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EXPLICIT SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEMS, CONTAINING BITSADZE-SAMARSKII CONSTRAINTS

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This paper is dedicated to the 70th anniversary of Professor Srivastava

Abstract

In this paper are found explicit solutions of four nonlocal boundary value problems for Laplace, heat and wave equations, with Bitsadze-Samarskii constraints based on non-classical one-dimensional convolutions. In fact, each explicit solution may be considered as a way for effective summation of a solution in the form of nonharmonic Fourier sine-expansion. Each explicit solution, may be used for numerical calculation of the solutions too.

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1. Introduction

Usually, the name of Bitsadze-Samarskii problem is associated with the following nonlocal BVP:

PROBLEM 1.

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = f(x), \quad u(x,1) = 0, \qquad (1)$$
$$u(0,y) = 0, \quad u(1,y) = u(c,y)$$

on the square $G = [0,1] \times [0,1]$ for a given $0 < c < 1$ (see [1] where $c = \frac{1}{2}$). Nevertheless, the Bitsadze-Samarskii condition $u(1,y) = u(c,y)$ may be consider in a larger context, not necessarily connected with the Laplace equation, but with other types of equations. Here, along with the original Bitsadze-Samarskii problem, we will consider also BVPs for the heat equation and for the equation of a vibrating string, when one of the BVCs is of the form $u(1,t) = u(c,t)$ with $0 < c < 1$.

2. Nonlocal boundary value problems, containing Bitsadze-Samarskii constraints

Further, along with the original Bitsadze-Samarskii problem (1), we consider also the problems:

Problem 2.

$$\begin{gathered} u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty, \\ u(x,0) = f(x), \quad u(0,t) = 0, \quad u(1,t) = u(c,t), \quad 0 \le x \le 1, \ 0 \le t, \end{gathered} \tag{2}$$

with $0 < c < 1$,

Problem 3.

$$\begin{gathered} u_{tt} = u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty, \\ u(x,0) = f(x), \quad u(0,t) = 0, \\ u_t(x,0) = 0, \quad u(1,t) = u(c,t), \quad 0 \le x \le 1, \ 0 \le t, \end{gathered} \tag{3}$$

with $0 < c < 1$,

and

Problem 4.

$$\begin{gathered} u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \\ u(x,0) = f(x), \quad u(x,b) = u(x,d), \quad 0 \le x \le a, \\ u(0,y) = 0, \quad u(a,y) = u(c,y), \quad 0 \le y \le b, \end{gathered} \tag{4}$$

with $0 < c < a$ and $0 < d < b$. Here we have two nonlocal conditions on x and y, respectively.

Our aim is to find explicit solutions of all four problems (1)-(4).

3. One-dimensional Bitsadze-Samarskii spectral problem

We start with the following one-dimensional elementary boundary value problem on $[0,1]$:

$$y'' + \lambda^2 y = f(x), \quad y(0) = 0, \quad y(1) - y(c) = 0, \quad 0 < c < 1.$$

Its solution $y = R_{-\lambda^2} f(x)$ determines the resolvent operator of $\frac{d^2}{dx^2}$ with boundary condition $y(0) = 0$ and $y(1) - y(c) = 0$. It has the form

$$R_{-\lambda^2} f(x) = \frac{1}{\lambda} \int_0^x \sin \lambda (x - \xi) f(\xi) d\xi$$

$$- \frac{\sin \lambda x}{\lambda E(\lambda)} \left(\int_0^1 \sin \lambda (1 - \xi) f(\xi) d\xi - \int_0^c \sin \lambda (c - \xi) f(\xi) d\xi \right),$$

with $E(\lambda) = \frac{\sin \lambda - \sin c\lambda}{\lambda}$.

$R_{-\lambda^2} f(x)$ is determined for all $\lambda \in \mathbb{C}$, except for the zeros of $E(\lambda)$ (the eigenvalues). This resolvent operator is defined for $\lambda = 0$ too, since $\lambda = 0$ is not a zero of $E(\lambda)$. Denoting $L_x f(x) = R_0 f(x)$, we have

$$L_x u(x,y) = \int_0^x (x - \xi) u(\xi, y) d\xi$$

$$- \frac{x}{1-c} \left(\int_0^1 (1 - \xi) u(\xi, y) d\xi - \int_0^c (c - \xi) u(\xi, y) d\xi \right).$$

The zeros of $E(\lambda)$ are $\lambda_n = \frac{(2n-1)\pi}{1+c}$ and $\mu_k = \frac{2k\pi}{1-c}$, $n, k \in \mathbb{N}$. There arise two cases :

1) The arithmetic progressions (λ_n) and (μ_k) have no common terms. This happens when, e.g. c is an irrational number;

2) For some rational c it may happen some λ_n to be equal to some μ_k, i.e. to exist dispersion relations of the form $\lambda_n = \mu_k$. For example, such a the cases $c = \frac{1}{5}$ and $c = \frac{3}{7}$.

4. The spectral projectors and their totality

1. Let all the eigenvalues be simple. Then the spectral Riesz' projectors ([3], p. 165) for λ_n are

$$P_{\lambda_n}\{f\} = \frac{1}{\pi i} \int_{\Gamma_n} R_{-\lambda^2} f(x) \lambda d\lambda =$$

$$\frac{4}{\cos\lambda_n - c\cos c\lambda_n}\left(\int_0^1 f(\xi)\sin\lambda_n(1-\xi)d\xi - \int_0^c f(\xi)\sin\lambda_n(c-\xi)d\xi\right)\sin\lambda_n x.$$

Here Γ_n is a contour in $\mathbb{C}$, containing only the zero λ_n of $E(\lambda)$. The same form have the projectors for the zeros μ_k:

$$P_{\mu_k}\{f\} = \frac{4}{\cos\mu_k - c\cos c\mu_k}$$
$$\times\left(\int_0^1 f(\xi)\sin\mu_k(1-\xi)d\xi - \int_0^c f(\xi)\sin\mu_k(c-\xi)d\xi\right)\sin\mu_k x.$$

2. If $\lambda_n = \mu_k$, then $E(\lambda_n) = 0$, $E'(\lambda_n) = 0$ but $E''(\lambda_n) \neq 0$. Indeed, assume that $E''(\lambda_n)=0$. From the last equality it follows $\sin\lambda_n = \sin c\lambda_n = 0$. Therefore, $\cos\lambda_n = (-1)^p$, $\cos c\lambda_n = (-1)^q$ where $p, q \in \mathbb{N}$ but $0 < c < 1$, and thus we find that $E'(\lambda_n) \neq 0$, which is a contradiction. In this case the eigenspace of λ_n is two-dimensional and the spectral projector is

$$P_{\lambda_n}\{f\} = C_n\left(\int_0^1 f(\xi)\sin\lambda_n(1-\xi)d\xi - \int_0^c f(\xi)\sin\lambda_n(c-\xi)d\xi\right) x\cos\lambda_n x$$
$$+\left[C_n\left(\int_0^1 (1-\xi)f(\xi)\cos\lambda_n(1-\xi)d\xi - \int_0^c (c-\xi)f(\xi)\cos\lambda_n(c-\xi)d\xi\right)\right.$$
$$\left.+\frac{G_n - C_n}{\lambda_n}\left(\int_0^1 f(\xi)\sin\lambda_n(1-\xi)d\xi - \int_0^c f(\xi)\sin\lambda_n(c-\xi)d\xi\right)\right]\sin\lambda_n x,$$

where

$$C_n = \frac{4}{(1-c^2)\sin\lambda_n},\quad G_n = \frac{4(\lambda_n\cos\lambda_n - c^3\lambda_n\cos\lambda_n c - 3(1-c^2)\sin\lambda_n)}{3(1-c^2)^2\sin^2\lambda_n}.$$

In both cases, the projectors $P_{\lambda_n}\{f(x)\}$ and $P_{\mu_k}\{f(x)\}$, considered together, form a total system of projectors, i.e. such that if $P_{\lambda_n}\{f\} = 0$ for all $n \in \mathbb{N}$ and $P_{\mu_k}\{f\} = 0$ for all $k \in \mathbb{N}$, then $f \equiv 0$ (see Bozhinov [2]).

4. Weak solutions of BVPs (1)-(4)

We introduce the notion of a weak solution of problems (1)-(4). In order to give an exact meaning of this notion, we introduce some notations. Let us consider BVP (4) in the domain $D = [0,a] \times [0,b]$ and denote

$$L_x u(x,y) = \int_0^x (x-\xi)u(\xi,y)d\xi$$

$$
\begin{aligned}
&\quad - \frac{x}{a-c}\left(\int_0^a (a-\xi)u(\xi,y)d\xi - \int_0^c (c-\xi)u(\xi,y)d\xi\right),\\
L_y u(x,y) &= \int_0^y (y-\eta)u(x,\eta)d\eta\\
&\quad - \frac{x}{b-d}\left(\int_0^b (b-\eta)u(x,\eta)d\eta - \int_0^d (d-\eta)u(x,\eta)d\eta\right).
\end{aligned}
$$

Definition 1. **a)** A function $u(x,y) \in C([0,1]\times[0,1])$ is said to be a weak solution of Bitsadze - Samarskii problem (1), iff $u(x,y)$ satisfies the integral equation (see [4])

$$(L_x + L_y)u = (1-y)L_x f(x). \tag{5}$$

b) A function $u(x,y) \in C([0,a]\times[0,b])$ is said to be a weak solution of problem (4), iff $u(x,y)$ satisfies the integral equation

$$(L_x + L_y)u = L_x f(x). \tag{6}$$

In order to define the notation of a weak solution of problems (2) and (3), we introduce the integration operator

$$lu(x,t) = \int_0^t u(x,\tau)d\tau.$$

Definition 2. A function $u(x,t) \in C([0,1]\times[0,\infty))$ is said to be a weak solution of problem (2) or (3), iff $u(x,t)$ satisfies the integral equation

$$\text{a)} \quad (L_x - l)u = L_x f(x), \quad \text{for problem (2)}, \tag{7}$$

or

$$\text{b)} \quad (L_x - l^2)u = L_x f(x), \quad \text{for problem (3)}. \tag{8}$$

Formally, (5) and (6) can be obtained from the equation $u_{xx} + u_{yy} = 0$ by applying the operator $L_x L_y$. Equation (7) is obtained from $u_t = u_{xx}$ by application the operator $l\ L_x$ to $u_t = u_{xx}$ and (8) by application of $l^2\ L_x$ to the equation $u_{tt} = u_{xx}$. We use also the corresponding boundary value conditions of (1)-(4).

Lemma 1. [4] *If $u(x,y) \in C([0,1]\times[0,1])$ satisfies (5), then $u(x,y)$ satisfies the boundary value conditions*

$$u(x,0) = f(x), (x,1) = 0, \qquad u(0,y) = 0, u(1,y) = u(c,y).$$

Analogical assertions are also true for integral relations (6), (7) and (8) and the corresponding initials and boundary conditions of the problems (2), (3) and (4).

5. A convolution

As a special case of a convolution considered in Dimovski [3], p. 119, it may be written explicitly a convolution $f{*}g$ in $C[0,1]$ such that $R_{-\lambda^2}\{f(x)\} = \left\{\frac{\sin\lambda x}{E(\lambda)}\right\} * f$. It has the form

$$(f*g)(x) = -\int_c^1 h(x,\eta)d\eta, \tag{9}$$

with

$$h(x,\eta) = \int_x^\eta f(x+\eta-\xi)g(\xi)d\xi \tag{10}$$
$$-\int_{-x}^\eta f(|\eta-x-\xi|)g(|\xi|)\ \mathrm{sgn}\ \xi(\eta-x-\xi)d\xi.$$

It is a bilinear, associative and commutative operation in $C[0,1]$.

LEMMA 2. ([2]) *If* $f,g \in C[0,1]$, *then* $f*g \in C^1[0,1]$.

P r o o f. It is ease to see, that $\frac{\partial h(x,\eta)}{\partial x} = \frac{\partial k(x,\eta)}{\partial \eta}$, where

$$k(x,\eta) = \int_x^\eta f(x+\eta-\xi)g(\xi)d\xi$$
$$+\int_{-x}^\eta f(|\eta-x-\xi|)g(|\xi|)\ \mathrm{sgn}\ \xi(\eta-x-\xi)d\xi.$$

Hence

$$\begin{aligned}\frac{d}{dx}((f*g)(x)) &= k(x,c)-k(x,1) \\ &= \int_x^c f(x+c-\xi)g(\xi)d\xi \\ &+ \int_{-x}^c f(|c-x-\xi|)g(|\xi|)\ \mathrm{sgn}\ \xi(c-x-\xi)d\xi \\ &- \int_x^1 f(x+1-\xi)g(\xi)d\xi \\ &+ \int_{-x}^1 f(|1-x-\xi|)g(|\xi|)\ \mathrm{sgn}\ \xi(1-x-\xi)d\xi.\end{aligned} \tag{11}$$

□

Let $\Omega(x,y)$ denote the solution of (1) and $\Omega(x,t)$ be the solution of (2) or (3) for

$$f(x) = L_x\{x\} = \frac{x^3}{6} - \frac{1+c+c^2}{6}x, \tag{12}$$

Then, we can represent the solutions of the problems (1)(see [4]), (2) and (3) by one and the same formula:

$$u = \frac{\partial^4}{\partial x^4}(\Omega * f), \tag{13}$$

where Ω is the solution of the corresponding problem for (12). The same is true for BVP (4) too, but with slightly changed convolution, instead of (9). If $f, g \in C[0,a]$, then the convolution is:

$$(f*g)(x) = -\int_c^a h(x,\eta)d\eta, \tag{14}$$

with $h(x,\eta)$, given by (12). In this case $\Omega(x,y)$ is the solution of (4) for

$$f(x) = L_x\{x\} = \frac{x^3}{6} - \frac{a^2+ac+c^2}{6}x. \tag{15}$$

6. Special solutions $\Omega(x,y)$ and $\Omega(x,t)$

6.1. The case when all the zeros are simple

LEMMA 3. *If all eigenvalues λ_n and μ_m, $m,n \in \mathbb{N}$ are simple, then*

$$\begin{aligned}\Omega(x,y) = 2(1-c)\sum_{n=1}^{\infty} \frac{\sinh \lambda_n(1-y)}{\lambda_n^3(\cos\lambda_n - c\ \cos c\ \lambda_n)\sinh\lambda_n}\sin\lambda_n x \\ + 2(1-c)\sum_{n=1}^{\infty} \frac{\sinh \mu_n(1-y)}{\mu_n^3(\cos\mu_n - c\ \cos c\ \mu_n)\sinh\mu_n}\sin\mu_n x\end{aligned}$$

is a weak solution of Problem 1 for $f(x) = \frac{x^3}{6} - \frac{1+c+c^2}{6}x$.

P r o o f. See [4].

Lemma 4. *If all eigenvalues* λ_n *and* μ_m, $m, n \in \mathbb{N}$ *are simple, then*

$$\Omega(x,t) = 2(1-c)\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t}}{\lambda_n^3(\cos\lambda_n - c\ \cos c\ \lambda_n)}\sin\lambda_n x + 2(1-c)\sum_{n=1}^{\infty}\frac{e^{-\mu_n^2 t}}{\mu_n^3(\cos\mu_n - c\ \cos c\ \mu_n)}\sin\mu_n x$$

is a weak solution of Problem 2 for $f(x) = \frac{x^3}{6} - \frac{1+c+c^2}{6}x$.

P r o o f. Substituting $\Omega(x,t)$ in (7) we verify the satisfying the equation (7) termwise. We should verify that the left-hand side of (7) is equal to $L_x x = f(x) = \frac{x^3}{6} - \frac{1+c+c^2}{6}x$. But $P_{\lambda_n}\{\Omega(x,0)\} = P_{\lambda_n}\{f(x)\}$ and $P_{\mu_k}\{\Omega(x,0)\} = P_{\mu_k}\{f(x)\}$ for $x \in [0,a]$. Since $a \in \sup\Phi$ according to a theorem of N. Bozhinov [2] we obtain that $\Omega(x,0) = f(x)$. □

Lemma 5. *If all eigenvalues* λ_n *and* μ_m, $m, n \in \mathbb{N}$ *are simple, then*

$$\Omega(x,t) = 2(1-c)\sum_{n=1}^{\infty}\frac{\cos\lambda_n t}{\lambda_n^3(\cos\lambda_n - c\ \cos c\ \lambda_n)}\sin\lambda_n x + 2(1-c)\sum_{n=1}^{\infty}\frac{\cos\mu_n t}{\mu_n^3(\cos\mu_n - c\ \cos c\ \mu_n)}\sin\mu_n x$$

is a weak solution of Problem 3 for $f(x) = \frac{x^3}{6} - \frac{1+c+c^2}{6}x$.

P r o o f. Analogical as in Lemma 4.

Lemma 6. *If all eigenvalues* λ_n *and* μ_m, $m, n \in \mathbb{N}$ *are simple, then*

$$\Omega(x,y) = 2(a-c)\sum_{n=1}^{\infty}\frac{\cosh\frac{1}{2}(b+d-2y)\lambda_n}{\lambda_n^3(a\cos a\lambda_n - c\cos c\lambda_n)\cosh\frac{1}{2}(b+d)\lambda_n}\sin\lambda_n x + 2(a-c)\sum_{n=1}^{\infty}\frac{\cosh\frac{1}{2}(b+d-2y)\mu_n}{\mu_n^3(a\cos a\mu_n - c\cos c\mu_n)\cosh\frac{1}{2}(b+d)\mu_n}\sin\mu_n x$$

is a weak solution of Problem 4 for $f(x) = \frac{x^3}{6} - \frac{a^2+ac+c^2}{6}x$.

P r o o f. Analogical as in Lemma 4.

6.2. The case of roots of multiplicity two

For definiteness, let we consider all the problems (1)-(3) for $c = \frac{1}{5}$ and for the problems (4) $a = 1$, $b = 1$ and $c = \frac{1}{5}$. In this case we find the zeros $\lambda_n = \frac{5}{6}(2n-1)\pi$ and $\mu_k = \frac{5k\pi}{2}$, $n, k \in \mathbb{N}$, where $\mu_{2k} = 5k\pi$ and $\lambda'_1 = \lambda_1, \lambda'_2 = \lambda_3$, $\lambda'_3 = \lambda_4$, $\lambda'_4 = \lambda_6$, $\lambda'_5 = \lambda_7$, $\lambda'_6 = \lambda_9$, $\lambda'_6 = \lambda_{10}, ...$ are the sequences of the simple roots and $\mu_{2k-1} = \frac{5(2k-1)}{2}k\pi$ is the sequence of the double roots.

LEMMA 7. *If* $c = \frac{1}{5}$ *and* $f(x) = \frac{x^3}{6} - \frac{31}{150}$, *then the weak solution of Problem 1 is:*

$$\Omega(x,y) = 2(1-c)\sum_{n=1}^{\infty} \frac{\sinh \lambda'_n(1-y)}{\lambda'^3_n(\cos\lambda'_n - c\ \cos c\ \lambda'_n)\sinh\lambda'_n}\sin\lambda'_n x$$

$$+ 2(1-c)\sum_{k=1}^{\infty} \frac{\sinh \mu_{2k}(1-y)}{\mu^3_{2k}(\cos\mu_{2k} - c\ \cos c\ \mu_{2k})\sinh\mu_{2k}}\sin\mu_{2k} x$$

$$+ \sum_{m=1}^{\infty}\Bigg\{ \frac{-4\sinh\mu_{2k-1}(1-y)}{(1+c)\mu^3_{2k-1}\sin\mu_{2k-1}\sinh\mu_{2k-1}}\ x\cos\mu_{2k-1}x$$

$$+ \frac{4e^{-\mu_{2k-1}y}}{3\mu^4_{2k-1}(1+c)(e^{2\mu_{2k-1}}-1)^2\sin\mu_{2k-1}}\Bigg[3\Big(e^{2\mu_{2k-1}(1+y)}((y-2)\mu_{2k-1}-3)$$

$$- e^{2\mu_{2k-1}}(3+(y-2)\mu_{2k-1}) + e^{2\mu_{2k-1}y}(3-\mu_{2k-1}y) + e^{4\mu_{2k-1}}(3+\mu_{2k-1}y)\Big)$$

$$+ 4e^{\mu_{2k-1}(2+y)}\mu_{2k-1}\sinh\mu_{2k-1}\sinh\mu_{2k-1}(1-y)\cot\mu_{2k-1}\Bigg]\sin\mu_{2k-1}x\Bigg\}.$$

P r o o f. Analogical as in Lemma 4.

LEMMA 8. *If* $c = \frac{1}{5}$ *and* $f(x) = \frac{x^3}{6} - \frac{31}{150}$, *then the weak solution of Problem 2 is:*

$$\Omega(x,t) = 2(1-c)\sum_{n=1}^{\infty} \frac{e^{-\lambda'^2_n t}}{\lambda'^3_n(\cos\lambda'_n - c\ \cos c\ \lambda'_n)}\sin\lambda'_n x$$

$$+ 2(1-c)\sum_{k=1}^{\infty} \frac{e^{-\mu^2_{2k} t}}{\mu^3_{2k}(\cos\mu_{2k} - c\ \cos c\ \mu_{2k})}\sin\mu_{2k} x$$

$$+\frac{4}{1+c}\sum_{k=1}^{\infty}\Big(\frac{-\,e^{-\mu_{2k-1}^2 t}}{\mu_{2k-1}^3\sin\mu_{2k-1}}x\cos\mu_{2k-1}x$$

$$+\frac{e^{-\mu_{2k-1}^2 t}(9+6\mu_{2k-1}^2 t+\mu_{2k-1}\cot\mu_{2k-1})}{3\mu_{2k-1}^4\sin\mu_{2k-1}}\sin\mu_{2k-1}x\Big).$$

LEMMA 9. *If $c=\frac{1}{5}$ and $f(x)=\frac{x^3}{6}-\frac{31}{150}$, then the weak solution of Problem 3 is:*

$$\Omega(x,t)=2(1-c)\sum_{n=1}^{\infty}\frac{\cos\lambda_n' t}{\lambda_n'^3(\cos\lambda_n'-c\ \cos c\ \lambda_n')}\sin\lambda_n' x$$

$$+\ 2(1-c)\sum_{k=1}^{\infty}\frac{\cos\mu_{2k}t}{\mu_{2k}^3(\cos\mu_{2k}-c\ \cos c\ \mu_{2k})}\sin\mu_{2k}x$$

$$+\frac{4}{1+c}\sum_{k=1}^{\infty}\Big(\frac{-\cos\mu_{2k-1}t}{\mu_{2k-1}^3\sin\mu_{2k-1}}x\cos\mu_{2k-1}x$$

$$+\frac{(\cos\mu_{2k-1}t(9+\mu_{2k-1}\cot\mu_{2k-1})+3t\mu_{2k-1}\sin\mu_{2k-1}t)}{3\mu_{2k-1}^4\sin\mu_{2k-1}}\sin\mu_{2k-1}x\Big).$$

LEMMA 10. *If $a=1$, $c=\frac{1}{5}$ and $f(x)=\frac{x^3}{6}-\frac{31}{150}$, then the weak solution of Problem 4 is:*

$$\Omega(x,y)=2(a-c)\sum_{n=1}^{\infty}\frac{\cosh\frac{1}{2}(b+d-2y)\lambda_n'}{\lambda_n'^3(a\cos a\lambda_n'-c\cos c\lambda_n')\cosh\frac{1}{2}(b+d)\lambda_n'}\sin\lambda_n' x$$

$$+\ 2(a-c)\sum_{k=1}^{\infty}\frac{\cosh\frac{1}{2}(b+d-2y)\mu_{2k}}{\mu_{2k}^3(a\cos a\mu_{2k}-c\cos c\mu_{2k})\cosh\frac{1}{2}(b+d)\mu_{2k}}\sin\mu_{2k}x$$

$$+\ \frac{4}{a+c}\sum_{k=1}^{\infty}\Bigg\{\frac{-\cosh\frac{1}{2}(b+d-2y)\mu_{2k-1}}{\mu_{2k-1}^3\sin a\mu_{2k-1}\cosh\frac{1}{2}(b+d)\mu_{2k-1}}\ x\cos\mu_{2k-1}x$$

$$+\ \frac{e^{-\mu_{2k-1}y}}{3\mu_{2k-1}^4(1+e^{(b+d)\mu_{2k-1}})^2\sin a\mu_{2k-1}}\Bigg[3\Big(e^{(b+d)\mu_{2k-1}}(3-(b+d-y)\mu_{2k-1})$$

$$+\ e^{(b+d+2y)\mu_{2k-1}}(3+(b+d-y)\mu_{2k-1})+e^{2\mu_{2k-1}y}(3-\mu_{2k-1}y)$$

$$+\ e^{2(b+d)\mu_{2k-1}}(3+\mu_{2k-1}y)\Big)$$

$$+\, a(1+e^{(b+d)\mu_{2k-1}})(e^{(b+d)\mu_{2k-1}}+e^{2\mu_{2k-1}y})\mu_{2k-1}\cot\mu_{2k-1}a\Big]\sin\mu_{2k-1}x\Big\}.$$

Here $\Omega(x,t)$ and $\Omega(x,y)$ are weak solutions of the corresponding problems in the sense of Definition 1.

7. Explicit weak and classical solutions of Problems 1 - 4

Representation (13) can be simplified using Lemma 2. In the case of Problem 4, we get

THEOREM 1. *Let $f \in C^2[0,a]$ be such that $f(0) = f(a) - f(c) = 0$. Then*

$$\begin{aligned} u = {} & \frac{\partial^4}{\partial x^4}(\Omega(x,y) * f(x)) \qquad (16)\\ = {} & -\frac{1}{2(a-c)}\Bigg(\int_0^x \Big(\Omega_x(\xi+a-x,y) - \Omega_x(x+a-\xi,y)\\ & -\Omega_x(\xi+c-x,y) + \Omega_x(x+c-\xi,y)\Big) f''(\xi)d\xi\\ & + \int_0^a \left(\Omega_x(x+a-\xi,y) - \Omega_x(a-x-\xi,y)\right) f''(\xi)d\xi\\ & - \int_0^c \left(\Omega_x(x+c-\xi,y) - \Omega_x(c-x-\xi,y)\right) f''(\xi)d\xi\Bigg) \end{aligned}$$

is a weak solution of (4).

Additionally, if $f \in C^3[0,a]$ and $f''(0) = f''(a) - f''(c) = 0$, then(16) is a classical solution of (4).

The proof may be accomplished by a direct check.

If we put $a = 1$, we get representation of the solution for Problem 1.

THEOREM 2. *Let $f \in C^2[0,1]$ be such that $f(0) = f(1) - f(c) = 0$. Then*

$$\begin{aligned} u = {} & \frac{\partial^4}{\partial x^4}(\Omega(x,t) * f(x)) \qquad (17)\\ = {} & -\frac{1}{2(1-c)}\Bigg(\int_0^x \Big(\Omega_x(\xi+1-x,t) - \Omega_x(x+1-\xi,t)\\ & -\Omega_x(\xi+c-x,t) + \Omega_x(x+c-\xi,t)\Big) f''(\xi)d\xi \end{aligned}$$

$$+\int_0^1 \left(\Omega_x(x+1-\xi,t)-\Omega_x(1-x-\xi,t)\right) f''(\xi)d\xi$$
$$\left.-\int_0^c \left(\Omega_x(x+c-\xi,t)-\Omega_x(c-x-\xi,t)\right) f''(\xi)d\xi\right)$$

is a weak solution of (2) and (3), correspondingly.

Additionally, if $f \in C^3[0,a]$ *and* $f''(0) = f''(a) - f''(c) = 0$, *then (17) is a classical solution of (2) and (3), respectively.*

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