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AN APPLICATION OF CONVOLUTION INTEGRAL

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We dedicate this paper to the 70th anniversary of Professor Srivastava

Abstract

Applying the Bernardi integral operator, an interesting convolution integral is introduced. The object of the present paper is to derive some convolution integral properties of functions $f(z)$ to be in the subclasses of the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ by making use of their coefficient inequalities.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$, if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < 1)$. Also a function $f(z) \in \mathcal{A}$ is said to be convex of order α in $\mathbb{U}$, if it satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some α $(0 \leqq \alpha < 1)$. The classes of starlike and convex functions $f(z)$ of order α are respectively denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, and $\mathcal{S}^*(0)\equiv\mathcal{S}^*$ and $\mathcal{K}(0)\equiv\mathcal{K}$. These classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [7].

It follows from the definitions for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ that $f(z)\in\mathcal{K}(\alpha)$ if and only if $zf'(z)\in\mathcal{S}^*(\alpha)$.

Silverman [10] showed the following coefficient inequalities for the the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

LEMMA 1.1. *If $f(z)\in\mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq 1-\alpha \tag{1.1}$$

for some α $(0\leqq\alpha<0)$, then $f(z)\in\mathcal{S}^*(\alpha)$.

LEMMA 1.2. *If $f(z)\in\mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n| \leqq 1-\alpha \tag{1.2}$$

for some α $(0\leqq\alpha<0)$, then $f(z)\in\mathcal{K}(\alpha)$.

We define the subclass $\mathcal{T}^*(\alpha)$ of $\mathcal{S}^*(\alpha)$ consisting of functions $f(z)$ which satisfy the coefficient inequality (1.1) and the subclass $\mathcal{C}(\alpha)$ of $\mathcal{K}(\alpha)$ consisting of functions $f(z)$ which satisfy the coefficient inequality (1.2).

For functions $f_j(z)\in\mathcal{A}$ $(j=1,2)$ given by

$$f_j(z)=z+\sum_{n=2}^{\infty}a_{n,j}z^n \qquad (z\in\mathbb{U}), \tag{1.3}$$

the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z+\sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n.$$

Furthermore, We also define the convolution integral of $f_1(z)$ and $f_2(z)$ below:

$$(f_1 \circledast f_2)(z)=\int_0^z\frac{(f_1*f_2)(t)}{t}dt=z+\sum_{n=2}^{\infty}\frac{a_{n,1}a_{n,2}}{n}z^n.$$

This convolution integral was studied by Duren [3].

In the present paper, we aim at presenting some interesting application of convolution integral.

2. An application of convolution integral

For functions $f_j(z) \in \mathcal{A}$ $(j = 1, 2, \cdots, m)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (z \in \mathbb{U}), \tag{2.1}$$

Bernardi [2] considered the integral operator

$$B_j(z) = \frac{1 + c_j}{z^{c_j}} \int_0^z t^{c_j - 1} f(t) dt \qquad (c_j > -1).$$

Libera [4] studied the above operator for $c_j = 1$, and Alexander [1] studied it for $c_j = 0$.

Using the Bernardi integral operator, we consider a new application of the convolution integral operator as follows:

$$(B_1 * B_2)(z) = \frac{1 + c_1}{z^{c_1}} \int_0^z t^{c_1 - 1} f(t) dt * \frac{1 + c_2}{z^{c_2}} \int_0^z t^{c_2 - 1} f(t) dt$$

$$= z + \sum_{n=2}^{\infty} \frac{(1 + c_1)(1 + c_2)}{(n + c_1)(n + c_2)} a_{n,1} a_{n,2} z^n.$$

Hence, we see the application of convolution integral of $f_1(z)$, $f_2(z)$, $\cdots$ and $f_m(z)$ below:

$$(B_1 * \cdots * B_m)(z) = \frac{1 + c_1}{z^{c_1}} \int_0^z t^{c_1 - 1} f(t) dt * \cdots * \frac{1 + c_m}{z^{c_m}} \int_0^z t^{c_m - 1} f(t) dt. \tag{2.2}$$

For functions $f_j(z) \in \mathcal{A}$ $(j = 1, 2, \cdots, m)$ given by the equality (2.1), the familiar Hölder-type inequality assumes the form

$$\sum_{n=2}^{\infty} \left(\prod_{j=1}^{m} |a_{n,j}| \right) \leqq \prod_{j=1}^{m} \left(\sum_{n=2}^{\infty} |a_{n,j}|^{p_j} \right)^{\frac{1}{p_j}}, \tag{2.3}$$

where $p_j > 1$ $(j = 1, 2, 3, \cdots, m)$ and $\sum\limits_{j=1}^{m} \dfrac{1}{p_j} \geqq 1$.

Our result for the application of convolution integral defined by (2.2) is contained in next theorem.

THEOREM 2.1. *If* $f_j(z) \in \mathcal{T}^*(\alpha_j)$ *for each* $j = 1, 2, \cdots, m$, *then* $(B_1 * \cdots * B_m)(z) \in \mathcal{T}^*(\beta)$ *with*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{\prod_{j=1}^{m}(2-\alpha_j)(2+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}.$$

P r o o f. For $f_j(z) \in \mathcal{T}^*(\alpha_j)$, Lemma 1.1 gives us that

$$\sum_{n=2}^{\infty} \frac{n-\alpha_j}{1-\alpha_j}|a_{n,j}| \leqq 1 \qquad (\, j = 1, 2, 3, \cdots, m)$$

which implies

$$\left(\sum_{n=2}^{\infty} \frac{n-\alpha_j}{1-\alpha_j}|a_{n,j}| \right)^{\frac{1}{m}} \leqq 1.$$

Applying the Hölder-type inequality (2.3), we arrive the following inequality

$$\sum_{n=2}^{\infty} \left\{ \prod_{j=1}^{m} \left(\frac{n-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{m}} |a_{n,j}|^{\frac{1}{m}} \right\} \leqq \left(\sum_{n=2}^{\infty} \frac{n-\alpha_j}{1-\alpha_j}|a_{n,j}| \right)^{\frac{1}{m}} \leqq 1.$$

Then we have to determine the largest β such that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} \prod_{j=1}^{m} \frac{1+c_j}{n+c_j}|a_{n,j}| \leqq 1,$$

that is,

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} \prod_{j=1}^{m} \frac{1+c_j}{n+c_j}|a_{n,j}| \leqq \sum_{n=2}^{\infty} \left\{ \prod_{j=1}^{m} \left(\frac{n-\alpha_j}{1-\alpha_j} \right) \right\}^{\frac{1}{m}} |a_{n,j}|^{\frac{1}{m}}.$$

Therefore, we need to find the largest β such that

$$\frac{n-\beta}{1-\beta} \prod_{j=1}^{m} \frac{1+c_j}{n+c_j}|a_{n,j}| \leqq \prod_{j=1}^{m} \left(\frac{n-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{m}} |a_{n,j}|^{\frac{1}{m}},$$

which is equivalent to

$$\frac{n-\beta}{1-\beta} \prod_{j=1}^{m} \frac{1+c_j}{n+c_j}|a_{n,j}|^{1-\frac{1}{m}} \leqq \prod_{j=1}^{m} \left(\frac{n-\alpha_j}{1-\alpha_j} \right)^{\frac{1}{m}} \tag{2.4}$$

for all n $(n = 2, 3, 4, \cdots)$. Since

$$\prod_{j=1}^{m} \left(\frac{n-\alpha_j}{1-\alpha_j} \right)^{1-\frac{1}{m}} |a_{n,j}|^{1-\frac{1}{m}} \leqq 1,$$

we see that

$$\prod_{j=1}^{m} |a_{n,j}|^{1-\frac{1}{m}} \leqq \frac{1}{\prod_{j=1}^{m} \left(\frac{n-\alpha_j}{1-\alpha_j} \right)^{1-\frac{1}{m}}}. \tag{2.5}$$

From the inequalities (2.4) and (2.5),

$$\frac{n-\beta}{1-\beta} \prod_{j=1}^{m} \frac{1+c_j}{n+c_j} \leqq \prod_{j=1}^{m} \frac{n-\alpha_j}{1-\alpha_j},$$

so that we find for β that

$$\beta \leqq 1 - \frac{(n-1)\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{\prod_{j=1}^{m}(n-\alpha_j)(n+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)} \tag{2.6}$$

Let $F(n)$ be the right hand side of the last inequality (2.6). Further, let us define $G(n)$ by the numerator of $F'(n)$. Then $G(n)$ gives us that

$$G(n) = -\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)\left\{\prod_{j=1}^{m}(n-\alpha_j)(n+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)\right\}$$

$$+(n-1)\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)\,\{(n-\alpha_2)(n-\alpha_3)\cdots(n-\alpha_m)(n+c_1)\cdots(n+c_m)$$

$$+(n-\alpha_1)(n-\alpha_3)\cdots(n-\alpha_m)(n+c_1)\cdots(n+c_m)$$

$$\vdots$$

$$+(n-\alpha_1)\cdots(n-\alpha_{m-1})(n+c_1)\cdots(n+c_m)$$

$$+(n-\alpha_1)\cdots(n-\alpha_m)(n+c_2)\cdots(n+c_m)$$

$$\vdots$$

$$+(n-\alpha_1)\cdots(n-\alpha_m)(n+c_1)\cdots(n+c_{m-1})\}$$

$$= -\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)\left\{\prod_{j=1}^{m}(n-\alpha_j)(n+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)\right\}$$

$$+(n-1)\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)\left\{\sum_{j=1}^{m}\frac{\prod_{j=1}^{m}(n-\alpha_j)(n+c_j)}{n-\alpha_j}\right.$$

$$+\sum_{j=1}^{m}\frac{\prod_{j=1}^{m}(n-\alpha_j)(n+c_j)}{n+c_j}\Bigg\}+\prod_{j=1}^{m}(1-\alpha_j)^2(1+c_j)^2$$

$$=\prod_{j=1}^{m}\{(1-\alpha_j)(1+c_j)(n-\alpha_j)(n+c_j)\}\ \times$$

$$\left\{(n-1)\sum_{j=1}^{m}\frac{1}{n-\alpha_j}+(n-1)\sum_{j=1}^{m}\frac{1}{n+c_j}-1\right\}+\prod_{j=1}^{m}(1-\alpha_j)^2(1+c_j)^2\geqq 0.$$

Thus we show that $F(n)$ is increasing function for integers n ($n=2,3,4,\cdots$). This means that

$$\beta=F(2)=1-\frac{\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{\prod_{j=1}^{m}(2-\alpha_j)(2+c_j)-\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}.$$

Therefore, we complete the proof of the theorem. ■

Setting $c_j=1$ in Theorem 2.1, we have

Corollary 2.1. *If $f_j(z)\in\mathcal{T}^*(\alpha_j)$ for each $j=1,2,\cdots,m$, then $(B_1*\cdots*B_m)(z)\in\mathcal{T}^*(\beta)$ with*

$$\beta=1-\frac{2^m\prod_{j=1}^{m}(1-\alpha_j)}{3^m\prod_{j=1}^{m}(2-\alpha_j)-2^m\prod_{j=1}^{m}(1-\alpha_j)}.$$

Furthermore, setting $c_j=0$ in Theorem 2.1, we obtain the next result.

Corollary 2.2. *If $f_j(z)\in\mathcal{T}^*(\alpha_j)$ for each $j=1,2,\cdots,m$, then $(B_1*\cdots*B_m)(z)\in\mathcal{T}^*(\beta)$ with*

$$\beta=1-\frac{\prod_{j=1}^{m}(1-\alpha_j)}{2^m\prod_{j=1}^{m}(2-\alpha_j)-\prod_{j=1}^{m}(1-\alpha_j)}.$$

Below, we derive the application of convolution integral for the class $\mathcal{C}(\alpha)$.

THEOREM 2.2. *If $f_j(z) \in \mathcal{C}(\alpha_j)$ for each $j = 1, 2, \cdots, m$, then $(B_1 * \cdots * B_m)(z) \in \mathcal{C}(\beta)$ with*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{2^{m-1}\prod_{j=1}^{m}(2-\alpha_j)(2+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}.$$

P r o o f. In view of the proof of Theorem 2.1, we obtain

$$\beta \leqq 1 - \frac{(n-1)\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{n^{m-1}\prod_{j=1}^{m}(n-\alpha_j)(n+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}, \tag{2.7}$$

and it is easily to show that the the right hand side of the inequality (2.7) is increasing function for all n ($n = 2, 3, 4, \cdots$). Thus we have completed the proof of the theorem. ■

If we take $c_j = 1$ in Theorem 2.2, we deduce

COROLLARY 2.3. *If $f_j(z) \in \mathcal{C}(\alpha_j)$ for each $j = 1, 2, \cdots, m$, then $(B_1 * \cdots * B_m)(z) \in \mathcal{C}(\beta)$ with*

$$\beta = 1 - \frac{2^m\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{2^{m-1}3^m\prod_{j=1}^{m}(2-\alpha_j)(2+c_j) - 2^m\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}.$$

Finally, by taking $c_j = 0$ in Theorem 2.2, we derive the following

COROLLARY 2.4. *If $f_j(z) \in \mathcal{C}(\alpha_j)$ for each $j = 1, 2, \cdots, m$, then $(B_1 * \cdots * B_m)(z) \in \mathcal{C}(\beta)$ with*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}{2^{2m-1}\prod_{j=1}^{m}(2-\alpha_j)(2+c_j) - \prod_{j=1}^{m}(1-\alpha_j)(1+c_j)}.$$

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