

**IMPULSIVE PARTIAL HYPERBOLIC FUNCTIONAL
 DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER
 WITH STATE-DEPENDENT DELAY**

Saïd Abbas * and Mouffak Benchohra **

Abstract

This paper deals with the existence and uniqueness of solutions of two classes of partial impulsive hyperbolic differential equations with fixed time impulses and state-dependent delay involving the Caputo fractional derivative. Our results are obtained upon suitable fixed point theorems.

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1. Introduction

In this paper, we start by studying the existence result for fractional order initial value problems (*IVP* for short), for the system

$$({}^c D_0^\alpha u)(x, y) = f(x, y, u(\rho_1(x, y, u(x, y))), \rho_2(x, y, u(x, y))), \text{ if } (x, y) \in J; x \neq x_k, \quad (1)$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \text{ if } y \in [0, b]; k = 1, \dots, m, \quad (2)$$

$$u(x, y) = \phi(x, y), \text{ if } (x, y) \in \tilde{J} := [-\alpha, a] \times [-\beta, b] \setminus (0, a] \times (0, b], \quad (3)$$

$$u(x, 0) = \varphi(x), u(0, y) = \psi(y), x \in [0, a], \text{ and } y \in [0, b], \quad (4)$$

where $J = [0, a] \times [0, b]$, $a, b, \alpha, \beta > 0$, ${}^c D_0^r$ is the fractional Caputo derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = a$, $\phi : \tilde{J} \rightarrow \mathbb{R}^n$ is a given function, $\varphi : [0, a] \rightarrow \mathbb{R}^n$, $\psi : [0, b] \rightarrow \mathbb{R}^n$ are given absolutely continuous functions such that $\varphi(x) = \phi(x, 0)$, $\psi(y) = \phi(0, y)$ for each $x \in [0, a]$ and $y \in [0, b]$, $f : J \times C \rightarrow \mathbb{R}^n$, $\rho_1, \rho_2 : J \times C \rightarrow \mathbb{R}$, $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k = 1, \dots, m$ are given functions and C is the space defined by

$$C := C_{(\alpha, \beta)} = \{u : [-\alpha, 0] \times [-\beta, 0] \rightarrow \mathbb{R}^n : \text{continuous and there exist } \tau_k \in (-\alpha, 0) \text{ with } u(\tau_k^-, \tilde{y}) \text{ and } u(\tau_k^+, \tilde{y}), k = 1, \dots, m, \text{ exist for any } \tilde{y} \in [-\beta, 0] \text{ with } u(\tau_k^-, \tilde{y}) = u(\tau_k, \tilde{y})\}.$$

C is a Banach space with norm

$$\|u\|_C = \sup_{(x, y) \in [-\alpha, 0] \times [-\beta, 0]} \|u(x, y)\|.$$

For any function u defined on $[-\alpha, a] \times [-\beta, b]$ and any $(x, y) \in J$, we denote by $u_{(x, y)}$ the element of C defined by

$$u_{(x, y)}(s, t) = u(x + s, y + t); (s, t) \in [-\alpha, 0] \times [-\beta, 0],$$

here $u_{(x, y)}(\cdot, \cdot)$ represents the history of the state from time $(x - \alpha, y - \beta)$ up to the present time (x, y) .

Next we consider the following system of partial hyperbolic differential equations of fractional order with infinite delay

$$({}^c D_0^r u)(x, y) = f(x, y, u_{(\rho_1(x, y, u_{(x, y)}), \rho_2(x, y, u_{(x, y)}))}), \text{ if } (x, y) \in J; x \neq x_k, \quad (5)$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \text{ if } y \in [0, b]; k = 1, \dots, m, \quad (6)$$

$$u(x, y) = \phi(x, y), \text{ if } (x, y) \in \tilde{J}' := (-\infty, a] \times (-\infty, b] \setminus (0, a] \times (0, b], \quad (7)$$

$$u(x, 0) = \varphi(x), u(0, y) = \psi(y), x \in [0, a], \text{ and } y \in [0, b], \quad (8)$$

where φ, ψ, I_k are as in problem (1)-(4), $f : J \times B \rightarrow \mathbb{R}^n$, $\rho_1, \rho_2 : J \times B \rightarrow \mathbb{R}$, $\phi : \tilde{J}' \rightarrow \mathbb{R}^n$ and B is called a phase space that will be specified in Section 4.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs [21, 22, 24, 25], the papers [1, 2, 7, 9, 12, 27], and the references therein.

Integer order impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs [8, 23, 26], and the references therein. Very recently, some extensions to impulsive fractional order differential equations have been obtained in [4, 5, 6, 10].

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last year, see, for instance, [16, 17] and the references therein. The literature related to partial functional differential equations with state-dependent delay is limited, see for instance [19]. The literature related to ordinary and partial functional differential equations with delay for which $\rho(t, \cdot) = t$ or $(\rho_1(x, y, \cdot), \rho_2(x, y, \cdot)) = (x, y)$ is very extensive, see for instance [1, 2, 15] and the references therein.

In this paper, we shall present existence and uniqueness results for our problems. These results initiate the study of hyperbolic fractional functional differential equations with state-dependent delay subject to impulsive effect. We present two results for each of our problems, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray-Schauder type.

2. Preliminaries

In this section, we introduce notations and definitions which are used throughout this paper. By $AC(J, \mathbb{R}^n)$ we denote the space of absolutely continuous functions from J into \mathbb{R}^n and $L^1(J, \mathbb{R}^n)$ is the space of Lebesgue-integrable functions $w : J \rightarrow \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x, y)\| dy dx,$$

where $\|\cdot\|$ denotes a suitable complete norm on \mathbb{R}^n .

Let $a_1 \in [0, a]$, $z^+ = (a_1, 0) \in J$, $J_z = [a_1, a] \times [0, b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $w \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I_{z^+}^r w)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} w(s, t) dt ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order r .

DEFINITION 2.1. ([27]) For $w \in L^1(J_z, \mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression

$$({}^c D_{z^+}^r w)(x, y) = (I_{z^+}^{1-r} \frac{\partial^2}{\partial x \partial y} w)(x, y).$$

Set

$$J_k := (x_k, x_{k+1}] \times [0, b]$$

and

$$J' := J \setminus \{(x_1, y), \dots, (x_m, y), y \in [0, b]\}.$$

Consider the space

$$PC := PC(J, \mathbb{R}^n)$$

$$= \{u : J \rightarrow \mathbb{R}^n : u \in C(J_k, \mathbb{R}^n); k = 1, \dots, m, \text{ and there exist } u(x_k^-, y) \text{ and } u(x_k^+, y); k = 1, \dots, m, \text{ with } u(x_k^-, y) = u(x_k, y)\}.$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y) \in J} \|u(x, y)\|.$$

3. Impulsive functional hyperbolic differential equations with finite delay

Set

$$\widetilde{PC} := PC([- \alpha, a] \times [- \beta, b], \mathbb{R}^n),$$

which is a Banach space with the norm

$$\|u\|_{\widetilde{PC}} = \sup\{\|u(x, y)\| : (x, y) \in [- \alpha, a] \times [- \beta, b]\}.$$

DEFINITION 3.1. A function $u \in \widetilde{PC}$ whose r -derivative exists on J' is said to be a solution of (1)-(4) if u satisfies the condition (3) on \widetilde{J} , the equation (1) on J' and conditions (2) and (4) are satisfied on J .

Let $h \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, and

$$\mu_k(x, y) = u(x, 0) + u(x_k^+, y) - u(x_k^+, 0), \quad k = 0, \dots, m.$$

For the existence of solutions for the problem (1)–(3), we need the following lemma.

LEMMA 3.2. ([3]) A function $u \in AC([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$; $k = 0, \dots, m$ is a solution of the differential equation

$$({}^c D_{z_k}^r u)(x, y) = h(x, y); (x, y) \in [x_k, x_{k+1}] \times [0, b],$$

if and only if $u(x, y)$ satisfies

$$u(x, y) = \mu_k(x, y) + (I_{z_k}^r h)(x, y); (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{9}$$

Let

$$\mu := \mu_0 = u(x, 0) + u(0, y) - u(0, 0) = \varphi(x) + \psi(y) - \varphi(0).$$

LEMMA 3.3. ([3]) Let $0 < r_1, r_2 \leq 1$ and let $h \in PC(J, \mathbb{R}^n)$. A function u is a solution of the fractional integral equation

$$u(x, y) = \begin{cases} \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} h(s, t) dt ds; \\ \text{if } (x, y) \in [0, x_1] \times [0, b], \\ \mu(x, y) + \sum_{i=1}^k (I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1} (y-t)^{r_2-1} h(s, t) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} h(s, t) dt ds; \\ \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b], k = 1, \dots, m, \end{cases} \tag{10}$$

if and only if u is a solution of the fractional IVP

$${}^c D_{z_k}^r u(x, y) = h(x, y), (x, y) \in J', \tag{11}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), y \in [0, b], k = 1, \dots, m. \tag{12}$$

Set $\mathcal{R} := \mathcal{R}_{(\rho_1^-, \rho_2^-)}$

$$= \{(\rho_1(s, t, u), \rho_2(s, t, u)) : (s, t, u) \in J \times C, \rho_i(s, t, u) \leq 0; i = 1, 2\}.$$

We always assume that $\rho_i : J \times C \rightarrow \mathbb{R}$; $i = 1, 2$ are continuous and the function $(s, t) \mapsto u_{(s,t)}$ is continuous from \mathcal{R} into C .

The first result is based on Banach fixed point theorem.

THEOREM 3.4. Let $f(\cdot, \cdot, u) \in PC(J, \mathbb{R}^n)$ for each $u \in C$. Assume that:

(H1) There exists a constant $l > 0$ such that

$$\|f(x, y, u) - f(x, y, \bar{u})\| \leq l \|u - \bar{u}\|_C, \text{ for each } (x, y) \in J \text{ and each } u, \bar{u} \in C.$$

(H2) There exists a constant $l^* > 0$ such that

$$\|I_k(u) - I_k(\bar{u})\| \leq l^* \|u - \bar{u}\|, \text{ for each } u, \bar{u} \in \mathbb{R}^n, k = 1, \dots, m.$$

If

$$2ml^* + \frac{2la^{r_1}b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1, \quad (13)$$

then (1)-(4) has a unique solution on $[-\alpha, a] \times [-\beta, b]$.

P r o o f. We transform the problem (1)-(4) into a fixed point problem. Consider the operator $F : \widetilde{PC} \rightarrow \widetilde{PC}$ defined by

$$F(u)(x, y) = \begin{cases} \phi(x, y), & (x, y) \in \tilde{J}, \\ \mu(x, y) + \sum_{0 < x_k < x} (I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0 < x_k < x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y - t)^{r_2-1} \\ \times f(s, t, u(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \\ \times f(s, t, u(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) dt ds, & (x, y) \in J. \end{cases}$$

Clearly, the fixed points of the operator F are solution of the problem (1)-(4). We shall use the Banach contraction principle to prove that F has a fixed point. For this, we show that F is a contraction. Let $u, v \in \widetilde{PC}$, then for each $(x, y) \in J$, we have

$$\begin{aligned} & \|F(u)(x, y) - F(v)(x, y)\| \\ & \leq \sum_{k=1}^m (\|I_k(u(x_k^-, y)) - I_k(v(x_k^-, y))\| + \|I_k(u(x_k^-, 0)) - I_k(v(x_k^-, 0))\|) \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y - t)^{r_2-1} \| \\ & \times f(s, t, u(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) - f(s, t, v(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t))))\| dt ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t, u_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))}) \\
 & - f(s,t, v_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))})\| dt ds \\
 & \leq \sum_{k=1}^m l^* (\|u(x_k^-, y) - v(x_k^-, y)\| + \|u(x_k^-, 0) - v(x_k^-, 0)\|) \\
 & + \frac{l}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} \\
 & \times \|u_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))} - v_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))}\|_C dt ds \\
 & + \frac{l}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
 & \times \|u_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))} - v_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))}\|_C dt ds \\
 & \leq \left[2ml^* + \frac{2la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \right] \|u-v\|_C.
 \end{aligned}$$

By the condition (13), we conclude that F is a contraction. As a consequence of Banach's fixed point theorem, we deduce that F has a unique fixed point which is a solution of the problem (1)-(4). ■

In the following theorem we give an existence result for the problem (1)-(4) by applying the nonlinear alternative of Leray-Schauder type [13].

THEOREM 3.5. *Let $f(\cdot, \cdot, u) \in PC(J, \mathbb{R}^n)$ for each $u \in C$. Assume that the following conditions hold:*

(H3) *There exists $\phi_f \in C(J, \mathbb{R}_+)$ and $\tilde{\psi} : [0, \infty) \rightarrow (0, \infty)$ continuous and nondecreasing such that*

$$\|f(x, y, u)\| \leq \phi_f(x, y)\tilde{\psi}(\|u\|_C) \quad \text{for all } (x, y) \in J, u \in C.$$

(H4) *There exists $\psi^* : [0, \infty) \rightarrow (0, \infty)$ continuous and nondecreasing such that*

$$\|I_k(u)\| \leq \psi^*(\|u\|) \quad \text{for all } u \in \mathbb{R}^n.$$

(H5) *There exists an number $\bar{M} > 0$ such that*

$$\frac{\bar{M}}{\| \mu \|_\infty + 2m\psi^*(\bar{M}) + \frac{2a^{r_1}b^{r_2}\phi_f^0\tilde{\psi}(\bar{M})}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1,$$

where $\phi_f^0 = \sup\{\phi_f(x, y) : (x, y) \in J\}$.

Then (1)-(4) has at least one solution on $[-\alpha, a] \times [-\beta, b]$.

P r o o f. Consider the operator F defined in Theorem 3.4. We shall show that the operator F is continuous and completely continuous.

A priori estimate.

For $\lambda \in [0, 1]$, let u be such that for each $(x, y) \in J$ we have $u(x, y) = \lambda(Fu)(x, y)$.

For each $(x, y) \in J$, then from (H3) and (H4) we have

$$\begin{aligned} \|u(x, y)\| &\leq \|\mu(x, y)\| + \sum_{k=1}^m (\|I_k(u(x_k^-, y))\| + \|I_k(u(x_k^-, 0))\|) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y - t)^{r_2-1} \\ &\times \|f(s, t, u_{(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))})\| dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \\ &\times \|f(s, t, u_{(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))})\| dt ds \\ &\leq \|\mu\|_\infty + 2m\psi^*(\|u\|) + \frac{2a^{r_1}b^{r_2}\phi_f^0\tilde{\psi}(\|u\|)}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)}. \end{aligned}$$

Thus,

$$\frac{\|u\|_{PC}}{\|\mu\|_\infty + 2m\psi^*(\|u\|_{PC}) + \frac{2a^{r_1}b^{r_2}\phi_f^0\tilde{\psi}(\|u\|_{PC})}{\Gamma(r_1+1)\Gamma(r_2+1)}} \leq 1.$$

By condition (H5), there exists \bar{M} such that $\|u\|_\infty \neq \bar{M}$.

Let

$$U = \{u \in \widetilde{PC} : \|u\|_{\widetilde{PC}} < \bar{M}\}.$$

The operator $F : \bar{U} \rightarrow \widetilde{PC}$ is continuous and completely continuous. From the choice of U , there is no $u \in \partial U$ such that $u = \lambda F(u)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [13], we deduce that F has a fixed point u in \bar{U} which is a solution of the problem (1)-(4).

4. The phase space B

The notation of the phase space B plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [14] (see also [15, 20]).

For any $(x, y) \in J$ denote $E_{(x,y)} := [0, x] \times \{0\} \cup \{0\} \times [0, y]$, furthermore in case $x = a, y = b$ we write simply E . Consider the space $(B, \|(\cdot, \cdot)\|_B)$ is a seminormed linear space of functions mapping $(-\infty, 0] \times (-\infty, 0]$ into \mathbb{R}^n , and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A₁) If $z : (-\infty, a] \times (-\infty, b] \rightarrow \mathbb{R}^n, z_{(x,y)} \in B$ for all $(x, y) \in E$ and $z \in PC$, then for every $(x, y) \in J$ the following conditions hold:

(i) $z_{(x,y)}$ is in B ;

(ii) There exists a positive constant H such that $\|z(x, y)\| \leq H\|z_{(x,y)}\|_B$,

(iii) There exist two functions $K, M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ independent of u , with K continuous and M locally bounded such that

$$\|z_{(x,y)}\|_B \leq K(x, y) \sup_{(s,t) \in [0,x] \times [0,y]} \|z(s, t)\| + M(x, y) \sup_{(s,t) \in E_{(x,y)}} \|z_{(s,t)}\|_B,$$

(A₂) The space B is complete.

Denote $K = \sup_{(x,y) \in J} K(x, y)$ and $M = \sup_{(x,y) \in J} M(x, y)$.

Now, we present some examples of phase spaces (see [11]).

EXAMPLE 4.1. Let B be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \rightarrow \mathbb{R}^n$ such that for each $\alpha, \beta \geq 0$ we define in C the semi-norms by

$$\|\phi\|_B = \sup_{(s,t) \in [-\alpha, 0] \times [-\beta, 0]} \|\phi(s, t)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{B} = B/\|\cdot\|_B$ is isometric to the space $PC([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all piecewise continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into \mathbb{R}^n with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

EXAMPLE 4.2. Let $\alpha, \beta, \gamma \geq 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t) \in [-\alpha, 0] \times [-\beta, 0]} \|\phi(s, t)\| + \int_{-\infty}^0 \int_{-\infty}^0 e^{\gamma(s+t)} \|\phi(s, t)\| dt ds.$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \rightarrow \mathbb{R}^n$ which are measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1, K = \int_{-\alpha}^0 \int_{-\beta}^0 e^{\gamma(s+t)} dt ds, M = 2.$$

5. Impulsive functional hyperbolic differential equations with infinite delay

Now we present two existence results for the problem (5)-(8). Let us start in this section by defining what we mean by a solution of the problem (5)-(8). Let the space

$$\Omega := \{u : (-\infty, a] \times (-\infty, b] \rightarrow \mathbb{R}^n : u_{(x,y)} \in B \text{ for } (x, y) \in E \text{ and } u|_J \in PC\}.$$

DEFINITION 5.1. A function $u \in \Omega$ whose r -derivative exists on J' is said to be a solution of (5)-(8) if u satisfies the condition (7) on \tilde{J}' , the equation (5) on J' and conditions (6) and (8) are satisfied on J .

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-, \rho_2^-)}$

$$= \{(\rho_1(s, t, u), \rho_2(s, t, u)) : (s, t, u) \in J \times B, \rho_i(s, t, u) \leq 0; i = 1, 2\}.$$

We always assume that $\rho_i : J \times B \rightarrow \mathbb{R}; i = 1, 2$ are continuous and the function $(s, t) \mapsto u_{(s,t)}$ is continuous from \mathcal{R}' into B .

We will need to introduce the following hypothesis:

(H_ϕ) There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-, \rho_2^-)} \rightarrow (0, \infty)$ such that

$$\|\phi_{(s,t)}\|_B \leq L(s, t) \|\phi\|_B, \text{ for any } (s, t) \in \mathcal{R}'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms ([19]).

LEMMA 5.2. *If $u \in \Omega$, then*

$$\|u_{(s,t)}\|_B = (M + L')\|\phi\|_B + K \sup_{(\theta,\eta) \in [0, \max\{0,s\}] \times [0, \max\{0,t\}]} \|u(\theta, \eta)\|,$$

where

$$L' = \sup_{(s,t) \in \mathcal{R}'} L(s, t).$$

Our first existence result for the IVP (5)-(8) is based on the Banach contraction principle.

THEOREM 5.3. *Assume that the following hypotheses hold:*

(H01) *There exists $\ell' > 0$ such that*

$$\|f(x, y, u) - f(x, y, v)\| \leq \ell' \|u - v\|_B, \text{ for any } u, v \in B \text{ and } (x, y) \in J.$$

(H02) *There exists a constant $l^* > 0$ such that*

$$\|I_k(u) - I_k(\bar{u})\| \leq l^* \|u - \bar{u}\|, \text{ for each } u, \bar{u} \in \mathbb{R}^n, k = 1, \dots, m.$$

If

$$2ml^* + \frac{2K\ell' a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1, \tag{14}$$

then there exists a unique solution for IVP (5)-(8) on $(-\infty, a] \times (-\infty, b]$.

P r o o f. Transform the problem (5)-(8) into a fixed point problem. Consider the operator $N : \Omega \rightarrow \Omega$ defined by

$$N(u)(x, y) = \begin{cases} \phi(x, y), & (x, y) \in \tilde{J}', \\ \mu(x, y) + \sum_{0 < x_k < x} (I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0 < x_k < x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y - t)^{r_2-1} \\ \times f(s, t, u_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))}) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \\ \times f(s, t, u_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))}) dt ds, & (x, y) \in J. \end{cases} \tag{15}$$

Let $v(\cdot, \cdot) : (-\infty, a] \times (-\infty, b] \rightarrow \mathbb{R}^n$ be a function defined by,

$$v(x, y) = \begin{cases} \phi(x, y), & (x, y) \in \tilde{J}', \\ \mu(x, y), & (x, y) \in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x, y) \in E$. For each $w \in C(J, \mathbb{R}^n)$ with $w(x, y) = 0$ for each $(x, y) \in E$ we denote by \bar{w} the function defined by

$$\bar{w}(x, y) = \begin{cases} 0, & (x, y) \in \tilde{J}', \\ w(x, y) & (x, y) \in J. \end{cases}$$

If $u(\cdot, \cdot)$ satisfies the integral equation

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) dt ds,$$

we can decompose $u(\cdot, \cdot)$ as $u(x, y) = \bar{w}(x, y) + v(x, y)$; $(x, y) \in J$, which implies $u_{(x,y)} = \bar{w}_{(x,y)} + v_{(x,y)}$, for every $(x, y) \in J$, and the function $w(\cdot, \cdot)$ satisfies

$$\begin{aligned} w(x, y) &= \sum_{0 < x_k < x} (I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0 < x_k < x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y - t)^{r_2-1} \\ &\times f(s, t, u(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \\ &\times f(s, t, u(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) dt ds. \end{aligned}$$

Set

$$C_0 = \{w \in \Omega : w(x, y) = 0 \text{ for } (x, y) \in E\},$$

and let $\|\cdot\|_{(a,b)}$ be the seminorm in C_0 defined by

$$\|w\|_{(a,b)} = \sup_{(x,y) \in E} \|w_{(x,y)}\|_B + \sup_{(x,y) \in J} \|w(x, y)\| = \sup_{(x,y) \in J} \|w(x, y)\|, \quad w \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_{(a,b)}$. Let the operator $P : C_0 \rightarrow C_0$ be defined by

$$\begin{aligned} P(x, y) &= \sum_{0 < x_k < x} (I_k(u(x_k^-, y)) - I_k(u(x_k^-, 0))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0 < x_k < x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y - t)^{r_2-1} \\ &\times f(s, t, \bar{w}(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t))) + v(\rho_1(s, t, u(s, t)), \rho_2(s, t, u(s, t)))) dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \end{aligned}$$

$$\times f(s, t, \bar{w}_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))} + v_{(\rho_1(s,t,u(s,t)), \rho_2(s,t,u(s,t)))}) dt ds, \quad (16)$$

for each $(x, y) \in J$. Then the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. We can easily show that $P : C_0 \rightarrow C_0$ is a contraction map, and hence it has a unique fixed point by Banach's contraction principle. ■

Now we give an existence result based on the nonlinear alternative of Leray-Schauder type [13]. We will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

LEMMA 5.4([18]). *Let $v : J \rightarrow [0, \infty)$ be a real function and $\omega(., .)$ be a nonnegative, locally integrable function on J . If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$v(x, y) \leq \omega(x, y) + c \int_0^x \int_0^y \frac{v(s, t)}{(x-s)^{r_1}(y-t)^{r_2}} dt ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$v(x, y) \leq \omega(x, y) + \delta c \int_0^x \int_0^y \frac{\omega(s, t)}{(x-s)^{r_1}(y-t)^{r_2}} dt ds,$$

for every $(x, y) \in J$.

THEOREM 5.5. *Assume (H_ϕ) and*

(H03) *There exist $p, q \in C(J, \mathbb{R}_+)$ such that*

$$\|f(x, y, u)\| \leq p(x, y) + q(x, y)\|u\|_B, \text{ for } (x, y) \in J \text{ and each } u \in B.$$

(H04) *There exist $c_k > 0; k = 1, \dots, m$ such that*

$$\|I_k(u)\| \leq c_k \text{ for all } u \in \mathbb{R}^n.$$

Then the IVP (5)-(8) has at least one solution on $(-\infty, a] \times (-\infty, b]$.

P r o o f. Let $P : C_0 \rightarrow C_0$ defined as in (16). As in Theorem 3.5, we can show that the operator P is continuous and completely continuous.

We now show there exists an open set $U \subseteq C_0$ with $w \neq \lambda P(w)$, for $\lambda \in (0, 1)$ and $w \in \partial U$. Let $w \in C_0$ and $w = \lambda P(w)$ for some $0 < \lambda < 1$. By (H03) and (H04) for each $(x, y) \in J$, we have

$$\|w(x, y)\| \leq \sum_{k=1}^m 2c_k + \frac{2\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)}$$

$$+\frac{2}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}q(s,t)\|\bar{w}_{(s,t)}+v_{(s,t)}\|_B dt ds.$$

But Lemma 5.2 implies that

$$\begin{aligned}\|\bar{w}_{(s,t)}+v_{(s,t)}\|_B &\leq \|\bar{w}_{(s,t)}\|_B+\|v_{(s,t)}\|_B \\ &\leq K\sup\{w(\tilde{s},\tilde{t}):(\tilde{s},\tilde{t})\in[0,s]\times[0,t]\} \\ &\quad +(M+L')\|\phi\|_B+K\|\phi(0,0)\|.\end{aligned}\tag{17}$$

If we name $z(s,t)$ the right hand side of (17), then we have

$$\|\bar{w}_{(s,t)}+v_{(s,t)}\|_B\leq z(x,y),$$

and therefore, for each $(x,y)\in J$ we obtain

$$\begin{aligned}\|w(x,y)\| &\leq 2\sum_{k=1}^m c_k+\frac{2\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad +\frac{2}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}q(s,t)z(s,t) dt ds.\end{aligned}\tag{18}$$

Using the above inequality and the definition of z for each $(x,y)\in J$ we have

$$\begin{aligned}z(x,y) &\leq (M+L')\|\phi\|_B+K\|\phi(0,0)\|+2\sum_{k=1}^m c_k+\frac{2\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad +\frac{2K\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}z(s,t) dt ds.\end{aligned}$$

Then by Lemma 5.4, there exists $\delta=\delta(r_1,r_2)$ such that we have

$$\|z(x,y)\|\leq R+\delta\frac{2K\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}R dt ds,$$

where

$$R=(M+L')\|\phi\|_B+K\|\phi(0,0)\|+2\sum_{k=1}^m c_k+\frac{2\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|z\|_\infty\leq R+\frac{2R\delta K\|q\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}:=\widetilde{M}.$$

Then, (18) implies that

$$\|w\|_\infty \leq 2 \sum_{k=1}^m c_k + \frac{2a^{r_1}b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} (\|p\|_\infty + \widetilde{M}\|q\|_\infty) := M^*.$$

Set

$$U = \{w \in C_0 : \|w\|_{(a,b)} < M^* + 1\}.$$

$P : \overline{U} \rightarrow C_0$ is continuous and completely continuous. By our choice of U , there is no $w \in \partial U$ such that $w = \lambda P(w)$, for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [13], we deduce that N has a fixed point which is a solution to problem (5)-(8). ■

6. Examples

EXAMPLE 1. As an application of our results we consider the following impulsive partial hyperbolic functional differential equations of the form

$$\begin{aligned}
 ({}^c D_0^r u)(x, y) &= \frac{e^{-x-y}}{9 + e^{x+y}} \\
 &\times \frac{|u(x - \sigma_1(u(x, y)), y - \sigma_2(u(x, y)))|}{1 + |u(x - \sigma_1(u(x, y)), y - \sigma_2(u(x, y)))|}, \quad \text{if } (x, y) \in [0, 1] \times [0, 1], \quad x \neq \frac{1}{2}, \\
 & \hspace{15em} (19)
 \end{aligned}$$

$$u((\frac{1}{2})^+, y) = u((\frac{1}{2})^-, y) + \frac{|u((\frac{1}{2})^-, y)|}{3 + |u((\frac{1}{2})^-, y)|}, \quad y \in [0, 1], \quad (20)$$

$$u(x, y) = x + y^2, \quad (x, y) \in [-1, 1] \times [-2, 1] \setminus (0, 1] \times (0, 1], \quad (21)$$

$$u(x, 0) = x, \quad u(0, y) = y^2, \quad \text{for each } x \in [0, 1] \text{ and } y \in [0, 1], \quad (22)$$

where $\sigma_1 \in C(\mathbb{R}, [0, 1])$, $\sigma_2 \in C(\mathbb{R}, [0, 2])$. Set

$$\rho_1(x, y, \varphi) = x - \sigma_1(\varphi(0, 0)), \quad (x, y, \varphi) \in J \times C,$$

$$\rho_2(x, y, \varphi) = y - \sigma_2(\varphi(0, 0)), \quad (x, y, \varphi) \in J \times C,$$

where $C := C_{(1,2)}$. Set

$$f(x, y, \varphi) = \frac{e^{-x-y}|\varphi|}{(9 + e^{x+y})(1 + |\varphi|)}, \quad (x, y) \in [0, 1] \times [0, 1], \quad \varphi \in C,$$

and

$$I_k(u) = \frac{u}{3 + u}, \quad u \in \mathbb{R}_+.$$

A simple computations show that conditions of Theorem 3.4 are satisfied which implies that problem (19)-(22) has a unique solution defined on $[-1, 1] \times [-2, 1]$.

EXAMPLE 2. We consider now the following impulsive fractional order partial hyperbolic differential equations with infinite delay of the form

$$({}^c D_0^r u)(x, y) = \frac{ce^{x+y-\gamma(x+y)}|u(x - \sigma_1(u(x, y)), y - \sigma_2(u(x, y)))|}{(e^{x+y} + e^{-x-y})(1 + |u(x - \sigma_1(u(x, y)), y - \sigma_2(u(x, y)))|)};$$

if $(x, y) \in J := [0, 1] \times [0, 1], x \neq \frac{k}{k+1}; k = 1, \dots, m,$ (23)

$$u\left(\left(\frac{k}{k+1}\right)^+, y\right) = u\left(\left(\frac{k}{k+1}\right)^-, y\right) + \frac{\left|u\left(\left(\frac{k}{k+1}\right)^-, y\right)\right|}{3mk + \left|u\left(\left(\frac{k}{k+1}\right)^-, y\right)\right|}; \quad y \in [0, 1], k = 1, \dots, m,$$
 (24)

$$u(x, 0) = x, \quad u(0, y) = y^2, \quad \text{for each } x \in [0, 1] \text{ and } y \in [0, 1],$$
 (25)

$$u(x, y) = x + y^2, \quad (x, y) \in \tilde{J} := (-\infty, 1] \times (-\infty, 1] \setminus (0, 1] \times (0, 1],$$
 (26)

where $c = \frac{10}{\Gamma(r_1+1)\Gamma(r_2+1)}$, γ a positive real constant and $\sigma_1, \sigma_2 \in C(\mathbb{R}, [0, \infty))$.
Let

$$B_\gamma = \{u \in C((-\infty, 0] \times (-\infty, 0], \mathbb{R}) : u|_J \in PC(J, \mathbb{R}) \text{ and } \lim_{\|(\theta, \eta)\| \rightarrow \infty} e^{\gamma(\theta+\eta)} u(\theta, \eta) \text{ exists in } \mathbb{R}\}.$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{(\theta, \eta) \in (-\infty, 0] \times (-\infty, 0]} e^{\gamma(\theta+\eta)} |u(\theta, \eta)|.$$

Let

$$E := [0, 1] \times \{0\} \cup \{0\} \times [0, 1],$$

and $u : (-\infty, 1] \times (-\infty, 1] \rightarrow \mathbb{R}$ such that $u_{(x,y)} \in B_\gamma$ for $(x, y) \in E$, then

$$\lim_{\|(\theta, \eta)\| \rightarrow \infty} e^{\gamma(\theta+\eta)} u_{(x,y)}(\theta, \eta) = \lim_{\|(\theta, \eta)\| \rightarrow \infty} e^{\gamma(\theta-x+\eta-y)} u(\theta, \eta)$$

$$= e^{\gamma(x+y)} \lim_{\|(\theta,\eta)\| \rightarrow \infty} u(\theta, \eta) < \infty.$$

Hence $u_{(x,y)} \in B_\gamma$. Finally, we prove that

$$\begin{aligned} \|u_{(x,y)}\|_\gamma &= K \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\} \\ &\quad + M \sup\{\|u_{(s,t)}\|_\gamma : (s,t) \in E_{(x,y)}\}, \end{aligned}$$

where $K = M = 1$ and $H = 1$.

If $x + \theta \leq 0, y + \eta \leq 0$, we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty, 0] \times (-\infty, 0]\},$$

and if $x + \theta \geq 0, y + \eta \geq 0$ then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\}.$$

Thus for all $(x + \theta, y + \eta) \in [0, 1] \times [0, 1]$, we get

$$\|u_{(x,y)}\|_\gamma = \sup_{(s,t) \in (-\infty, 0] \times (-\infty, 0]} |u(s,t)| + \sup_{(s,t) \in [0,x] \times [0,y]} |u(s,t)|.$$

Then

$$\|u_{(x,y)}\|_\gamma = \sup_{(s,t) \in E} \|u_{(s,t)}\|_\gamma + \sup_{(s,t) \in [0,x] \times [0,y]} |u(s,t)|.$$

$(B_\gamma, \|\cdot\|_\gamma)$ is a Banach space. We conclude that B_γ is a phase space.

Set

$$\rho_1(x, y, \varphi) = x - \sigma_1(\varphi(0, 0)), \quad (x, y, \varphi) \in J \times B_\gamma,$$

$$\rho_2(x, y, \varphi) = y - \sigma_2(\varphi(0, 0)), \quad (x, y, \varphi) \in J \times B_\gamma,$$

$$f(x, y, \varphi) = \frac{ce^{x+y-\gamma(x+y)}|\varphi|}{(e^{x+y} + e^{-x-y})(1 + |\varphi|)}, \quad (x, y) \in [0, 1] \times [0, 1], \varphi \in B_\gamma$$

and

$$I_k(u) = \frac{u}{3mk + u}; \quad u \in \mathbb{R}_+, \quad k = 1, \dots, m.$$

We can easily show that conditions of Theorem 5.3 are satisfied, and hence problem (23)-(26) has a unique solution defined on $(-\infty, 1] \times (-\infty, 1]$.

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* *Laboratoire de Mathématiques*
Université de Saïda
B.P. 138, 20000 – Saïda, ALGÉRIE
e-mail: abbas_said_dz@yahoo.fr

** *(Corresponding author)*
Laboratoire de Mathématiques
Université de Sidi Bel-Abbès
B.P. 89, 22000 – Sidi Bel-Abbès, ALGÉRIE
e-mail: benchohra@univ-sba.dz