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# HERMITE POLYNOMIALS AND THE ZERO-DISTRIBUTION OF RIEMANN'S $\zeta$-FUNCTION

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**Abstract**

Let $\Phi$ be the meromorphic function defined e.g. in [2, Ch. XI, Sect. 3] by

$$\Phi(s) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}, \quad s = \sigma + it.$$

Necessary and sufficient conditions for absence of zeros of $\zeta(s)$ in the half-plane $\sigma > \theta, 1/2 \leq \theta < 1$ are proposed in terms of the growth of Fourier-Hermite's coefficients of the function $\Phi(1+ix), -\infty < x < \infty$ as well as of the growth of the Fourier transform of the function $\exp(-x^2/4)\Phi(1+ix/2)$.

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## 1. Expansion of holomorphic functions in series of Hermite polynomials

The region of convergence of a series in Hermite polynomials

$$\sum_{n=0}^{\infty} a_n H_n(z) \tag{1.1}$$

is, in general, a strip of the kind $S(\tau_0) := \{z \in \mathbb{C} : |\Im z| < \tau_0\}, 0 < \tau_0 \leq \infty$, [9, 9.2]. More precisely, let

$$\tau_0 = \max\{0, -\limsup(2n+1)^{-1/2} \log|(2n/e)^{n/2} a_n|\}, \tag{1.2}$$

then:

*If $\tau_0 = 0$, then the series* (1.1) *diverges at each point of the open set $\mathbb{C} \setminus \mathbb{R}$. If $0 < \tau_0 \leq \infty$, then it is absolutely uniformly convergent on each compact subset of the strip $S(\tau_0)$ and diverges in the open set $\mathbb{C} \setminus \overline{S(\tau_0)}$*, see e.g. [8,(IV.3.1), (b)].

The equality (1.2) can be regarded as a formula of Cauchy-Hadamard type for series in Hermite polynomials. It is a corollary of the asymptotic formula for these polynomials in the complex plane [9, Th. 8.22.7].

Denote by $\mathcal{E}(\tau_{\prime}), {\prime} < \tau_{\prime} \leq \infty$ the $\mathbb{C}$-vector space of the complex functions holomorphic in the strip $S(\tau_0)$ and having there a representation by a series of the kind (1.1). This space is completely characterized by E. Hille [5]. In fact, he has proved the following theorem:

*Let $0 \leq \tau < \infty$ and define*

$$\eta(\tau; x, y) = x^2/2 - |x|(\tau^2 - y^2)^{1/2}, -\infty < x < \infty, |y| \leq \tau. \tag{1.3}$$

*Then, a complex function $f$ holomorphic in the strip $S(\tau_0), 0 < \tau_0 \leq \infty$ is in the space $\mathcal{E}(\tau_{\prime})$ iff for each $\tau \in [0, \tau_0)$,*

$$|f(z)| = O(\exp(\eta(\tau; x, y))), \tag{1.4}$$

*provided $z = x + iy \in \overline{S(\tau)} (S(0) = \mathbb{R})$. Moreover, if* (1.1) *is the Hermite polynomial expansion of the function $f$ in the strip $S(\tau_0)$, then*

$$a_n = (I_n)^{-1} \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) f(x)\, dx, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

*where*

$$I_n = \sqrt{\pi} n! 2^n = \int_{-\infty}^{+\infty} \exp(-x^2)\{H_n(x)\}^2 dx, \quad n = 0, 1, 2, \ldots. \tag{1.6}$$

Another approach to the representation of holomorphic functions by series in Hermite polynomials is proposed in [8]. It is based on the class $G(\lambda), -\infty < \lambda \leq \infty$ of entire functions $F$ such that

$$\limsup_{|w| \to \infty} (2\sqrt{|w|})^{-1} (\log|F(w)| - |w|) \leq -\lambda. \tag{1.7}$$

The above assumption is equivalent to the requirement the estimate

$$|F(w)| = O(\exp(|w| - 2(\lambda - \varepsilon)\sqrt{|w|})), \quad w \in \mathbb{C} \tag{1.8}$$

to hold whatever the positive $\varepsilon$ may be and, hence, $G(\lambda)$ is a $\mathbb{C}$-vector space. The role of this space is cleared up by the following assertion [8,(VI.4.1)]:

*Suppose that* $0 < \tau_0 \leq \infty$. *A complex function* $f$ *is in the space* $\mathcal{E}(\tau_{\prime})$ *iff the representation*

$$f(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} E(u) \exp(-(u - iz)^2)\, du \tag{1.9}$$

*holds in the strip* $S(\tau_0)$, *where*

$$E(w) = U(w^2) + wV(w^2), w \in \mathbb{C} \tag{1.10}$$

*and the functions* $U, V$ *are in the space* $G(\tau_0)$.

Denote by $\mathcal{R}(\lambda), -\infty < \lambda \leq \infty$ the $\mathbb{C}$-vector space of the entire functions of the form (1.11) provided the entire functions $U, V$ are in $G(\lambda)$. If $E \in \mathcal{R}(\lambda)$, then from (1.8) it immediately follows that

$$\limsup_{|w| \to \infty} (2|w|)^{-1} (\log |E(w)| - |w|^2) \leq -\lambda. \tag{1.11}$$

Conversely, if the entire function $E$ satisfies (1.10), then it is in the space $\mathcal{R}(\lambda)$. Indeed, if we define $U(w) = (1/2)(E(w^{1/2}) + E(-w^{1/2})), V(w) = (1/2)w^{-1/2}(E(w^{1/2}) - E(-w^{1/2}))$, then the entire functions $U, V$ are in the space $G(\lambda)$ and $E(w) = U(w^2) + wV(w^2)$. Therefore, the space $\mathcal{R}(\lambda)$ consists of the entire functions $E$ satisfying (1.11) and the above assertion can be reformulated as follows:

*Suppose* $0 < \tau_0 \leq \infty$. *A complex function* $f$ *is in the space* $\mathcal{E}(\tau_0)$ *iff the representation* (1.9) *holds in the strip* $S(\tau_0)$ *with function* $E \in \mathcal{R}(\tau_0)$.

Let us note that as a corollary of (1.9), the inversion formula for the Fourier transform and the identity theorem for holomorphic functions we obtain that

$$E(w) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) \exp(-(x + iw)^2)\, dx, \quad w \in \mathbb{C}. \tag{1.12}$$

## 2. Holomorphic extension by means of series in Hermite polynomials

For a complex function $\omega$ defined in an interval $(a,b), -\infty \le a < b \le \infty$ is said that it admits a holomorphic extension if there exist a domain $D \subset \mathbb{C}$ and a function $\Omega$ holomorphic in $D$ such that $\Omega(x) = \omega(x)$ a.e. (almost everywhere) in $(a,b)$. It is quite evident that the holomorphic extension is unique if it exists.

Let $W(r,\delta)(r > 0, \delta < 1)$ be the class of measurable complex functions $\omega(x), -\infty < x < \infty$ such that $\int_{-r}^{r} |\omega(x)|\,dx < \infty$ and, moreover, the function $\exp(-\delta x^2)\omega(x)$ is essentially bounded when $|x| \ge r$.

A criterion for existence of holomorphic extension of functions from the class $W(r,\delta)$ is announced without proof, as Theorem 3 in [7]. It says:

*Suppose that* $\omega \in W(r,\delta)$ *and define*

$$a_n(\omega) = \int_{-\infty}^{\infty} exp(-x^2)H_n(x)\omega(x)\,dx, \quad n = 0,1,2,\ldots.$$

*If*

$$\tau_0(\omega) = -\limsup(2n+1)^{-1/2}\log|(2n/e)^{-n/2}a_n(\omega)| > 0,$$

*then* $\omega$ *has a holomorphic extension. More precisely, there exists a function* $\Omega \in \mathcal{E}(\tau_0(\omega))$ *such that* $\Omega(x) = \omega(x)$ *a.e. in* $(-\infty,\infty)$.

In order to justify the validity of the above assertion, first define

$$a_n^*(\omega) = (I_n)^{-1}a_n(\omega), n = 0,1,2,\ldots,$$

where $I_n$, $n = 0,1,2,\ldots$ are given by the equalities (1.6). Then, Stirling's formula yields that

$$-\limsup(2n+1)^{-1/2}\log|(2n/e)^{n/2}a_n^*(\omega)|$$

$$= -\limsup(2n+1)^{-1/2}\log|(2n/e)^{-n/2}a_n(\omega)| = \tau_0(\omega).$$

Further, from the Cauchy-Hadamard formula (1.2) it follows that the series

$$\Omega(z) = \sum_{n=0}^{\infty} a_n^*(\omega)H_n(z)$$

is absolutely uniformly convergent on each compact subset of the strip $S(\tau_0(\omega))$. That means the function $\Omega$ is in the space $\mathcal{E}(\tau_0(\omega))$. Moreover,

$$a_n^*(\omega) = (I_n)^{-1} \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) \Omega(x)\, dx, \quad n = 0, 1, 2, \ldots.$$

Define $a(x) = \exp(-x^2)(\Omega(x) - \omega(x)), -\infty < x < \infty$, then

$$\int_{-\infty}^{\infty} a(x) H_n(x)\, dx = 0, \quad n - 0, 1, 2, \ldots.$$

Since $\deg H_n = n$ for $n = 0, 1, 2, \ldots$, the system of Hermite's polynomials is linearly independent and, hence, it is a basis in the space of algebraic polynomials. Then, the above equalities yield that

$$\int_{-\infty}^{\infty} a(x) x^n\, dx = 0, \quad n = 0, 1, 2. \ldots.$$

Further, since $\Omega \in \mathcal{E}(\tau_0(\omega))$, (1.4) yields that $|\Omega(x)| = O(\exp(x^2/2))$, $-\infty < x < \infty$, hence $|a(x)| = O(\exp(-\gamma x^2))$ a.e. in $(-\infty, -r) \cup (r, \infty)$, where $\gamma = \min(1/2, 1 - \delta)$. Therefore, the Fourier transform

$$\hat{a}(w) = \int_{-\infty}^{\infty} a(x) \exp(iwx)\, dx$$

is in fact an entire function. Moreover, the function $\hat{a}$ and all its derivatives vanish at the point $w = 0$. Then, the identity theorem for holomorphic functions gives that $\hat{a}(w) = 0$ for each $w \in \mathbb{C}$ and the uniqueness property of the Fourier transform yields that $a(x) = 0$ a.e. in $(-\infty, \infty)$, i.e $\Omega(x) = \omega(x)$ a.e. in $(-\infty, \infty)$.

## 3. The results

A well-known fact is that Riemann's function $\zeta(s), s = \sigma + it$ has no zeros on the closed half-plane $\sigma \geq 1$. More precisely, there exists a region $B \subset \mathbb{C}$ containing this half-plane and such that $\zeta(s) \neq 0$ for $s \in B$. Hence, the function

$$\Phi(s) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} \tag{3.1}$$

is holomorphic in the region $B$. Moreover, the integral representation

$$\Phi(s) = \int_1^{\infty} \frac{\psi(x) - x}{x^{s+1}}\, dx \tag{3.2}$$

holds on the closed half-plane $\sigma \geq 1$, where $\psi$ is one of the Chebyshev functions [2, Sect. 3]. Let us note that the integral in (3.2) is in fact absolutely uniformly convergent in this half-plane and, moreover, the function $\Phi$ is bounded there. Indeed, since $\psi(x)-x = O(x\exp(-c(\log x)^{1/2}))$, $\quad c>0$, as $x \to \infty$, see e.g. [4, Sect. 18, (1)], we have that for $\sigma \geq 1$ and $-\infty < t < \infty$,

$$|\Phi(s)| \leq \int_1^\infty \frac{|\psi(x)-x|}{x^{\sigma+1}}\,dx = O\left(\int_1^\infty x^{-1}\exp(-c(\log x)^{1/2})\,dx\right)$$
$$= O\left(\int_0^\infty \exp(-cx^{1/2})\,dx < \infty\right) = O(1).$$

It turns out that the function

$$\Phi(1+iz) = \int_1^\infty \frac{\psi(t)-t}{t^{2+iz}}\,dt, \quad z = x+iy, \tag{3.3}$$

is holomorphic on the closed half-plane $y \leq 0$. Moreover, it is bounded there and, in particular, on the real axis. Hence, there exist

$$a_n(\Phi) = \int_{-\infty}^\infty \exp(-x^2)H_n(x)\Phi(1+ix)\,dx, \quad n = 0,1,2,\ldots. \tag{3.4}$$

Define

$$A_n(\psi) = \int_0^\infty t^n \exp(-t^2/4-t)(\psi(\exp t)-\exp t)\,dt, \quad n = 0,1,2,\ldots, \tag{3.5}$$

then the equalities

$$a_n(\Phi) = \sqrt{\pi}(-i)^n A_n(\psi), \quad n = 0,1,2,\ldots \tag{3.6}$$

hold. Indeed,

$$\Phi(1+ix) = \int_1^\infty t^{-2}\exp(-ix\log t)(\psi(t)-t)\,dt,$$

and after changing the order of integrations, we obtain that for $n = 0,1,2,\ldots$,

$$a_n(\Phi) = \int_1^\infty t^{-2}(\psi(t)-t)\,dt \int_{-\infty}^\infty \exp(-x^2-ix\log t)H_n(x)\,dx.$$

Further, Rodrigues' formula for Hermite's polynomials gives that

$$\int_{-\infty}^\infty \exp(-x^2-ix\log t)H_n(x)\,dx = \int_{-\infty}^\infty \exp(-ix\log t)(\exp(-x^2))^{(n)}\,dx$$

$$= (-i)^n (\log t)^n \int_{-\infty}^{\infty} \exp(-x^2 - ix\log t)\,dx$$

$$= (-i)^n (\log t)^n \exp(-(\log t)^2/4)) \int_{-\infty}^{\infty} \exp(-(x + i(\log t)/2)^2)\,dx.$$

But

$$\int_{-\infty}^{\infty} \exp(-(x + i(\log t)/2)^2)\,dx = \int_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi},$$

hence

$$a_n(\Phi) = \sqrt{\pi}(-i)^n \int_1^{\infty} (\log t)^n \exp(-(\log t)^2/4)t^{-2}(\psi(t) - t)\,dt, n = 0, 1, 2, \dots.$$

Then, changing $t$ by $\exp t$, we come to the equalities (3.6).

Define

$$\tau_0(\Phi) = -\limsup (2n+1)^{-1/2} \log |(2n/e)^{-n/2} a_n(\Phi)|,$$

and

$$T_0(\psi) = -\limsup (2n+1)^{-1/2} \log |(2n/e)^{-n/2} A_n(\psi)|,$$

then, (3.6) yields that

$$\tau_0(\Phi) = T_0(\psi) \tag{3.7}$$

The first of our results is the following assertion:

(I) *The function $\zeta(s)$ has no zeros in the half-plane $\sigma > \theta, 1/2 \le \theta < 1$ iff $T_0(\psi) \ge 1 - \theta$.*

Suppose that $T_0(\psi) \ge 1 - \theta$, then (3.7) yields that $\tau_0(\Phi) \ge 1 - \theta$ and, hence, the function $\Phi(1 + ix), -\infty < x < \infty$ has a holomorphic extension at least in the strip $S(1 - \theta)$. That means $\Phi(s)$ has no poles in the half-plane $\sigma > \theta$, i.e. $\zeta(s) \ne 0$ in this half-plane.

The assumption that $\zeta(s) \ne 0$ when $\sigma > \theta, 1/2 \le \theta < 1$ has as a corollary that $\psi(x) = x + O(x^\theta \log^2 x)$ as $x \to \infty$ [4, Sect. 18], i.e.

$$\psi(x) = x + O(x^{\theta+\varepsilon}), \quad x \to \infty \tag{3.8}$$

whatever $\varepsilon \in (0, 1 - \theta)$ may be. Hence, the integral in (3.2) is absolutely uniformly convergent on the closed half-plane $\sigma \ge \theta + \varepsilon$. That means the function $\Phi(s)$ is analytically continuable in the half-plane $\sigma > \theta + \varepsilon$ and, moreover, it is bounded when $\sigma \ge \theta + \varepsilon$. Hence, the function $\Phi(1 + iz)$

is holomorphic in the half-plane $y < 1-\theta-\varepsilon$ and bounded on its closure. By Hille's theorem, already mentioned, $\Phi(1+iz)$ has an expansion in series of Hermite polynomials in the strip $S(1-\theta-\varepsilon)$ with coefficients $(I_n)^{-1}a_n(\Phi), n = 0,1,2,\ldots$. Then, Cauchy-Hadamard's as well as Stirling's formula yield that

$$-\limsup(2n+1)^{-1/2}\log|(2n/e)^{n/2}(I_n)^{-1}a_n(\Phi)|$$
$$= -\limsup(2n+1)^{-1/2}\log|(2n/e)^{-n/2}a_n(\Phi)| = \tau_0(\Phi) \geq 1-\theta-\varepsilon,$$

i.e. $T_0(\psi) \geq 1-\theta-\varepsilon$ whatever the positive $\varepsilon < 1-\theta$ may be and, hence, $T_0(\psi) \geq 1-\theta$.

Now we are going to prove more directly the validity of the inequality $T_0(\psi) \geq 1-\theta$ provided that $\zeta(s) \neq 0$ when $\sigma > \theta$ and thus, to avoid the whole "machinary" of Hermite's series representation of holomorphic functions including Hille's theorem. Indeed, from (3.5) and (3.8) it follows that

$$|A_n(\psi)| = O\left(\int_0^\infty t^n \exp(-t^2/4 - (1-\theta-\varepsilon)t)\,dt\right)$$
$$= O\left(2^{n/2}\int_0^\infty t^n \exp(-t^2/2 - \sqrt{2}(1-\theta-\varepsilon)t)\,dt\right)$$

and the integral representation [1, 8.3,(3)]

$$D_\nu(z) = \frac{\exp(-z^2/4)}{\Gamma(-\nu)}\int_0^\infty t^{-\nu-1}\exp(-t^2/2 - zt)\,dt, \quad \Re\nu < 0,$$

of Weber-Hermite's function $D_\nu(z)$ gives that

$$|A_n(\psi)| = O\left(2^{n/2}\Gamma(n+1)D_{-n-1}(\sqrt{2}(1-\theta-\varepsilon))\right).$$

Further, Stirling's formula as well as T.M. Cherry's asymptotic formula, [1, 8.4,(5)],

$$D_\nu(z) = \frac{1}{\sqrt{2}}\exp((\nu/2)\log(-\nu) - \nu/2 - (-\nu)^{1/2}z)(1+O(|\nu|^{-1/2})), \quad (3.9)$$

$$|\arg(-\nu)| \leq \pi/2, \quad |\nu| \to \infty$$

yield that

$$(2n/e)^{-n/2}|A_n(\psi)| = O(\exp(-(2n+2)^{1/2}(1-\theta-\varepsilon)))$$

as $n \to \infty$ and, hence, the inequality $T_0(\psi) \geq 1 - \theta - \varepsilon$ holds for each positive $\varepsilon < 1 - \theta$, i.e. $T_0(\psi) \geq 1 - \theta$.

It is clear that $T_0(\psi) \leq 1/2$. Indeed, if $T_0(\psi) > 1/2$, then $\tau_0(\Phi) > 1/2$, i.e. the function $\Phi(1 + ix), -\infty < x < \infty$ would have a holomorphic extension in the strip $S(\tau_0(\Phi))$ which is impossible. Hence, we can allow us to formulate the following assertion:

(II) *Riemann's hypothesis is true iff* $T_0(\psi) = 1/2$.

The next assertion we are going to prove is "inspired" by the integral representation (1.9) of the functions from the space $\mathcal{E}(\tau_0), 0 < \tau_0 \leq \infty$. More precisely:

(III) *The function* $\zeta(s)$ *has no zeros in the half-plane* $\sigma > \theta, 1/2 \leq \theta < 1$ *iff the Fourier transform of the function*

$$\exp(-x^2/4)\Phi(1 + ix/2), -\infty < x < \infty \tag{3.10}$$

*is of the form*

$$\sqrt{2}\exp(-u^2)E(u), \quad E \in \mathcal{R}(1 - \theta). \tag{3.11}$$

Suppose that $\zeta(s) \neq 0$ when $\sigma > \theta$, then the function $\Phi(1 + iz) \in \mathcal{E}(\tau_0(\Phi))$. Hence, the representation

$$\Phi(1 + iz) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} E(u)\exp(-(u - iz)^2)\, du,$$

holds in the strip $S(\tau_0(\Phi))$ with $E \in \mathcal{R}(\tau_0(\Phi))$. Further, if $z = x \in (-\infty, \infty)$, then (1.12) yields that

$$E(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Phi(1 + ix)\exp(-(x + iu)^2)\, dx$$

and, hence,

$$\sqrt{2}\exp(-u^2)E(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2/4)\Phi(1 + ix/2)\exp(iux)\, dx. \tag{3.12}$$

From (1.12) it follows that $\lambda \geq \mu$ implies $\mathcal{R}(\lambda) \subset \mathcal{R}(\mu)$. Since $T_0(\psi) \geq 1 - \theta$ and $\mathcal{R}(\tau_0(\Phi)) = \mathcal{R}(T_0(\psi))$, the entire function $E \in \mathcal{R}(1 - \theta)$.

Conversely, let the Fourier transform of the function (3.10) be of the form (3.11) with $E \in \mathcal{R}(1-\theta)$, i.e. (3.12) holds. Then, the inversion formula for this transform yields that

$$\Phi(1+ix) = \int_{-\infty}^{\infty} E(u) \exp(-(u+ix)^2)\, du, -\infty < x < \infty. \tag{3.13}$$

Further, since $E \in \mathcal{R}(1-\theta)$, from (1.12) it follows that the integral

$$\int_{-\infty}^{\infty} E(u) \exp(-(u+iz)^2)\, du$$

is, in fact, absolutely and uniformly convergent on the closed strip $\overline{S(1-\theta-\varepsilon)}$ whatever the positive $\varepsilon < 1-\theta$ may be. That means the functions $\Phi(1+ix), -\infty < x < \infty$ has a holomorphic extension in the strip $S(1-\theta)$ and, hence the function $\zeta(s)$ has no zeros in the half-plane $\sigma > \theta$. Now, as a corollary of assertion (III), we can formulate the following one:

(IV) *Riemann's hypothesis is true iff the Fourier transform of the function* $\exp(-x^2/4)\Phi(1+ix/2), -\infty < x < \infty$ *is of the form* $\sqrt{2}\exp(-u^2)E(u)$ *with* $E \in \mathcal{R}(1/2)$.

### Comments

• There is a coefficient criterion an entire function $F(w) = \sum_{n=0}^{\infty}(n!)^{-1}c_n w^n$ to be in the space $G(\lambda)$ This is true iff $\limsup(2\sqrt{n})^{-1}\log|c_n| \le -\lambda$, [8, (VI.1.2)]. Further, the representation (1.10) and the coefficient criterion just mentioned as well as Stirling's formula yield that the entire function $E(w) = \sum_{n=0}^{\infty}(n!)^{-1}c_n w^n$ is in the space $\mathcal{R}(\lambda)$ iff

$$\limsup(2n)^{-1}\log(2n/e)^{n/2}|c_n| \le -\lambda.$$

• The asymptotic formula [9, (8.22.7)] for the Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ is proved by Liouville-Stekloff's method when $z = x$ is real. For the complex case at the end of [9, 8.65] is only mentioned that: "The proof of Theorem 8.22.7 can be given along these same lines".

• An asymptotic formula of Szegö's type for the Hermite polynomials in the complex plane is obtained in [6, 3.] as a corollary of a more general asymptotic formula of T.M. Cherry's type for the Weber-Hermite functions [6, (2.41)].

• The asymptotic formula (3.9) is given in T.M. Cherry's paper [3], without any proof or reference.

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