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AN ANALOG OF THE TRICOMI PROBLEM FOR A MIXED TYPE EQUATION WITH A PARTIAL FRACTIONAL DERIVATIVE

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Abstract

The paper deals with an analog of Tricomi boundary value problem for a partial differential equation of mixed type involving a diffusion equation with the Riemann-Liouville partial fractional derivative and a hyperbolic equation with two degenerate lines. By using the properties of the Gauss hypergeometric function and of the generalized fractional integrals and derivatives with such a function in the kernel, the uniqueness and existence of a solution of the considered problem are proved, and its explicit solution is established in terms of the new special function.

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1. Introduction

The fractional calculus is widely applied to investigation of partial differential equations of mixed type and hyperbolic type with generations; see [13], [16] and [18, Sections 41-42]. A series of papers [1], [3], [6], [7], [8] were devoted to study of various modifications of parabolic-hyperbolic equations in which hyperbolic equation has a generate line and an equation of parabolic type is replaced by the equation

$$u_{xx} - D^{\alpha}_{0+,\,y} u = 0 \quad (y > 0; \;\; 0 < \alpha < 1). \tag{1.1}$$

Here $D^{\alpha}_{0+,\,y}$ is the partial Riemann-Liouville fractional derivative of order α of a function $u(x,\, y)$ with respect to the second variable [18, Section 24.1]:

$$\left(D^{\alpha}_{0+,y} u\right)(x,y) = \frac{\partial}{\partial y} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^y \frac{u(x,t)}{(y-t)^{\alpha}} dt \;\; (0 < \alpha < 1, \;\; y > 0). \tag{1.2}$$

Our article deals with the mixed type equation involving equation (1.1) and a hyperbolic equation of the second kind having two generate lines:

$$x u_{xx} + y u_{yy} + p u_x + q u_y = 0 \quad \left(y < 0; \;\; 0 < p < \frac{1}{2}, \;\; q < \frac{1}{2}, \;\; q \le p \right). \tag{1.3}$$

We consider equation (1.1) in the quadrate domain $D^+ = \{(x,y) : \; 0 < x < 1, \;\; 0 < y < 1\}$, and equation (1.3) in the domain D^- lying in the lower half-plane $y < 0$ and bounded by the characteristics $AC : \;\; x + y = 0$ and $BC : \;\; \sqrt{x} + \sqrt{-y} = 1$ of equation (1.3) and by the segment $(0,1)$ of the line $y = 0$, with $A(0,0)$, $B(1,0)$ and $C\left(\frac{1}{4}, -\frac{1}{4}\right)$. Let D be the union of D^+, the segment $J = (0,1)$ and D^-: $D = D^+ \bigcup J \bigcup D^-$.

For equations (1.1) and (1.3) we study the following boundary value problem being an analog of the Tricomi problem: find a solution $u(x,y)$ of equations (1.1) and (1.3) satisfying the boundary conditions

$$u(0,y) = \varphi_0(y), \;\; u(1,y) = \varphi_1(y) \;\; (0 < y < 1), \tag{1.4}$$

$$u|_{AC} = 0 \quad \left(0 \le x \le \frac{1}{4} \right), \tag{1.5}$$

and the transmission conditions

$$\lim_{y \to 0+} y^{1-\alpha} u(x,y) = \lim_{y \to 0-} u(x,y) \;\; (0 \le x \le 1), \tag{1.6}$$

$$\lim_{y \to 0+} y^{1-\alpha} \left(y^{1-\alpha} u(x,y) \right)_y = - \lim_{y \to 0-} (-y)^q u_y(x,y) \;\; (0 < x < 1). \tag{1.7}$$

Here $\varphi_0(y)$ and $\varphi_1(y)$ are given functions such that

$$y^{1-\alpha} \varphi_0(y), \;\; y^{1-\alpha} \varphi_1(y) \in C([0,1]), \qquad \varphi_0(0) = \varphi_1(0) = 0. \tag{1.8}$$

We shall seek a solution $u(x,\, y)$ of the above problem in the space of two times differentiable functions $u(x,y)$ on the domain D such that

$$y^{1-\alpha} u(x,y) \in C(\overline{D^+}), \;\; u(x,y) \in C(\overline{D^-}), \;\; u_{xx} \in C(D^+ \cup D^-),$$

$$u_{yy} \in C(D^-), \quad y^{1-\alpha}\left(y^{1-\alpha}u\right)_y \in C\left(D^+ \cup \{(x,y):\ 0<x<1,\ y=0\}\right). \tag{1.9}$$

For solving the above problem we need the following generalized fractional integro-differential operators with the Gauss hypergeometric function $F(a,b;c;z)$, defined for real α, β, η and $x>0$ by

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x)=\begin{cases} \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int\limits_0^x (x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt & (\alpha>0),\\ \left(\frac{d}{dx}\right)^n\left(I_{0+}^{\alpha+n,\beta-n,\eta-n}f\right)(x) & (\alpha\le 0,\ n=[-\alpha]+1); \end{cases} \tag{1.10}$$

in particular,

$$\left(I_{0+}^{0,0,\eta}f\right)(x)=f(x). \tag{1.11}$$

The operators in (1.10) were introduced in [17] (see also [18, Section 23.2, 18.1]). If $\alpha>0$, then

$$\left(I_{0+}^{\alpha,-\alpha,\eta}f\right)(x)=\left(I_{0+}^{\alpha}f\right)(x), \quad \left(I_{0+}^{-\alpha,\alpha,\eta}f\right)(x)=\left(D_{0+}^{\alpha}f\right)(x), \tag{1.12}$$

where I_{0+}^{α} and D_{0+}^{α} are the operators of the Riemann-Liouville fractional integration and differentiation of order $\alpha>0$ [18, Section 2.3]:

$$\left(I_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int\limits_0^x (x-t)^{\alpha-1}f(t)dt \quad (\alpha>0,\ x>0), \tag{1.13}$$

$$\left(D_{0+}^{\alpha}g\right)(x)=\left(\frac{d}{dx}\right)^n\frac{1}{\Gamma(n-\alpha)}\int\limits_0^x (x-t)^{n-\alpha-1}g(t)dt \quad (\alpha>0,\ n=[\alpha]+1), \tag{1.14}$$

and $[\alpha]$ means the integral part of α.

2. Uniqueness of the solution

Suppose that the above original problem has the solution. We introduce the notation

$$\lim_{y\to 0+} y^{1-\alpha}u(x,y)=\tau_1(x), \quad \lim_{y\to 0-} u(x,y)=\tau_2(x), \tag{2.1}$$

$$\lim_{y\to 0+} y^{1-\alpha}\left(y^{1-\alpha}u(x,y)\right)_y=\nu_1(x), \quad \lim_{y\to 0-}(-y)^q u_y(x,y)=\nu_2(x). \tag{2.2}$$

It is known [15, Section 4.2.1] that the solution of equation (1.1) in the domain D^+, satisfying conditions in (1.4) and the condition

$$\lim_{y\to 0+} y^{1-\alpha}u(x,y) = \tau_1(x) \quad (0 \le x \le 1) \tag{2.3}$$

is given by the formula

$$u(x,y) = \int_0^y G_t(x,y;1,s)\varphi_0(s)ds$$

$$-\int_0^y G_t(x,y;1,s)\varphi_1(s)ds + \Gamma(\alpha)\int_0^1 G(x,y;t,0)\tau_1(t)dt, \tag{2.4}$$

where

$$G(x,y;t,s) = \frac{(y-s)^{\beta-1}}{2}$$

$$\times \sum_{n=-\infty}^{\infty}\left[e_{1,\beta}^{1,\beta}\left(-\frac{|x-t+2n|}{(y-s)^\beta}\right) - e_{1,\beta}^{1,\beta}\left(-\frac{|x+t+2n|}{(y-s)^\beta}\right)\right], \quad \beta = \frac{\alpha}{2}, \tag{2.5}$$

$$e_{1,\beta}^{1,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\beta-\beta k)} \quad (\beta < 1).$$

Remark 1. Function $e_{1,\beta}^{1,\beta}(z)$ can be expressed in terms of the Wright function $\varphi(\gamma,\delta;z)$ defined for $\gamma > 0$ and complex $\delta \in \mathbf{C}$ by [5, 18.1(27)]

$$\varphi(\gamma,\delta;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\gamma k+\delta)},$$

namely

$$e_{1,\beta}^{1,\beta}(z) = \varphi(-\beta,\beta;z).$$

Note that for $\gamma > -1$, $\varphi(\gamma,\delta;z)$ is an entire function of $z \in \mathbf{C}$; see [11, Section 1.11].

It is also known (for example, see [6], [7]) that the functional relation between $\tau_1(x)$ and $\nu_1(x)$ transferred from the parabolic part D^+ to the line $y = 0$ has the form

$$\nu_1(x) = \frac{1}{\Gamma(1+\alpha)}\tau_1''(x). \tag{2.6}$$

Let us find the relation between $\tau_2(x)$ and $\nu_2(x)$ transferred from the hyperbolic part D^- to the line $y = 0$.

The solution of the second Darboux problem for equation (1.3) in the hyperbolic part D^- with the data

$$u|_{AC} = 0 \quad \left(0 \le x \le \frac{1}{4}\right), \quad \lim_{y\to 0-} (-y)^q u_y(x,y) = \nu_2(x) \quad (0 < x < 1) \quad (2.7)$$

is given by the formula (see [12])

$$u(x,y) = \gamma(\eta+\xi)^{\frac{1}{2}-p} \int_0^{\xi} \frac{\tilde{\nu_2}(t) t^{p-\frac{1}{2}} F\left(p-\frac{1}{2},\, \frac{3}{2}-p;\, \frac{3}{2}-q;\, \sigma\right)}{[(\xi-t)(\eta-t)]^{q-\frac{1}{2}}} dt. \quad (2.8)$$

Here

$$\xi = \sqrt{x} - \sqrt{-y}, \quad \eta = \sqrt{x} + \sqrt{-y}, \quad \tilde{\nu_2}(t) = \nu_2(t^2), \quad (2.9)$$

$$\sigma = \frac{(t-\xi)(\eta-t)}{2t(\xi+\eta)}, \quad \gamma = \frac{2^{2q+p-\frac{1}{2}}\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(2q)\Gamma\left(\frac{3}{2}-q\right)}. \quad (2.10)$$

By using the relation [4, 2.11(25)]

$$F(a,\, 1-a;\, c;\, -z) = (1+z)^{c-1}(\sqrt{1+z}+\sqrt{z})^{2-2a-2c}$$

$$\times F\left[c+a-1, c-\frac{1}{2}; 2c-1; 4\sqrt{z(1+z)}(\sqrt{1+z}+\sqrt{z})^{-2}\right]$$

with $a = p-\frac{1}{2}$, $b = \frac{3}{2}-p$, $c = \frac{3}{2}-q$ and $z = -\sigma$ and setting $\eta = \xi$, we have

$$F\left(p-\frac{1}{2},\, \frac{3}{2}-p;\, \frac{3}{2}-q;\, \sigma\right)$$

$$= \left[\frac{(\xi+t)^2}{4t\xi}\right]^{\frac{1}{2}-q} \left(\frac{\xi}{t}\right)^{q-p} F\left(p-q, 1-q; 2-2q; \frac{\xi^2-t^2}{\xi^2}\right). \quad (2.11)$$

Setting $\eta = \xi$ in (2.8) and using (2.9)-(2.11), we find

$$u(x,0) \equiv u(\xi^2, \xi^2)$$

$$= \gamma 2^{2q-p-1/2} \xi^{2q-2p} \int_0^{\xi} \tilde{\nu_2}(t) t^{2p-1} (\xi^2-t^2)^{1-2q} F\left(p-q, 1-q; 2-2q; \frac{\xi^2-t^2}{\xi^2}\right) dt$$

$$= \gamma 2^{2q-p-1/2} x^{q-p} \int_0^{\sqrt{x}} \nu_2\left(t^2\right) t^{2p-1} \left(x-t^2\right)^{1-2q} F\left(p-q, 1-q; 2-2q; 1-\frac{t^2}{x}\right) dt.$$

Making the change $s = t^2$, we obtain

$$u(x,0) = \gamma 2^{2q-p-3/2} x^{q-p} \int_0^x (x-s)^{1-2q} F\left(p-q, 1-q; 2-2q; 1-\frac{s}{x}\right) s^{p-1} \nu_2(s) ds,$$

which according to (1.10) with $\alpha = 2-2q > 0$, $\beta = p+q-2$ and $\eta = q-1$ yields

$$u(x,0) = \gamma 2^{2q-p-3/2}\Gamma(2-2q)\left(I_{0+}^{2-2q,p+q-2,q-1}t^{p-1}\nu_2(t)\right)(x). \tag{2.12}$$

Using the Gauss-Legendre duplication formula for the gamma function [4, 1.2(15)]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

and taking into account the meaning of γ in (2.10) we have

$$\gamma 2^{2q-p-3/2}\Gamma(2-2q) = \frac{\Gamma(1-q)}{\Gamma(q)} \equiv k_1. \tag{2.13}$$

By (2.1) $u(x,0) = \tau_2(x)$, and thus, in accordance with (2.13), equation (2.12) take the form

$$\tau_2(x) = k_1\left(I_{0+}^{2-2q,p+q-2,q-1}\nu_2(t)t^{p-1}\right)(x). \tag{2.14}$$

By applying the equality [4, 2.9(2)]

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z), \tag{2.15}$$

it is directly verified the formula

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\beta-\eta}\varphi\right)(x) = \left(I_{0+}^{\alpha,\eta,\beta}\varphi\right)(x) \quad (\alpha > 0).$$

By this relation, (2.14) can be represented as

$$\tau_2(x) = k_1\left(I_{0+}^{2-2q,q-1,p+q-2}\nu_2(t)\right)(x). \tag{2.16}$$

Differentiating both sides of (2.16) with respect to x, we have

$$\tau_2'(x) = k_1\frac{d}{dx}\left(I_{0+}^{2-2q,q-1,p+q-2}\nu_2(t)\right)(x),$$

or, according to (1.10) with $\alpha = 1-2q < 0$, $\beta = q$ and $\eta = p+q-2$,

$$\tau_2'(x) = k_1\left(I_{0+}^{1-2q,q,p+q-1}\nu_2(t)\right)(x).$$

Applying operator $I_{0+}^{2q-1,\,-q,\,p-q}$ to both sides of this relation, on the basis of the formula [18, Section 23.2, 18.2]

$$\left(I_{0+}^{\alpha,\beta,\eta}\left(I_{0+}^{\gamma,\delta,\alpha+\eta}\varphi\right)(t)\right)(x) = \left(I_{0+}^{\alpha+\gamma,\beta+\delta,\eta}\varphi\right)(x) \quad (\gamma > 0) \tag{2.17}$$

and (1.10), we have

$$k_1\nu_2(x) = \left(I_{0+}^{2q-1,-q,p-q}\tau_2'(t)\right)(x) = \frac{d}{dx}\left(I_{0+}^{2q,-q-1,p-q-1}\tau_2'(t)\right)(x)$$

$$= \frac{d}{dx}\frac{x^{1-q}}{\Gamma(2q)}\int_0^x (x-t)^{2q-1}F\left(1+q-p,q-1;2q;1-\frac{t}{x}\right)\tau_2'(t)dt. \quad (2.18)$$

By using the relation [4, 2.9(4)]

$$F(a,b;c;z) = (1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right), \quad (2.19)$$

we have

$$F\left(1+q-p,q-1;2q;1-\frac{t}{x}\right) = \left(\frac{t}{x}\right)^{1-q}F\left(q+p-1,q-1;2q;\frac{t-x}{t}\right).$$

Therefore equation (2.18) can be rewritten in the form

$$k_1\nu_2(x)$$

$$= \frac{1}{\Gamma(2q)}\lim_{\varepsilon\to 0}\frac{d}{dx}\int_0^{x-\varepsilon}(x-t)^{2q-1}F\left(q+p-1,q-1;2q;\frac{t-x}{t}\right)t^{1-q}\tau_2'(t)dt. \quad (2.20)$$

There holds the following preliminary assertion.

LEMMA 1. *If a function $\tau_2(x)$ has a positive maximum (respectively a negative minimum) at the point $x=\xi\in(0,1)$, then $\nu_2(\xi)<0$ (respectively $\nu_2(\xi)>0$).*

P r o o f. Following A.V. Bitsadze, chose an arbitrary point x_0 such that $0<x_0<x$, and represent the integral in (2.20) as a sum of two integrals:

$$\frac{d}{dx}\int_0^{x-\varepsilon}(x-t)^{2q-1}F\left(q+p-1,q-1;2q;\frac{t-x}{t}\right)t^{1-q}\tau_2'(t)dt$$

$$= \frac{d}{dx}\left[\int_0^{x_0}(x-t)^{2q-1}F\left(q+p-1,q-1;2q;\frac{t-x}{t}\right)t^{1-q}\tau_2'(t)dt+\right.$$

$$\left.+\int_{x_0}^{x-\varepsilon}(x-t)^{2q-1}F\left(q+p-1,q-1;2q;\frac{t-x}{t}\right)t^{1-q}\tau_2'(t)dt\right].$$

Taking a differentiation with respect to x and using the formula [4, 11.2.8(22)]

$$\frac{d}{dz}\left[z^{c-1}F(a,\,b;\,c;\,z)\right] = (c-1)z^{c-2}F(a,\,b;\,c-1;\,z),$$

we have

$$\frac{d}{dx}\int\limits_0^{x-\varepsilon}(x-t)^{2q-1}F\left(q+p-1,q-1;2q;\frac{t-x}{t}\right)t^{1-q}\tau_2'(t)dt$$

$$= -(2q-1)\int\limits_0^{x_0}(x-t)^{2q-2}F\left(q+p-1,q-1;2q-1;\frac{t-x}{t}\right)t^{-q}\tau_2'(t)dt$$

$$+\varepsilon^{2q-1}(x-\varepsilon)^{1-q}\tau_2'(x-\varepsilon)F\left(q+p-1,q-1;2q;\frac{\varepsilon}{\varepsilon-x}\right)$$

$$-(2q-1)\int_{x_0}^{x-\varepsilon}(x-t)^{2q-2}F\left(q+p-1,q-1;2q-1;\frac{t-x}{t}\right)t^{-q}\tau_2'(t)dt$$

$$\equiv I_1 + I_2 + I_3. \tag{2.21}$$

We transform I_1 and I_3. By integrating by parts and applying (2.19), we have

$$I_1 = (2q-1)\int_0^{x_0}(x-t)^{2q-2}F\left(q+p-1,q-1;2q-1;\frac{t-x}{t}\right)t^{1-q}\tau_2'(t)dt$$

$$= \frac{(2q-1)\Gamma(2q-1)\Gamma(p)}{\Gamma(p+q-1)\Gamma(q)}x^{q-1}\tau_2(x) + (2q-1)[\tau_2(x_0)-\tau_2(x)]$$

$$\times(x-x_0)^{2q-2}x_0^{1-q}F\left(q+p-1,q-1;2q-1;\frac{x_0-x}{x_0}\right)$$

$$-(2q-1)(2q-2)\int\limits_0^{x_0}\frac{\tau_2(x)-\tau_2(t)}{(x-t)^{3-2q}}F\left(p+q-2,q-1;2q-2;\frac{t-x}{t}\right)t^{1-q}dt, \tag{2.22}$$

and

$$I_3 = (1-2q)\int_{x_0}^{x-\varepsilon}\frac{\tau_2'(x)-\tau_2'(t)}{(x-t)^{2-2q}}t^{1-q}F\left(q+p-1,q-1;2q-1;\frac{t-x}{t}\right)dt$$

$$+\tau_2'(x)x_0^{1-q}(x-x_0)^{2q-1}F\left(p+q,1+q;2q;\frac{x_0-x}{x_0}\right)$$

$$- \tau_2'(x-\varepsilon)(x-\varepsilon)^{1-q}\varepsilon^{2q-1}F\left(p+q, 1+q; 2q; \frac{\varepsilon}{\varepsilon-x}\right). \tag{2.23}$$

Substitute (2.22) and (2.23) into (2.21) and take a limit as $\varepsilon \to 0$, then put $x = \xi$ and make a limit as $x_0 \to \xi$, from (2.20) we deduce the function $\nu_2(x)$ at the point $x = \xi$ in the form

$$\nu_2(\xi) = \frac{1}{k_1}\left(I_{0+}^{2q-1,\,-q,\,p-q}\tau_2'(t)\right)(\xi) = \frac{2q-1}{k_1\Gamma(2q)}\left[\frac{\Gamma(2q-1)\Gamma(p)}{\Gamma(p+q-1)\Gamma(q)}\xi^{q-1}\tau_2(\xi)\right.$$

$$\left. - (2q-2)\int_0^\xi \frac{\tau_2(\xi)-\tau_2(t)}{(\xi-t)^{3-2q}}F\left(p+q-2, q-1; 2q-2; \frac{t-\xi}{t}\right)t^{1-q}dt\right]. \tag{2.24}$$

This relation, in accordance with (2.15) and the formula [4.2.11(29)]

$$F(a,\, b;\, 2b;\, z) = (1-z)^{b-a}\left(1-\frac{z}{2}\right)^{a-2b}F\left(b-\frac{a}{2}, b+\frac{1}{2}-\frac{a}{2}; b+\frac{1}{2}; \frac{z^2}{(2-z)^2}\right),$$

can be represented in the form

$$\nu_2(\xi) = \frac{(2q-1)}{k_1\Gamma(2q)}\left[\frac{\Gamma(2q-1)\Gamma(p)}{\Gamma(p+q-1)\Gamma(q)}\xi^{q-1}\tau_2(\xi)\right.$$

$$- (2q-2)\int_0^\xi \frac{\tau_2(\xi)-\tau_2(t)}{(\xi-t)^{3-2q}}\left(\frac{t+\xi}{2}\right)^{2-p-q}t^{p-1}$$

$$\left.\times F\left(\frac{p+q-1}{2}, \frac{p+q-2}{2}; q-\frac{1}{2}; \frac{(t-\xi)^2}{(t+\xi)^2}\right)dt\right]. \tag{2.25}$$

Consider the function

$$g(t) = F\left(\frac{p+q-1}{2}, \frac{p+q-2}{2}; q-\frac{1}{2}; \frac{(t-\xi)^2}{(t+\xi)^2}\right)$$

for $t \in [0, \xi]$ $(0 < \xi < 1,\ q \le p)$. Since $\frac{(\xi-t)^2}{(\xi+t)^2} \le 1$, then $g(t)$ is absolutely convergent series, and $g(t)$ is continuous for $t \in [0,\, \xi]$. By [4, 2.1(14)],

$$g(0) = F\left(\frac{p+q-1}{2}, \frac{p+q-2}{2}; q-\frac{1}{2}; 1\right) = \frac{\Gamma\left(q-\frac{1}{2}\right)\Gamma(1-p)}{\Gamma\left(\frac{q-p}{2}\right)\Gamma\left(\frac{1+q-p}{2}\right)} = M \ge 0,$$

and $g(\xi) = 1$.

Investigating the behavior of $g(t)$ on $(0,\xi)$. Using the differentiation formula for the Gauss hypergeometric function [4, 2.1(14)]

$$\left(\frac{d}{dz}\right)^m F(a,b;c;z) = \frac{(a)_m(b)_m}{(c)_m} F(a+m,b+m;c+m;z) \quad (m=0,1,2,\cdots),$$

where $(a)_k$ $(a \in \mathbf{C},\ k=0,1,2,\cdots)$ is the Pochhammer symbol:

$$(a)_0 = 1, \quad (a)_m = a(a+1)\cdots(a+m-1) \quad (m=1,2,\cdots), \tag{2.26}$$

we have

$$g'(t) = \frac{2(p+q-1)(2-p-q)}{2q-1} \frac{\xi(\xi-t)}{(t+\xi)^3}\, F\left(\frac{p+q+1}{2}, \frac{p+q}{2}; q+\frac{1}{2}; \left(\frac{t-\xi}{t+\xi}\right)^2\right).$$

It follows from here that $g'(t) > 0$ for $t \in (0,\xi)$, and thus $g(t)$ is continuous on $[0,\xi]$ and monotonically increase from $M \geq 0$ to the unite. Hence $g(t) \geq 0$ for $t \in [0,\xi]$.

It follows from (2.25) that $\nu_2(\xi) < 0$, and the first statement of lemma is proved. Similarly the proof in the case when $\tau_2(x)$ has an negative minimum at $x = \xi$. This completes the proof of the lemma. ∎

Lemma 2. *If $\tau_1(x)$ has a positive maximum (respectively negative minimum) at the point $x = \xi \in (0,1)$, then $\nu_1(\xi) \leq 0$ (respectively $\nu_1(\xi) \geq 0$).*

P r o o f. Lemma 2 follow from relation (2.6). ∎

Using Lemmas 1 and 2, applying the extreme principal for nonlocal parabolic equation [14] and take transmission conditions (1.7) into account, we deduce the following statement.

Theorem 1. *If there exists a solution $u(x,y)$ of the analog of the Tricomi problem for equations (1.1) and (1.3) with boundary conditions (1.4)-(1.5) and transmission conditions (1.6)-(1.7) in the space defined in (1.8) and (1.9), then this solution $u(x,y)$ is unique.*

3. Existence of the solution

We prove the existence of a solution of the original problem for equations (1.1) and (1.3) in the case $p = q$. By the first formula in (1.10) equation (2.14) takes the form

$$\tau_2(x) = k_1 \left(I_{0+}^{2-2q,2q-2,q-1} \nu_2(t) t^{q-1}\right)(x),$$

or, according to the first formula in (1.12),

$$\tau_2(x) = k_1 \left(I_{0+}^{2-2q} \nu_2(t) t^{q-1} \right)(x), \tag{3.1}$$

where I_{0+}^{2-2q} is the Riemann-Liouville operator (1.13) and k_1 is given by (2.13). By differentiating both sides of (3.1) twice with respect to x and using (1.14), we have

$$\tau_2''(x) = k_1 \left(D_{0+}^{2q} \nu_2(t) t^{q-1} \right)(x). \tag{3.2}$$

Let $\tau_1(x) = \tau_2(x) = \tau(x)$ and $\nu_2(t) t^{q-1} = \nu(t)$. Then, on the basis of (1.7) and (2.2), $\nu_1(t) = -t^{1-q}\nu(t)$. Thus, taking (2.6) into account, (3.2) lead to the differential equation of fractional order $2q$ $(0 < 2q < 1)$

$$D_{0+}^{2q} \nu(t) = \lambda t^{1-q} \nu(t), \quad \lambda = -\frac{\Gamma(1+\alpha)}{k_1}. \tag{3.3}$$

It was proved in [10] (see also [11, Section 2.4.3]) that the explicit solution of the homogeneous differential equation of fractional order

$$D_{0+}^{\alpha} y(x) = \lambda x^{\beta} y(x) \quad (0 < \alpha < 1, \quad \beta > -\alpha; \quad \lambda \neq 0, \quad \beta \in R) \tag{3.4}$$

is given by

$$y(x) = x^{\alpha-1} E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{(\beta-1)}{\alpha}} \left(\lambda x^{\alpha+\beta} \right). \tag{3.5}$$

Here $E_{\alpha,m,l}(z)$ is a special function of the form

$$E_{\alpha,m,l}(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{3.6}$$

$$c_0 = 1, \quad c_n = \prod_{i=0}^{n-1} \frac{\Gamma\left[\alpha(im+l)+1\right]}{\Gamma\left[\alpha(im+l+1)+1\right]} \quad (n = 1, 2, \cdots), \tag{3.7}$$

with

$$\alpha > 0, \quad m > 0, \quad l \in \mathbf{R}; \quad \alpha(jm+l) \neq 0, -1, -2, \cdots \ (j = 0, 1, 2 \cdots). \tag{3.8}$$

This function was introduced in [9]. $E_{\alpha,m,l}(z)$ with $\alpha > 0$ is an entire function of z of order $1/\alpha$ and type m; for example, see [11, Section 1.9]. In particular, if $m = 1$, the condition in (3.8) takes the form

$$\alpha > 0, \quad l \in \mathbf{R}; \quad \alpha(j+l) \neq 0, -1, -2, \cdots,$$

and (3.6) is reduced to the classical Mittag-Leffler function [2], [5, Sect.18.1]

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\quad(\alpha>0,\quad\beta\in\mathbf{R}),\tag{3.9}$$

apart from a constant multiplier $\Gamma(\alpha l+1)$:

$$E_{\alpha,1,l}(z)=\Gamma(\alpha l+1)\,E_{\alpha,\alpha l+1}(z).\tag{3.10}$$

Equation (3.3) is the equation (3.4) with

$$y(x)=\nu(x),\ \alpha=2q,\ \lambda=-\frac{\Gamma(1+\alpha)}{k_1}\ \text{ and }\ \beta=1-q.$$

Therefore, by (3.5) its explicit solution is given by

$$\nu(x)=x^{2q-1}E_{2q,\frac{1+q}{2q},\frac{1}{2}}\left(\lambda x^{1+q}\right).\tag{3.11}$$

By substituting (3.11) into (3.1) with $\tau_2(t)=\tau_1(t)$ and $\nu_2(t)=t^{1-q}\nu(t)$, we deduce the explicit expression for $\tau_1(x)$:

$$\tau_1(x)=k_1\left(I_{0+}^{2-2q}t^{2q-1}E_{2q,\frac{1+q}{2q},\frac{1}{2}}\left(\lambda t^{1+q}\right)\right)(x).\tag{3.12}$$

Lemma 3. *If $0<q<1$, then there holds*

$$\left(I_{0+}^{2-2q}t^{2q-1}E_{2q,\frac{1+q}{2q},\frac{1}{2}}\left(\lambda t^{1+q}\right)\right)(x)=xF_q\left(\lambda x^{q+1}\right),\tag{3.13}$$

where

$$F_q(z)=\sum_{n=0}^{\infty}d_n z^n,\tag{3.14}$$

$$d_0=\Gamma(2q),\quad d_n=\frac{\Gamma[n(q+1)+2q]}{\Gamma[n(q+1)+2]}\prod_{i=0}^{n-1}\frac{\Gamma[i(q+1)+q+1]}{\Gamma[i(q+1)+3q+1]}\quad(n=1,2,\cdots).\tag{3.15}$$

P r o o f. The lemma is proved directly by using (3.6)-(3.7), changing the orders of integration and summation (being possible because of uniform convergence of the series) and applying the formula [18, (2.44)]

$$\left(I_{0+}^{\alpha}t^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{\alpha+\beta-1}\quad(x>0;\quad\alpha>0,\quad\beta>0).\tag{3.16}$$

■

By (3.13), $\tau_1(x)$ in (3.12) is given by

$$\tau_1(x) = k_1 x F_q\left(\lambda x^{q+1}\right). \tag{3.17}$$

By substituting this relation into (2.4) we obtain the explicit solution $u(x,y)$ of the original problem

$$u(x,y) = \int_0^y \varphi_0(s) G_t(x,y;1,s)ds - \int_0^y \varphi_1(s) G_t(x,y;1,s)ds$$

$$+\Gamma(\alpha)k_1 \int_0^1 G(x,y;t,0) t F_q\left(\lambda t^{q+1}\right) dt, \tag{3.18}$$

where k_1 and λ are defined in (2.13) and (3.3), respectively.

By using formula (3.18), it is directly verified the validity of boundary conditions (1.4)-(1.5) and transmission conditions (1.6)-(1.7), and also that the solution $u(x,y)$ of the original problem given by (3.13) belongs to the space of functions defined in (1.8) and (1.9). This completes the proof of the existence of the solution of an analog of the original Tricomi problem. This yields the following result.

THEOREM 2. *The analogue of the Tricomi problem for equations (1.1) and (1.3) (for $p = q$) with boundary conditions (1.4)-(1.5) and transmission conditions (1.6)-(1.7) has an unique solution $u(x,y)$ in the space defined by (1.8) and (1.9), and this solution is given by (3.18).*

REMARK 2. The function $F_q(z)$ defined by (3.14)-(3.15) in Lemma 3 for $0 < q < 1$, exists for any $q > 0$. It yields an example of a new special entire function of z. Namely, there holds the following assertion.

LEMMA 4. *If $q > 0$, then $F_q(z)$ defined by (3.14)-(3.15) is an entire function of $z \in \mathbf{C}$.*

P r o o f. By (3.15), we have

$$\frac{d_n}{d_{n+1}} = \frac{\Gamma[n(q+1)+2q]}{\Gamma[n(q+1)+2]} \frac{\Gamma[n(q+1)+q+3]}{\Gamma[n(q+1)+3q+1]} \frac{\Gamma[n(q+1)+3q+1]}{\Gamma[n(q+1)+q+1]}. \tag{3.19}$$

By [4, 1.9(4)]],

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \quad (|z| \to \infty, \quad |arg(z)| < \pi).$$

By this formula with $z = n(q+1)$,

$$\frac{d_n}{d_{n+1}} \sim [n(q+1)]^{2q} \quad (n \to \infty).$$

Therefore, if $q > 0$, then

$$\lim_{n\to\infty} \frac{|d_n|}{|d_{n+1|}} = +\infty.$$

and, in accordance with known convergence principle for power series, the series in (3.14) is absolutely convergence for any $z \in \mathbf{C}$. This completes the proof of the lemma. ∎

In conclusion, we indicate that $F_q(z)$ with $q = 1$ and $q = 1/2$ yields some special cases of the generalized hypergeometric series ${}_pF_q(z)$ defined for complex $a_i,\ b_j$ $(i = 1, \cdots, p;\ j = 1, \cdots, q)$ $(b_j \neq 0, -1, -2, \cdots)$ by [4, Sect. 4.1]:

$${}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z] = \sum_{n=0}^{\infty} \frac{(a)_1 \cdots (a)_p}{(b)_1 \cdots (b)_q} \frac{z^k}{k!}, \tag{3.20}$$

where $(a)_i$ and $(b)_j$ $(i = 1, \cdots, p;\ j = 1, \cdots, q)$ are given by (2.26), and an empty product in (3.20), if it occurs, is taken to be one. Using (3.14)-(3.15), it is directly verified that

$$F_1(z) = {}_0F_1\left[-; \frac{3}{2}; \frac{z}{4}\right], \quad F_{1/2}(z) = {}_1F_1\left[\frac{2}{3}; \frac{5}{3}; \frac{2}{3}z\right]. \tag{3.21}$$

Note that the first formula in (3.21) can be also rewritten in terms of the sine and hyperbolic sine functions:

$$F_1\left(-z^2\right) = \frac{\sin z}{z}, \quad F_1\left(z^2\right) = \frac{\sinh z}{z}. \tag{3.22}$$

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