uchajkin and Zolotarev [24], Gorenflo, Iskenderov and Luchko [6], Scalas, Gorenflo and Mainardi [23], Metzler and Klafter [13]. The results obtained in [6] are complemented in [9], where the space-time fractional diffusion equation expressed by the Riesz-Feller space-fractional derivative and the Caputo time-fractional derivative is considered. The fundamental solution (the Green function) of the corresponding Cauchy problem is found in the cited paper by means of Fourier-Laplace transform. Based on Mellin-Barnes integral representation, the fundamental solutions of the Cauchy problem are also expressed in terms of proper Fox H-functions, [10].

In this paper we develop some properties of the FRFT introduced in [7].

We apply it and the Laplace transform

$$L[f(t);s] = \int_0^{\infty} e^{-st} f(t)dt \tag{1}$$

for solving the space-time fractional diffusion equation as well as a generalization of it.

2. Preliminaries

For a function u of the class S of a rapidly decreasing test functions on the real axis $\mathbb{R}$, the Fourier transform is defined as

$$\hat{u}(\omega) = F[u(x);\omega] = \int_{-\infty}^{+\infty} e^{i\omega x} u(x)dx, \quad \omega \in \mathbb{R}, \tag{2}$$

whereas the inverse Fourier transform has the form

$$u(x) = F^{-1}[\hat{u}(\omega);x] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}\hat{u}(\omega)d\omega, \quad x \in \mathbb{R}. \tag{3}$$

Denote by $V(R)$ the set of functions $v(x) \in S$ satisfying the conditions:

$$\left.\frac{d^n v}{dx^n}\right|_{x=0} = 0, \quad n = 0, 1, 2, ...$$

The Fourier pre-image of the space $V(R)$, i.e.

$$\Phi(R) = \{\varphi \in S; \hat{\varphi} \in V(R)\}$$

is called the Lizorkin space. As it is stated in [7], the Lizorkin space is invariant with respect to the fractional integration and differentiation operators.

In this paper we adopt the following FRFT as introduced in [7].

Definition 2.1. For a function $u \in \Phi(R)$, the FRFT $\hat{u}_\alpha$ of the order α $(0 < \alpha \leq 1)$ is defined as

$$\hat{u}_\alpha(\omega) = F_\alpha[u(x);\omega] = \int_{-\infty}^{+\infty} e_\alpha(\omega, x)u(x)dx, \quad \omega \in \mathbb{R}, \tag{4}$$

where

$$e_\alpha(\omega, x) := \begin{cases} e^{-i|\omega|^{\frac{1}{\alpha}}x}, & \omega \leq 0 \\ e^{i|\omega|^{\frac{1}{\alpha}}x}, & \omega > 0 \end{cases}. \tag{5}$$

Evidently if $\alpha = 1$, the kernel (5) reduces to the kernel of (2). The relation between the FRFT (4) and the classical Fourier transform (2) is given by the equality

$$\hat{u}_\alpha(\omega) = F_\alpha[u(x);\omega] = F_1[u(x);\mathrm{w}] = \hat{u}(\mathrm{w}), \tag{6}$$

where

$$\mathrm{w} := \begin{cases} -|\omega|^{\frac{1}{\alpha}}, & \omega \le 0 \\ |\omega|^{\frac{1}{\alpha}}, & \omega > 0 \end{cases}. \tag{7}$$

Thus, if

$$F_\alpha[u(x);\omega] = F_1[u(x);\mathrm{w}] = \varphi(\mathrm{w}),$$

then

$$u(x) := F_\alpha^{-1}[\hat{u}_\alpha(\omega);x] = F_1^{-1}[\varphi(\mathrm{w});x]. \tag{8}$$

In this paper we study the fractional diffusion equation in terms of the Caputo fractional derivative [4]:

$$D_*^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \\ \dfrac{d^n f(t)}{dt^n}, & \alpha = n \end{cases}, \tag{9}$$

where n is positive integer. The method we follow makes the rule of the Laplace transform of Caputo derivative very important, see e.g. [21]:

$$L[D_*^\alpha f(x);s] = s^\alpha L[f(t);s] - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-1-k}, \quad n-1 < \alpha \le n. \tag{10}$$

For generalization of the time-space diffusion equation we use the fractional derivative operator of the form

$$D_\beta^\alpha u(x) := (1-\beta) D_+^\alpha u(x) - \beta D_-^\alpha u(x), \quad 0 < \alpha \le 1, \ \beta \in \mathbb{R}, \tag{11}$$

where D_+^α and D_-^α are the Riemann-Liouville fractional derivatives on the real axis given as

$$D_+^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^x (x-\tau)^{\alpha-1} u(\tau) d\tau$$

and

$$D_-^\alpha u(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty (\tau - x)^{\alpha-1} u(\tau) d\tau.$$

A key role in our considerations is given to a relation established in [7] according to which for $0 < \alpha \leq 1$, any value of β and a function $u(x) \in \Phi(\mathbb{R})$,

$$F_\alpha[D^\alpha_\beta u(x);\omega] = (-ic_\alpha\omega)F_\alpha[u(x);\omega], \quad \omega \in \mathbb{R}, \tag{12}$$

where $c_\alpha = \sin(\alpha\pi/2) + i\text{sign}\omega(1-2\beta)\cos(\alpha\pi/2)$.

The one-parameter generalization of the exponential function was introduced by Mittag-Leffler [14] as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}.$$

Its further generalization was done by Agarwal [2] who defined the two-parameter function of the Mittag-Leffler type in the form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \beta > 0. \tag{13}$$

Let us remind also that the effect of the application of the Laplace transform (1) on the functions (13) is described by the formulas [21, 1.2.2, (1.80)]:

$$L[t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^\alpha);s] = \frac{k!s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \quad \text{Re}\, s > |a|^{\frac{1}{\alpha}}. \tag{14}$$

3. Some properties of the FRFT

Due to the relations (7) and (8), it is not difficult to prove the validity of the following statement.

Theorem 3.1. *If $0 < \alpha \leq 1$, $u(x) \in \Phi(R)$ and $F_\alpha[u(x);\omega] = \hat{u}_\alpha(\omega)$, then:*

(a) $F_\alpha[u(x-a);\omega] = e_\alpha(\omega,a)\hat{u}_\alpha(\omega)$ *(Shifting)*

(b) $F_\alpha[u(ax);\omega] = \frac{1}{|a|}\hat{u}_\alpha(\frac{|\omega|}{a}),\ a \neq 0$ *(Scaling)*

(c) $F_\alpha[\overline{u(-x)};\omega] = \overline{\hat{u}_\alpha(\omega)}$ *(Conjugate)*

(d) $F_\alpha[\hat{u}_\alpha(x);\omega] = u(-\omega)$ *(Duality)*

(e) *If $g(x) \in \Phi(R)$ and $F_\alpha[g(x);\omega] = \hat{g}_\alpha(\omega)$, then*

$$\int_{-\infty}^{+\infty} \hat{u}_\alpha(\omega)g(\omega)e_\alpha(\omega,t)d\omega = \int_{-\infty}^{+\infty} u(x)\hat{g}_\alpha(x+t)dx \quad (\textit{Composition}).$$

Theorem 3.2. *If* $0 \le \alpha < 1$ *and* $u^{(n)}(x) \in \Phi(R)$, *then*

$$F_\alpha[u^{(n)}(x);\omega] = (-i\mathrm{sign}\omega|\omega|^{\frac{1}{\alpha}})^n\,\hat{u}_\alpha(\omega), \quad \omega \in \mathbb{R}.$$

P r o o f. Let us take $n = 1$ and $\omega \le 0$. Then according to (4) and (5), the integration by parts leads to

$$F_\alpha[u'(x);\omega] = \int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}x}u'(x)dx = u(x)e^{-i|\omega|^{\frac{1}{\alpha}}x}\Big|_{-\infty}^{+\infty}$$
$$+i|\omega|^{\frac{1}{\alpha}}\int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}x}u(x)dx = (-i\mathrm{sign}\omega|\omega|^{\frac{1}{\alpha}})\hat{u}_\alpha(\omega). \tag{15}$$

The case $\omega > 0$ is considered similarly. By induction, formula (15) yields the desired result. ∎

Theorem 3.3. (Convolution theorem) *If* $a < \alpha \le 1$ *and* $u(x)$, $v(x) \in \Phi(R)$, *then*

$$F_\alpha[(u*v)(x);\omega] = \hat{u}_\alpha(\omega)\hat{v}_\alpha(\omega),$$

where

$$(u*v)(x) = \int_{-\infty}^{+\infty} u(x-\xi)v(\xi)d\xi,$$

and $F_\alpha[u(x);\omega] = \hat{u}_\alpha(\omega)$, $F_\alpha[v(x);\omega] = \hat{v}_\alpha(\omega)$.

P r o o f. Consider first the case $\omega \le 0$. According to (4) and (5) we have

$$F_\alpha[(u*v)(x);\omega] = \int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}x}\left[\int_{-\infty}^{+\infty} u(x-\xi)v(\xi)d\xi\right]dx$$
$$= \int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}\xi}v(\xi)\left[\int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}(x-\xi)}u(x-\xi)dx\right]d\xi$$
$$= \int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}\xi}v(\xi)d\xi\int_{-\infty}^{+\infty} e^{-i|\omega|^{\frac{1}{\alpha}}\eta}u(\eta)d\eta = \hat{u}_\alpha(\omega)\hat{v}_\alpha(\omega).$$

Likewise, if $\omega > 0$,

$$F_\alpha[(u*v)(x);\omega] = \int_{-\infty}^{+\infty} e^{i|\omega|^{\frac{1}{\alpha}}x}\left[\int_{-\infty}^{+\infty} u(x-\xi)v(\xi)d\xi\right]dx$$
$$= \int_{-\infty}^{+\infty} e^{i|\omega|^{\frac{1}{\alpha}}\xi}v(\xi)\left[\int_{\infty}^{+\infty} e^{i|\omega|^{\frac{1}{\alpha}}(x-\xi)}u(x-\xi)dx\right]d\xi$$
$$= \int_{-\infty}^{+\infty} e^{i|\omega|^{\frac{1}{\alpha}}\xi}v(\xi)d\xi\int_{-\infty}^{+\infty} e^{i|\omega|^{\frac{1}{\alpha}}\eta}u(\eta)d\eta = \hat{u}_\alpha(\omega)\hat{v}_\alpha(\omega).$$

This accomplishes the proof. ∎

4. Fractional diffusion equation

In this section we apply the FRFT (4) for solving the Cauchy-type problem for the fractional diffusion equation

$$D_*^\alpha u(x,t) = \mu \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \tag{16}$$

subject to the initial condition

$$u(x,t)|_{t=0} = f(x), \tag{17}$$

where D_*^α is the Caputo time-fractional derivative (9) of order α, $f(x) \in \Phi(\mathbb{R})$ and μ is a diffusivity constant.

Theorem 4.1. *If $0 < \alpha \leq 1$, the Cauchy-type problem (16)–(17) is solvable and its solution $u(x,t)$ is given by the integral*

$$u(x,t) = \int_{-\infty}^{+\infty} G(x-\xi,t) f(\xi) d\xi, \tag{18}$$

where

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\mathrm{w}x} E_\alpha(-\mu|\mathrm{w}|^{\frac{2}{\alpha}} t^\alpha) d\mathrm{w}.$$

P r o o f. Denote $F_\alpha[u(x,t);\omega] = \hat{u}_\alpha(\omega,t)$. According to Theorem 3.2, the application of the FRFT (4) to the equation (16) and the initial condition (17) transforms the Cauchy-type problem (16)–(17) to the equation

$$D_*^\alpha \hat{u}_\alpha(\omega,t) = -\mu|\omega|^{\frac{2}{\alpha}} \hat{u}_\alpha(\omega,t) \tag{19}$$

subject to the initial condition

$$\hat{u}_\alpha(\omega,t)|_{t=0} = \hat{f}_\alpha(\omega). \tag{20}$$

The formulas (10) and (20) make clear that applying the Laplace transform (1) to (19), we obtain

$$L[\hat{u}_\alpha(\omega,t);s] = \frac{s^{\alpha-1}}{s^\alpha + \mu|\omega|^{\frac{2}{\alpha}}} \hat{f}_\alpha(\omega). \tag{21}$$

Taking into account that by the formula (14),

$$\frac{s^{\alpha-1}}{s^\alpha + \mu|\omega|^{\frac{2}{\alpha}}} = L[E_\alpha(-\mu|\omega|^{\frac{2}{\alpha}} t^\alpha);s],$$

we apply the convolution theorem for the Laplace transform and thus obtain

$$\hat{u}_\alpha(\omega,t) = E_\alpha(-\mu|\omega|^{\frac{2}{\alpha}}t^\alpha)\hat{f}_\alpha(\omega).$$

Because of (6) and (7), the latest equality gives:

$$\hat{u}(\mathrm{w},t) = E_\alpha(-\mu|\mathrm{w}|^{\frac{2}{\alpha}}t^\alpha)\hat{f}(\mathrm{w}).$$

Now the validity of the statement follows from the above equation and Theorem 3.3 taken for the particular case $\alpha = 1$. ■

By means of the formula [5, p.611, (5)],

$$F_1^{-1}\left[\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a};x\right] = e^{-ax^2},$$

it might be seen that the solution in Theorem 4.1 occurs as a generalization of the fundamental solution of the classical diffusion problem.

Corollary 4.1. *If $\alpha = 1$, the solution of the Cauchy-type problem (16)–(17) is given by the integral*

$$u(x,t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4\mu t}f(\xi)d\xi.$$

5. Generalized fractional diffusion equation

Consider now the Cauchy-type problem that refers to the generalized fractional diffusion equation

$$D_*^\gamma u(x,t) = \mu D_\beta^{\alpha+1}u(x,t) \tag{22}$$

subject to the initial condition (17), where D_*^γ is the Caputo time-fractional derivative (9) of order γ, whereas $D_\beta^{\alpha+1}$ is the space-fractional derivative (11) we can refer to as generalized Riemann-Liouville space-fractional derivative. Evidently the equation (22) reduces to (16) if $\alpha = 1$.

Theorem 5.1. *If $f(x) \in \Phi(\mathbb{R})$, $0 < \gamma \leq 1$, $0 < \alpha \leq 1$ and for every value of $\beta \in \mathbb{R}$, the Cauchy-type problem (22)–(17) is solvable and its solution $u(x,t)$ is given by the integral (18), where*

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\mathrm{w}x}E_\gamma(-i\mu c_{\alpha+1}\mathrm{w}t^\gamma)d\mathrm{w}.$$

P r o o f. According to (12) it is clear that the application of the FRFT $F_{\alpha+1}$ to the equation (22) and the initial condition (17) results to the equation

$$D_*^\gamma \hat{u}_{\alpha+1}(\omega,t) = (-i\mu c_{\alpha+1}\omega)\hat{u}_{\alpha+1}(\omega,t) \tag{23}$$

subject to the condition

$$\hat{u}_{\alpha+1}(\omega,t)|_{t=0} = \hat{f}_{\alpha+1}(\omega). \tag{24}$$

The Laplace transform (1) applied then to (23) and (24) implies

$$L[\hat{u}_{\alpha+1}(\omega,t);s] = \frac{s^{\gamma-1}}{s^\gamma + i\mu c_{\alpha+1}\omega}\hat{f}_{\alpha+1}(\omega). \tag{25}$$

The formula (14) enables us to conclude from (25) that

$$\hat{u}(\mathrm{w},t) = E_\gamma(-i\mu c_{\alpha+1}\mathrm{w}t^\gamma)\hat{f}(\mathrm{w}).$$

By Theorem 3.3 (for $\alpha = 1$) we obtain from the latest equation that the solution desired is indeed given by (18), where

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mathrm{w}x}E_\gamma(-i\mu c_{\alpha+1}\mathrm{w}t^\gamma)d\mathrm{w}.$$

It is worth to notice that the solution of the Cauchy-type problem (22)–(17) reduces for $\mu = 1$ and $\beta = \frac{1}{2}$ to the solution of the space-time diffusion equation studied in [7]. ∎

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[1,2] *Faculty of Applied Mathematics and Informatics*
Technical University of Sofia
Sofia 1156, BULGARIA

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e-mails: [1] *jvr@abv.bg* , [2] *boyadjievl@yahoo.com*