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MIXED FRACTIONAL INTEGRATION OPERATORS IN MIXED WEIGHTED HÖLDER SPACES

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Abstract

We study mixed Riemann-Liouville integrals of functions of two variables in Hölder spaces of different orders in each variables. We consider Hölder spaces defined both by first order differences in each variable and also by the mixed second order difference, the main interest being in the evaluation of the latter for the mixed fractional integral in both the cases where the density of the integral belongs to the Hölder class defined by usual or mixed differences. The obtained results extend the well known theorem of Hardy-Littlewood for one-dimensional fractional integrals to the case of mixed Hölderness. We cover also the weighted case with power weights.

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Key Words and Phrases: functions of two variables, Riemann-Liouville integrals, mixed fractional integrals, mixed finite differences, Hölder spaces of mixed order

1. Introduction

The mapping properties of the one-dimensional fractional Riemann-Liouville operator

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \qquad x > a, \tag{1.1}$$

are well studied both in weighted Hölder spaces or in generalized Hölder spaces. A non-weighted statement on action of the fractional integral operator from H_0^{β} into $H_0^{\beta+\alpha}$ is due to Hardy and Littlewood ([1], see [11], Theorems 3.1 and 3.2), and it is known that the operator I_{a+}^{α} with $0 < \alpha < 1$ establishes an isomorphism between the Hölder spaces $H_0^{\lambda}([a,b])$ and $H_0^{\lambda+\alpha}([a,b])$ of functions vanishing at the point $x = a$, if $\lambda + \alpha < 1$. The weighted results with power weights were obtained in [9], [10], see their presentation in [11], Theorems 3.3, 3.4 and 13.13). For weighted generalized Hölder spaces $H_0^{\omega}(\rho)$ of functions φ with a given dominant of continuity modulus of $\rho\varphi$, mapping properties in the case of power weight were studied in [8], [7], [12], see also their presentation in [11], Section 13.6. Different proofs were suggested in [3], [4], where the case of complex fractional orders was also considered, the shortest proof being given in [3].

The case of weights more general than power ones, including in particular power-logarithmic type weights, in the spaces $H_0^{\omega}(\rho)$ was considered in [13], where operators more general than just fractional integrals were treated. We refer also to paper [2] where the mapping properties of fractional integration operators were reconsidered in terms of the Matuszewska-Orlich indices of the characteristic ω defining the generalized Hölder space H^{ω}. Finally, we mention also the papers [5], [6], where fractional integrals were studied in spaces of Nikolsky type.

In the multidimensional case, statements on mapping properties in generalized Hölder spaces are known ([14]) for the Riesz fractional integrals

$$\int\limits_{\mathbb{R}^n} \frac{\varphi(y)\,dy}{|x-y|^{n-\alpha}}, \qquad x \in \mathbb{R}^n$$

see [11], Theorem 25.5. Mixed Riemann-Liouville fractional integrals of order (α, β):

$$\left(I_{0+,0+}^{\alpha,\,\beta}\varphi\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_0^x \int\limits_0^y \frac{\varphi(t,\,\tau)dtd\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}, \quad x > 0,\ y > 0, \tag{1.2}$$

were not studied either in the usual Hölder spaces, or in the Hölder spaces defined by mixed differences. Meanwhile, there arise "points of interest" related to the investigation of the above mixed differences of fractional integrals (1.2). For operators (1.2) in Hölder spaces of mixed order there arise some questions to be answered in relation to the usage of these or those differences in the definition of Hölder spaces. Such mapping properties in

Hölder spaces of mixed order were not studied. This paper is aimed to fill in this gap. We deal with both non-weighted and weighted spaces.

We consider the operator (1.2) in the rectangle

$$Q = \{(x,y) : 0 < x < b,\ 0 < y < d\}.$$

2. Preliminaries

2.1. Notation and a technical lemma

For a continuous function $\varphi(x,y)$ on $\mathbb{R}^2$ we introduce the notation

$$\left(\Delta_h^{1,0}\varphi\right)(x,y) = \varphi(x+h,y)-\varphi(x,y), \quad \left(\Delta_\eta^{0,1}\varphi\right)(x,y) = \varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\Delta_{h,\eta}^{1,1}\varphi\right)(x,y) = \varphi(x+h,y+\eta) - \varphi(x+h,y) - \varphi(x,y+\eta) + \varphi(x,y),$$

so that

$$\varphi(x+h,y+\eta) = \left(\Delta_{h,\eta}^{1,1}\varphi\right)(x,y) + \left(\Delta_h^{1,0}\varphi\right)(x,y) + \left(\Delta_\eta^{0,1}\varphi\right)(x,y) + \varphi(x,y). \tag{2.1}$$

Everywhere in the sequel by C, C_1, C_2 etc we denote positive constants which may different values in different occurrences, and even in the same line.

We introduce two types of mixed Hölder spaces by the following definitions.

Definition 2.1. **I**. Let $\lambda, \gamma \in (0,1]$. We say that $\varphi \in H^{\lambda,\gamma}(Q)$, if

$$|\varphi(x_1,y_1) - \varphi(x_2,y_2)| \le C_1 \left|x_1 - x_2\right|^\lambda + C_2 \left|y_1 - y_2\right|^\gamma \tag{2.2}$$

for all $(x_1,y_1),(x_2,y_2) \in Q$. Condition (2.2) is equivalent to the couple of the separate conditions

$$\left|\left(\Delta_h^{1,0}\varphi\right)(x,y)\right| \le C_1|h|^\lambda, \qquad \left|\left(\Delta_\eta^{0,1}\varphi\right)(x,y)\right| \le C_2|\eta|^\gamma \tag{2.3}$$

uniform with respect to another variable. By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $f \in H^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

II. Let $\lambda = 0$ and\or $\mu = 0$. We put $H^{0,0}(Q) = L^\infty(Q)$ and

$$H^{\lambda,0}(Q) = \{\varphi \in L^\infty(Q) : \left|\left(\Delta_h^{1,0}\varphi\right)(x,y)\right| \le C_1|h|^\lambda\}, \quad \lambda \in (0,1]$$

$$H^{0,\gamma}(Q)=\{\varphi\in L^{\infty}(Q):\left|\left(\Delta_h^{0,1}\varphi\right)(x,y)\right|\le C_1|h|^{\gamma}\},\quad \gamma\in(0,1].$$

Definition 2.2. We say that $\varphi(x,y)\in\widetilde{H}^{\lambda,\gamma}(\Omega)$, where $\lambda,\gamma\in(0,1]$, if

$$\varphi\in H^{\lambda,\gamma}(Q)\quad\text{and}\quad\left|\left(\Delta_{h,\eta}^{1,1}\varphi\right)(x,y)\right|\le C_3|h|^{\lambda}|\eta|^{\gamma}.$$

We say that $\varphi\in\widetilde{H}_0^{\lambda,\gamma}(Q)$, if $\varphi\in\widetilde{H}^{\lambda,\gamma}(Q)$ and $\varphi(0,y)\equiv\varphi(x,0)\equiv 0$.

These spaces become Banach spaces under the standard definition of the norms:

$$\|\varphi\|_{H^{\lambda,\gamma}}:=\|\varphi\|_{C(Q)}+\sup_{\substack{x,x+h\in[0,b]\\ y\in[0,d]}}\frac{\left|\left(\Delta_h^{1,0}\varphi\right)(x,y)\right|}{|h|^{\lambda}}+\sup_{\substack{y,y+\eta\in[0,d]\\ x\in[0,b]}}\frac{\left|\left(\Delta_\eta^{0,1}\varphi\right)(x,y)\right|}{|\eta|^{\gamma}},$$

$$\|\varphi\|_{\widetilde{H}^{\lambda,\gamma}}=\|\varphi\|_{H^{\lambda,\gamma}}+\sup_{\substack{x,x+h\in[0,b],\\ y,y+\eta\in[0,d]}}\frac{\left|\left(\Delta_{h,\eta}^{1,1}\varphi\right)(x,y)\right|}{|h|^{\lambda}|\eta|^{\gamma}}.$$

Note that

$$\varphi\in H^{\lambda,\gamma}(Q)\implies\left|\left(\Delta_{h,\eta}^{1,1}\varphi\right)(x,y)\right|\le C_\theta|h|^{\theta\lambda}|\eta|^{\gamma(1-\theta)}\tag{2.4}$$

for any $\theta\in[0,1]$, where $C_\theta=2C_1^{\theta}C_2^{1-\theta}$, so that

$$\widetilde{H}^{\lambda,\gamma}(Q)\hookrightarrow H^{\lambda,\gamma}(Q)\hookrightarrow\bigcap_{0\le\theta\le1}\widetilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q),\tag{2.5}$$

where $\hookrightarrow$ stands for the continuous embedding, and the norm for $\bigcap\limits_{0\le\theta\le1}\widetilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$ is introduced as the maximum in θ of norms for $\widetilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$. Since $\theta\in[0,1]$ is arbitrary, it is not hard to see that the inequality in (2.4) is equivalent (up to the constant factor C) to

$$\left|\left(\Delta_{h,\eta}^{1,1}\varphi\right)(x,y)\right|\le C\min\{|h|^{\lambda},|\eta|^{\gamma}\}.\tag{2.6}$$

We will also make use of the following weighted spaces. Let $\varrho(x,y)$ be a non-negative function on Q (we will only deal with degenerate weights $\varrho(x,y)=\varrho_1(x)\varrho_2(y)$).

Definition 2.3. By $H^{\lambda,\gamma}(Q,\varrho)$ and $\widetilde{H}^{\lambda,\gamma}(Q,\varrho)$ we denote the spaces of functions $\varphi(x,y)$ such that $\varrho\varphi\in H^{\lambda,\gamma}(Q)$, $\varrho\varphi\in\widetilde{H}^{\lambda,\gamma}(Q,\varrho)$, respectively, equipped with the norms

$$\|\varphi\|_{H^{\lambda,\gamma}(Q,\varrho)}=\|\varrho\varphi\|_{H^{\lambda,\gamma}(Q)},\quad\|\varphi\|_{\widetilde{H}^{\lambda,\gamma}(Q,\varrho)}=\|\varrho\varphi\|_{\widetilde{H}^{\lambda,\gamma}(Q)}$$

By $H_0^{\lambda,\gamma}(\varrho)$ and $\widetilde{H}_0^{\lambda,\gamma}(Q,\varrho)$ we denote the corresponding subspaces of functions φ such that $\varrho\varphi|_{x=a} \equiv \varrho\varphi|_{y=c} \equiv 0$.

Below we follow some technical estimations suggested in [3] for the case of one-dimensional Riemann-Liouville fractional integrals. We denote

$$B(x,y;t,\tau) = \frac{\varrho(x,y) - \varrho(t,\tau)}{\varrho(t,\tau)(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}\,, \tag{2.7}$$

where $0 < \alpha, \beta < 1,\ a < t < x < b,\ c < \tau < y < d$, and

$$B_1(x,t) = \frac{\varrho_1(x) - \varrho_1(t)}{\varrho_1(t)(x-t)^{1-\alpha}}, \qquad B_2(y,\tau) = \frac{\varrho_2(y) - \varrho_2(\tau)}{\varrho_2(\tau)(y-\tau)^{1-\beta}}. \tag{2.8}$$

In the case $\varrho(x,y) = \varrho_1(x)\varrho_2(y)$ we have

$$B(x,y;t,\tau) = B_1(x,t)B_2(y,\tau) + \frac{B_1(x,t)}{(y-\tau)^{1-\beta}} + \frac{B_2(y,\tau)}{(x-t)^{1-\alpha}},$$

Let also

$$D_1(x,h,t) = B_1(x+h,t) - B_1(x,t), \quad t,x,x+h \in [0,b],\ h > 0,$$

$$D_2(y,\eta,\tau) = B_2(y+\eta,\tau) - B_2(y,\tau), \quad \tau,y,y+\eta \in [0,d],\ \eta > 0.$$

Remark 2.4. All the weighted estimations of fractional integrals in the sequel are based on inequalities (2.9)-(2.10). Note that the right-hand sides of these inequalities have the exponent $\max(\mu-1,0)$, which means that in the proof it suffices to consider only the case $\mu \geq 1$, evaluations for $\mu < 1$ being the same as for $\mu = 1$.

Lemma 2.5. ([3]) *Let* $\varrho_1(x) = x^\mu,\ \mu \in \mathbf{R}^1, 0 < \alpha < 1$. *Then*

$$|B_1(x,t)| \leq \left(\frac{x}{t}\right)^{\max(\mu-1,0)} \frac{(x-t)^\alpha}{t}, \tag{2.9}$$

$$|D_1(x,h,t)| \leq \left(\frac{x+h}{t}\right)^{\max(\mu-1,0)} \frac{h}{t(x+h-t)^{1-\alpha}}. \tag{2.10}$$

Similar estimates hold for $B_2(y,\tau)$ *and* $D_2(y,\eta,\tau)$ *with* $\varrho(y) = y^\nu$.

2.2. A one-dimensional statement

The following statement is known, being first proved in [9], see also the presentation of this proof in [11], p. 57; a shorter proof was given in [3]. Nevertheless we recall the scheme of the proof from [3] to make the presentation easier for the two-dimensional case.

THEOREM 2.6. *Let* $0 < \lambda < 1$, $\lambda + \alpha < 1$ *and* $\varrho(x) = x^{\mu}$. *The operator* $\left(I^{\alpha}_{a+} f\right)(x)$ *maps the space* $H^{\lambda}_{0}([0,b];\varrho)$ *into the space* $H^{\lambda+\alpha}_{0}([0,b];\varrho)$, *if* $\mu < \lambda + 1$.

P r o o f. Let $f \in H^{\lambda}_{0}([0,b];\varrho)$ and $\psi = \varrho(x)f(x)$, where $\psi(x) \in H^{\lambda}([0,b])$, $\psi(0) = 0$. We have

$$\left(\varrho I^{\alpha}_{0+} f\right)(x) = \left(I^{\alpha}_{0+}\psi\right)(x) + \left(J^{\alpha}_{a+}\psi\right)(x), \quad \left(J^{\alpha}_{0+}\psi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{0}^{x} B(x,t)\psi(t)dt.$$

Let $h > 0$ and $x, x+h \in [0,b]$. The estimation of $\left(I^{\alpha}_{0+}\psi\right)(x+h) - \left(I^{\alpha}_{0+}\psi\right)(x)$ is the same as in [11], pp. 54-55. For $J^{\alpha}_{0+}\psi$ we have

$$\left(J^{\alpha}_{0+}\psi\right)(x+h) - \left(J^{\alpha}_{0+}\psi\right)(x) = F_1(x,h) - F_2(x,h),$$

where

$$F_1(x,h) = \int_{x}^{x+h} B(x+h,t)\psi(t)dt, \quad F_2(x,h) = \int_{x}^{x+h} D_1(x,h,t)\psi(t)dt.$$

By Remark 2.4, it suffices to consider only the case $\mu \geq 1$. By (2.9) we have

$$|F_1| \leq C(x+h)^{\mu-1}\int_{x}^{x+h} \frac{(x+h-t)^{\alpha}}{t^{\mu-\lambda}}dt \leq Ch^{\alpha}(x+h)^{\mu-1}\int_{x}^{x+h} t^{\lambda-\mu}dt$$
$$\leq Ch^{\alpha+1}(x+h)^{\lambda-1} \leq Ch^{\alpha+\lambda}.$$

Making use of (2.10), we obtain

$$|F_2| \leq Ch(x+h)^{\mu-1}\int_{0}^{x} \frac{t^{\lambda-\mu}}{(x+h-t)^{1-\alpha}}dt$$
$$= Ch(x+h)^{\lambda+\alpha-1}\int_{0}^{\frac{x}{x+h}} \frac{\xi^{\lambda-\mu}d\xi}{(1-\xi)^{1-\alpha}} \leq Ch^{\alpha+\lambda}\int_{0}^{1} \frac{\xi^{\lambda-\mu}d\xi}{(1-\xi)^{1-\alpha}} = Ch^{\lambda+\alpha},$$

which completes the proof. ∎

3. Mapping properties of the mixed fractional integration operator in the mixed type Hölder spaces

LEMMA 3.1. *Let* $\varphi(x,y) \in H^{\lambda,\gamma}(Q)$, $0 \leq \lambda, \gamma \leq 1$, $0 < \alpha, \beta < 1$. *Then for the mixed fractional integral operator* (1.2) *the representation*

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y) = \frac{\varphi(0,0)x^\alpha y^\beta}{\Gamma(1+\alpha)\Gamma(1+\beta)} + \frac{\psi_1(x)y^\beta}{\Gamma(1+\beta)} + \frac{x^\alpha\psi_2(y)}{\Gamma(1+\alpha)} + \psi(x,y) \quad (3.1)$$

holds, where

$$\psi_1(x) = \frac{1}{\Gamma(\alpha)}\int\limits_0^x \frac{\varphi(t,0)-\varphi(0,0)}{(x-t)^{1-\alpha}}dt, \quad \psi_2(y) = \frac{1}{\Gamma(\beta)}\int\limits_c^y \frac{\varphi(0,\tau)-\varphi(0,0)}{(y-\tau)^{1-\beta}}d\tau,$$

$$\psi(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y \frac{\left(\Delta^{1,1}_{t,\tau}\varphi\right)(0,0)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau,$$

and

$$|\psi_1(x)| \le C_1 x^{\lambda+\alpha}, \qquad |\psi_2(y)| \le C_2 y^{\gamma+\beta}, \quad (3.2)$$

$$|\psi(x,y)| \le C \min_{\theta\in[0,1]} x^{\alpha+\theta\lambda}y^{\beta+(1-\theta)\gamma} = Cx^\alpha y^\beta \min\{x^\lambda, y^\gamma\}. \quad (3.3)$$

P r o o f. Representation (3.1) itself is easily obtained by means of (2.1). Since $\varphi \in H^{\lambda,\gamma}(Q)$, inequalities (3.2) are obvious. Estimate (3.3) is obtained by means of (2.4) and (2.6). ∎

THEOREM 3.2. *Let $0 \le \lambda, \gamma < 1$. The operator $I^{\alpha,\beta}_{0+,c+}$ is bounded from $H^{\lambda,\gamma}_0(Q)$ to $H^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\lambda+\alpha < 1$ and $\gamma+\beta < 1$.*

P r o o f. Since $\varphi(x,y) \in H^{\lambda,\gamma}_0(Q)$, by (3.1) we have

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y) = \psi(x,y).$$

We denote

$$g(t,\tau) = \left(\Delta^{1,1}_{t,\tau}\varphi\right)(0,0) \quad (3.4)$$

for brevity. Note that

$$\left(\Delta^{1,1}_{t,\tau}\varphi\right)(0,0) = \varphi(t,\tau)$$

for $\varphi \in H^{\lambda,\gamma}_0$, but we prefer to keep the notation for $g(t,\tau)$ via the mixed difference as in (3.4). By (2.4) we have

$$|g(t,\tau)| \le Ct^{\theta\lambda}\tau^{(1-\theta)\gamma} \le \min\{t^\lambda, \tau^\gamma\}. \quad (3.5)$$

For $h > 0$, $x, x+h \in Q_1 = [0,b]$, we consider the difference

$$\psi(x+h,y)-\psi(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left(\int\limits_x^h\int\limits_0^y\frac{g(x+t,y-\tau)}{(h-t)^{1-\alpha}\tau^{1-\beta}}dtd\tau\right.$$

$$\left.-\int\limits_0^x\int\limits_0^y\frac{g(x-t,y-\tau)}{t^{1-\alpha}\tau^{1-\beta}}dtd\tau\right)=\frac{(x+h)^\alpha-x^\alpha}{\Gamma(1+\alpha)\Gamma(\beta)}\int\limits_0^y\frac{g(x,y-\tau)}{\tau^{1-\beta}}d\tau$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^h\int\limits_0^y\frac{g(x+t,y-\tau)-g(x,y-\tau)}{(h-t)^{1-\alpha}\tau^{1-\beta}}dtd\tau$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y[g(x-t,y-\tau)-g(x,y-\tau)]\cdot\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]\tau^{\beta-1}dtd\tau$$

$$=\Delta_1+\Delta_2+\Delta_3. \tag{3.6}$$

We make use of (3.5) with $\theta=1$ and obtain

$$|\Delta_1|\le C\left|(x+h)^\alpha-x^\alpha\right|x^\lambda\le Ch^{\alpha+\lambda}.$$

For Δ_2 in view of (2.4), we have

$$|g(x-t,y-\tau)-g(x,y-\tau)|=\left|\left(\Delta^{1,1}_{-t,y-\tau}\varphi\right)(x,0)\right|\le C|t|^\lambda, \tag{3.7}$$

and then

$$\Delta_2\le Ch^\lambda.$$

For Δ_3 by (3.7) and (2.4) we obtain

$$\Delta_3\le C\int\limits_0^{x-a}t^\lambda|t^{\alpha-1}-(t+h)^{\alpha-1}|dt\le C_0h^{\alpha+\lambda},\ C_0=\int\limits_0^\infty t^\lambda|t^{\alpha-1}-(t+1)^{\alpha-1}|dt<\infty.$$

Gathering the estimates for $\Delta_1,\Delta_2,\Delta_3$ we obtain

$$|\psi(x+h,y)-\psi(x,y)|\le Ch^{\lambda+\alpha}.$$

Rearranging symmetrically representation (3.6), we can similarly obtain that

$$|\psi(x,y+h)-\psi(x,y)|\le Ch^{\gamma+\beta},$$

which proves the theorem. ■

THEOREM 3.3. *The mixed fractional integral operator* $I^{\alpha,\beta}_{0+,0+}$ *is bounded from the space* $\widetilde{H}_0^{\lambda,\gamma}(Q)$, $0 \le \lambda, \gamma \le 1$ *into the space* $\widetilde{H}_0^{\alpha+\lambda,\,\beta+\gamma}(Q)$, *if* $\lambda+\alpha,\ \gamma+\beta \le 1$.

P r o o f. Let $\varphi \in \widetilde{H}_0^{\lambda,\gamma}(Q)$. By Theorem 3.2 and embedding (2.5), for $f(x,y) = \left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)$ it satisfies to estimate the difference $\left(\Delta^{1,1}_{h,\eta}f\right)(x,y)$. Since $\varphi(x,y)|_{x=0,y=0} = 0$, according to (3.1) we have $f(x,y) = \psi(x,y)$, where $\psi(x,y)$ is the function from (3.1). The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losing the corresponding information in another variable. The suggested splitting runs as follows

$$\left(\Delta^{1,1}_{h,\eta}f\right)(x,y) = \left(\Delta^{1,1}_{h,\eta}\psi\right)(x,y) = \sum_{k=1}^{9} T_k \ :$$

$$= \frac{g(x,y)}{\Gamma(1+\alpha)\Gamma(1+\beta)}\left[(x+h)^\alpha - x^\alpha\right]\left[(y+\eta)^\beta - y^\beta\right] + \frac{(y+\eta)^\beta - y^\beta}{\Gamma(\alpha)\Gamma(1+\beta)}$$

$$\times \int_0^h \frac{g(x+t,y)-g(x,y)}{(h-t)^{1-\alpha}}dt + \frac{(x+h)^\alpha - x^\alpha}{\Gamma(1+\alpha)\Gamma(\beta)}\int_0^\eta \frac{g(x,y+\tau)-g(x,y)}{(\eta-\tau)^{1-\beta}}d\tau$$

$$+ \frac{(y+\eta)^\beta - y^\beta}{\Gamma(\alpha)\Gamma(1+\beta)}\int_0^x \left[g(x-t,y)-g(x,y)\right]\cdot\left[(t+h)^{\alpha-1} - t^{\alpha-1}\right]dt$$

$$+ \frac{(x+h)^\alpha - x^\alpha}{\Gamma(1+\alpha)\Gamma(\beta)}\int_0^y \left[g(x,y-\tau)-g(x,y)\right]\cdot\left[(\tau+\eta)^{\beta-1} - \tau^{\beta-1}\right]d\tau$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^h\int_0^\eta \frac{\left(\Delta^{1,1}_{t,\tau}g\right)(x,y)}{(h-t)^{1-\alpha}(\eta-\tau)^{1-\beta}}dtd\tau$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^h\int_0^y \frac{\left(\Delta^{1,1}_{t,-\tau}g\right)(x,y)}{(h-t)^{1-\alpha}}\left[(\tau+\eta)^{\beta-1} - \tau^{\beta-1}\right]dtd\tau$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^\eta \frac{\left(\Delta^{1,1}_{-t,\tau}g\right)(x,y)}{(\eta-\tau)^{1-\beta}}\left[(t+h)^{\alpha-1} - t^{\alpha-1}\right]dtd\tau$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y \left(\Delta^{1,1}_{-t,-\tau}g\right)(x,y)\left[(t+h)^{\alpha-1} - t^{\alpha-1}\right]\cdot\left[(\tau+\eta)^{\beta-1} - \tau^{\beta-1}\right]dtd\tau,$$

where $h > 0,\ \eta > 0;\ x,\ x+h \in [0,b];\ y, y+\eta \in [0,d]$ and $g(x,y)$ is the function from (3.4). The validity of this representation may be checked directly.

Since $\varphi \in \widetilde{H}^{\lambda,\mu}$, we have $|g(x,y)| = |\Delta_{x,y}\varphi(0,0)| \le Cx^\lambda y^\gamma$ and then

$$|T_1| \le Cx^\lambda y^\gamma \left|(x+h)^\alpha - x^\alpha\right| \left|(y+\eta)^\beta - y^\beta\right|,$$

$$|T_2| \le Cy^\gamma \left|(y+\eta)^\beta - y^\beta\right| \int\limits_0^h \frac{t^\lambda dt}{(h-t)^{1-\alpha}},$$

$$|T_3| \le Cx^\lambda \left|(x+h)^\alpha - x^\alpha\right| \int\limits_0^h \frac{\tau^\gamma d\tau}{(\eta-\tau)^{1-\beta}},$$

$$|T_4| \le Cy^\gamma \left|(y+\eta)^\beta - y^\beta\right| \int\limits_0^x t^\lambda \left|(h+t)^{\alpha-1} - t^{\alpha-1}\right| dt,$$

$$|T_5| \le Cx^\lambda \left|(x+h)^\alpha - x^\alpha\right| \int\limits_0^y \tau^\gamma \left|(\eta+\tau)^{\beta-1} - \tau^{\beta-1}\right| dt.$$

For $T_6 - T_9$ we similarly, make use of

$$\left|\left(\Delta^{1,1}_{-t,-\tau} g\right)(x,y)\right| = \left|\left(\Delta^{1,1}_{-t,-\tau}\varphi\right)(x,y)\right| \le c|t|^\lambda |\tau|^\gamma,$$

and obtain

$$|T_6| \le C \int\limits_0^h \int\limits_0^\eta \frac{t^\lambda \tau^\gamma dt d\tau}{(h-t)^{1-\alpha}(\eta-\tau)^{1-\beta}},$$

$$|T_7| \le C \int\limits_0^h \int\limits_0^y \frac{t^\lambda \tau^\gamma \left|(\eta+\tau)^{\beta-1} - \tau^{\beta-1}\right|}{(h-t)^{1-\alpha}} dt d\tau,$$

$$|T_8| \le C \int\limits_0^x \int\limits_0^\eta \frac{t^\lambda \tau^\gamma \left|(h+t)^{\alpha-1} - t^{\alpha-1}\right|}{(\eta-\tau)^{1-\beta}} dt d\tau,$$

$$|T_9| \le C \int\limits_0^x \int\limits_0^y t^\lambda \tau^\gamma \left|(h+t)^{\alpha-1} - t^{\alpha-1}\right| \left|(\eta+\tau)^{\beta-1} - \tau^{\beta-1}\right| dt d\tau,$$

after which every term is estimated in the standard way, and we get

$$\left|\left(\Delta^{1,1}_{h,\eta} f\right)(x,y)\right| \le C_3 h^{\lambda+\alpha} \eta^{\gamma+\beta}.$$

This completes the proof. ∎

4. Extension to the weighted case

In this section we give a generalization of Theorem 3.3 to the weighted case with the weight

$$\varrho(x,y) = x^{\mu}y^{\nu}, \qquad \mu < \lambda + 1,\ \nu < \gamma + 1 \tag{4.1}$$

THEOREM 4.1. *Let $\alpha, \beta \in (0,1)$, $\lambda, \gamma \in [0,1)$, $\lambda + \alpha < 1, \gamma + \beta < 1$ and ϱ be weight (4.1). Then the mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from the space $\widetilde{H}^{\lambda,\gamma}_0(\varrho)$ to $\widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0(\varrho)$.*

P r o o f. By Remark 2.4, it suffices to deal with the case $\mu, \nu \geq 1$. Let $\varphi \in \widetilde{H}^{\lambda,\gamma}_0(\varrho)$, so that $\varphi(x,y) = \frac{\varphi_0(x,y)}{\varrho(x,y)}$, where $\varphi_0(x,y) \in \widetilde{H}^{\lambda,\gamma}$ and $\varphi_0(x,y)|_{x=0,y=0} = 0$. For

$$\Phi(x,y) := \int\limits_a^x \int\limits_c^y \frac{\varrho(x,y)\varphi_0(t,\tau)dtd\tau}{\varrho(t,\tau)(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}$$

we have to show that $\Phi \in \widetilde{H}^{\lambda+\alpha,\gamma+\beta}_0$ and $\|\Phi\|_{\widetilde{H}^{\lambda+\alpha,\gamma+\beta}} \leq C\|\varphi_0\|_{\widetilde{H}^{\lambda,\gamma}}$. We represent $\Phi(x,y)$ in the form

$$\Phi(x,y) = \Phi_1(x,y) + \Phi_2(x,y)$$

$$:= \int\limits_a^x \int\limits_c^y \frac{\varphi_0(t,\tau)dtd\tau}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}} + \int\limits_a^x \int\limits_c^y B(x,y;t,\tau)\varphi_0(t,\tau)dtd\tau, \tag{4.2}$$

where notation (2.7) has been used. Here $\Phi_1(x,y) \in \widetilde{H}^{\lambda+\alpha,\beta+\gamma}_0(Q)$ by Theorem 3.3. To estimate the term $\Phi_2(x,y)$, we note that the weight being degenerate, we have

$$\varrho(x,y) - \varrho(t,\tau) = [\varrho(x) - \varrho(t)]\,[\varrho(y) - \varrho(\tau)] + \varrho(\tau)\,[\varrho(x) - \varrho(t)] + \varrho(t)\,[\varrho(y) - \varrho(\tau)],$$

which leads to the following representation

$$\Phi_2(x,y) = \int\limits_0^x \int\limits_0^y B_1(x,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau + \int\limits_0^x \int\limits_0^y B_1(x,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau$$

$$+ \int\limits_0^x \int\limits_0^y B_2(y,\tau)\frac{\varphi_0(t,\tau)}{(x-t)^{1-\alpha}}dtd\tau,$$

where the notation (2.8) has been used. For the difference $\left(\Delta^{1,0}_h \Phi_2\right)(x,y)$ with $h > 0$ and $x, x+h \in (0,b)$, we have

$$\left(\Delta_h^{1,0}\Phi_2\right)(x,y) = \int\limits_x^{x+h}\int\limits_0^y B_1(x+h,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_0^x\int\limits_0^y D(x,h,t)B_2(y,\tau)\varphi_0(t,\tau)dtd\tau + \int\limits_x^{x+h}\int\limits_0^y B_1(x+h,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau$$

$$+\int\limits_0^x\int\limits_0^y D_1(x,h,t)\frac{\varphi_0(t,\tau)}{(y-\tau)^{1-\beta}}dtd\tau + \int\limits_x^{x+h}\int\limits_0^y B_2(y,\tau)\frac{\varphi_0(t,\tau)dtd\tau}{(x+h-t)^{1-\alpha}}$$

$$+\int\limits_0^x\int\limits_0^y B_2(y,\tau)\left[(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right]\varphi_0(t,\tau)dtd\tau.$$

Since $\varphi \in \widetilde{H}_0^{\lambda,\gamma}$, we have

$$|\varphi_0(t,\tau)| \le Ct^\lambda\tau^\gamma, \quad |\varphi_0(t,\tau)-\varphi_0(x,0)| \le C(t-x)^\lambda\tau^\gamma\,,$$

and then

$$\left|\left(\Delta_h^{1,0}\Phi_2\right)(x,y)\right| \le C\Bigg\{\Bigg(\int\limits_x^{x+h}|B_1(x+h,t)|t^\lambda dt + \int\limits_0^x|D_1(x,h,t)|t^\lambda dt$$

$$+\int\limits_x^{x+h}\frac{(t-x)^\lambda dt}{(x+h-t)^{1-\alpha}} + \int\limits_0^x\left|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\right|(t-x)^\lambda dt\Bigg)y^{\nu-1}$$

$$\times\int\limits_0^y\frac{(y-\tau)^\beta}{\tau^{\nu-\gamma}}d\tau+\Bigg(\int\limits_x^{x+h}B_1(x+h,t)t^\lambda dt+\int\limits_0^x|D_1(x,h,t)t^\lambda dt\Bigg)\int\limits_0^y\frac{\tau^\gamma}{(y-\tau)^{1-\beta}}d\tau\Bigg\}.$$

Hence, by inequalities (2.9)-(2.10), via standard estimations can easily arrive at

$$\left|\left(\Delta_h^{1,0}\Phi_2\right)(x,y)\right| \le Ch^{\alpha+\lambda}.$$

The estimate

$$\left|\left(\Delta_h^{1,0}\Phi_2\right)(x,y)\right| \le C\eta^{\beta+\gamma}$$

is symmetrically obtained.

For the mixed difference with $\left(\Delta_{h,\eta}^{1,1}\Phi_2\right)(x,y)$ with $h,\eta>0,\ x,x+h\in[0,b],\ y,y+\eta\in[0,d]$ the appropriate representation leading to the separate evaluation in each variable without losses in another variable is as follows:

$$\left(\Delta^{1,1}_{h,\eta}\Phi_2\right)(x,y) = \int\limits_x^{x+h}\int\limits_y^{y+\eta} B_1(x+h,t)B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_a^x\int\limits_c^y D_1(x,h,t)D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau + \int\limits_x^{x+h}\int\limits_0^y B_1(x+h,t)D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_0^x\int\limits_y^{y+\eta} D_1(x,h,t)B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau + \int\limits_x^{x+h}\int\limits_y^{y+\eta}\frac{B_1(x+h,t)}{(y+\eta-\tau)^{1-\beta}}$$

$$\times\varphi_0(t,\tau)dtd\tau + \int\limits_x^{x+h}\int\limits_0^y B(x+h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_0^x\int\limits_y^{y+\eta} D_1(x,h,t)(y+\eta-\tau)^{\beta-1}\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_0^x\int\limits_0^y D_1(x,h,t)\left[(y+\eta-\tau)^{\beta-1}-(y-\tau)^{\beta-1}\right]\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_x^{x+h}\int\limits_y^{y+\eta}(x+h-t)^{\alpha-1}B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_0^x\int\limits_y^{y+\eta}\left[(x+h-t)^{\alpha-1}-x^{\alpha-1}\right]B_2(y+\eta,\tau)\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_x^{x+h}\int\limits_0^y(x+h-t)^{\alpha-1}D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau$$

$$+\int\limits_0^x\int\limits_0^y\left[(x+h-t)^{\alpha-1}-x^{\alpha-1}\right]D_2(y,\eta,\tau)\varphi_0(t,\tau)dtd\tau.$$

We omit the details of evaluation of each term in the above representation, it is standard via Lemma 2.5 and yields

$$\left|\left(\Delta^{1,1}_{h,\eta}\Phi_2\right)(x,y)\right| \le Ch^{\lambda+\alpha}\eta^{\gamma+\beta}.$$

Finally, it remains to note that

$$\Phi(x,0)=\Phi(0,y)\equiv 0, \text{ since } |\Phi(x,y)|\le Cx^{\lambda+\alpha}y^{\gamma+\beta}$$

under the conditions $\mu<\lambda+1, \nu<\gamma+1$. ■

References

[1] H.G. Hardy and J.E. Littlewood, Some properties of fractional integrals, I. *Math. Z.* **27**, No 4 (1928), 565-606.

[2] N.K. Karapetiants and N.G. Samko, Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω. *Fract. Calc. Appl. Anal.* **7**, No 4 (2004), 437-458.

[3] N.K. Karapetiants and L.D. Shankishvili, A short proof of Hardy-Littlewood-type theorem for fractional integrals in weighted Hölder spaces. *Fract. Calc. Appl. Anal.* **2**, No 2 (1999), 177-192.

[4] N.K. Karapetiants and L.D. Shankishvili, Fractional integro-differentiation of the complex order in generalized Holder spaces $H_0^\omega([0,1],\rho)$. *Integral Transforms Spec. Funct.* **13**, No 3 (2002), 199-209.

[5] N. K. Karapetyants, Kh. M. Murdaev, and A. Ya. Yakubov, The isomorphism realized by fractional integrals in generalized Hölder classes. *Dokl. Akad. Nauk SSSR* **314**, No 2 (1990), 288-291.

[6] N. K. Karapetyants, Kh. M. Murdaev, and A. Ya. Yakubov, On isomorphism provided by fractional integrals in generalized Nikolskiy classes. *Izv. Vuzov. Matematika* **(9)** (1992), 49-58.

[7] Kh. M. Murdaev and S.G. Samko, Mapping properties of fractional integro-differentiation in weighted generalized Hölder spaces $H_0^\omega(\rho)$ with the weight $\rho(x) = (x-a)^\mu(b-x)^\nu$ and given continuity modulus (Russian), *Deponierted in VINITI*, Moscow, 1986: No 3350-B, 25 p.

[8] Kh. M. Murdaev and S.G. Samko, Weighted estimates of continuity modulus of fractional integrals of function having a prescribed continuity modulus with weight (Russian). *Deponierted in VINITI*, Moscow, 1986: No 3351-B, 42 p.

[9] B.S. Rubin, Fractional integrals in Hölder spaces with weight, and operators of potential type. *Izv. Akad. Nauk Armjan. SSR Ser. Mat.* **9**, No 4 (1974), 308-324.

[10] B.S. Rubin, Fractional integrals and Riesz potentials with radial density in spaces with power weight. *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* **21**, No 5 (1986), 488-503, 511.

[11] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Gordon & Breach. Sci. Publ., N. York - London, 1993, 1012 pp. (Russian Ed. - *Fractional Integrals and Derivatives and Some of Their Applications*, Nauka i Tekhnika, Minsk, 1987.)

[12] S.G. Samko and Kh.M. Murdaev, Weighted Zygmund estimates for fractional differentiation and integration, and their applications. *Trudy Matem. Inst. Steklov* **180** (1987), 197-198; English transl. in: *Proc. Steklov Inst. Math. (AMS)* **1989**, Issue 3 (1989), 233-235.

[13] S.G. Samko and Z. Mussalaeva, Fractional type operators in weighted generalized Hölder spaces. *Proc. Georgian Acad. Sci., Math.* **1**, No 5 (1993), 601-626.

[14] B.G. Vakulov, Potential type operator on a sphere in generalized Hölder classes. *Izv. Vuzov. Matematika* **(11)** (1986), 66-69; English transl.: *Soviet Math. (Izv. VUZ)* **30**, No 11 (1986), 90-94.

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