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INVESTIGATION OF A STOLARSKY TYPE INEQUALITY FOR INTEGRALS IN PSEUDO-ANALYSIS

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Abstract

In this paper, we prove a Stolarsky type inequality for pseudo-integrals. More precisely, we show that:

$$\int_{[0,1]}^{sup} f(x^{\frac{1}{a+b}})dx \geq \Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{a}})dx\Big) \odot \Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{b}})dx\Big),$$

where $a, b > 0, f : [0,1] \to [0,1]$ is a continuous and strictly decreasing function (strictly increasing function) and μ is the *sup*-measure the same as Theorem 2.4. Also $\odot$ is represented by an increasing multiplicative generator g.

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Key Words and Phrases: fuzzy measure, Stolarsky type inequality, pseudo-addition, pseudo-multiplication, pseudo-integral

1. Introduction

Not long ago, H. Román-Flores et al. in [2] analyzed an intersting type of geometric inequalities for the Sugeno integral with some applications to convex geometry. More precisely, a Prékopa-Leindler type inequality for fuzzy integrals was proven, and subsequently used for the characterization of some convexity properties of fuzzy measures.

We know that the Sugeno integral is not an extension of the Lebesgue integral. The difference between the Sugeno integral and the Lebesgue integral is that addition and multiplication in the definition of the Lebesgue integral are replaced respectively by the operations *max* and *min* when the Sugeno integral is considered. In this paper, we use of pseudo-analysis for the generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval $[a, b] \subset [-\infty, +\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Thus it would be an interesting topic to generalize an inequality from the from work of the classical analysis as special cases. In this paper, we investigate a Stolarsky type inequality for pseudo-integrals where $\oplus = max$ and $\odot$ generate by a continuous monotonic generator function g. The paper is organized as follows: Section 2 contains some of preliminaries, such as pseudo-operations, pseudo-analysis and pseudo-additive measures as well as integrals. In Section 3 we have main results.

2. Preliminaries

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (natural) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a *semiring* (see [8, 19]).

There are semirings with the following continuous operations:

Case I: The pseudo-addition is an idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = sup(x, y)$, $\odot$ is not arbitrary idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $sup(x, y) = y$.

(b) $x \oplus y = inf(x, y), \odot$ is not arbitrary idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$. If the zero element of the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = 1$. If the zero element of the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = 1$. If the generator g is increasing or decreasing, then the operation $\oplus$ induces the usual order or opposite to the usual order respectively on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have

(a) $x \oplus y = sup(x, y), x \odot y = inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $sup(x, y) = y$).

(b) $x \oplus y = inf(x, y), x \odot y = sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an opposite order to the usual order ($x \preceq y$ if and only if $inf(x, y) = y$).

But in this paper we consider semirings with the following continuous operations:

Case I: When $x \oplus y = max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$.

Case II: When $\oplus = Sup$ and $\odot = inf$.

Let X be a non-empty set. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X.

Definition 2.3. Let $m : \mathcal{A} \to [a, b]_+$ be a $\oplus$-*measure*.

(i) The pseudo-integral of an elementary function $e : X \to [a, b]$ with respect to m is defined by

$$\int_X^{\oplus} e \odot dm = \bigoplus_{i=1}^{n} a_i \odot m(A_i).$$

(ii) The pseudo-integral of a bounded measurable function $f : X \to [a, b]$, (if $\oplus$ is not idempotent we suppose that for each $\epsilon > 0$ there exists a monotone ϵ-net in $f(X)$) is defined by

$$\int_X^{\oplus} f(x) \odot dm = lim_{n\to\infty} \int_X^{\oplus} e_n(x) \odot dm,$$

where $(e_n)_{n\in N}$ is a sequence of elementary functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$.

The pseudo-integral for a function $f : \mathbf{R} \to [a, b]$ is given by

$$\int_{\mathbf{R}}^{\oplus} f \odot dm = sup\Big(f(x) \odot \psi(x)\Big), \tag{2.1}$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [10]. For any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [10].

We denote by μ the usual Lebesgue measure on $\mathbf{R}$. We have

$$m(A) = esssup(x|x \in A) = sup\{a|\mu(\{x|x \in A, x > a\}) > 0\}.$$

THEOREM 2.4. ([10]) *Let m be a sup-measure on $([0, \infty], \mathcal{B}([0, \infty]))$, where $\mathcal{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = esssup_{\mu}(\psi(x)|x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$–measure on $([0, \infty), \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$, such that $lim_{\lambda\to\infty} m_\lambda = m$.*

THEOREM 2.5. ([10]) *Let $([0, \infty]$, sup, $\odot)$ be a semiring with $\odot$ with a generator g, i.e.,we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0, \infty)$. Let m be the same as in Theorem 2.4. Then there exists a family m_λ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by $g^\lambda, \lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$,*

$$\begin{aligned} \int^{sup} f \odot dm &= lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda \\ &= lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int g^\lambda(f(x))dx\Big). \end{aligned} \tag{2.2}$$

3. Main results

THEOREM 3.1. *Let $a, b > 0$, if $f : [0, 1] \to [0, 1]$ is a continuous and strictly increasing function and let m be the same as in the Theorem 2.4. if $\odot$ is represented by a decreasing multiplicative generator g, then the inequality*

$$\int_{[0,1]}^{sup} f(x^{\frac{1}{a+b}})dm \geq \Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{a}})dm\Big)\Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{b}})dm\Big)$$

holds.

P r o o f. Theorem 2.5 implies that

$$\begin{aligned}
\int_{[0,1]}^{sup} f(x^{\frac{1}{a+b}})dm &= lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} f(x^{\frac{1}{a+b}})dm_\lambda \\
&= lim_{\lambda\to\infty}(g^\lambda)^{-1} \int_0^1 g^\lambda(f(x^{\frac{1}{a+b}}))dx \\
&= lim_{\lambda\to\infty}(g^\lambda)^{-1} \int_0^1 (g^\lambda f)(x^{\frac{1}{a+b}})dx. \qquad (3.1)
\end{aligned}$$

Since $g^\lambda f$ is nonincreasing, if we get

$$\int_{[0,1]}^{sup} f(x^{\frac{1}{a+b}})dm = a,$$

then from (3.1) we have

$$\begin{aligned}
a &\geq lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(\int_0^1 (g^\lambda f)(x^{\frac{1}{a}})dx \int_0^1 (g^\lambda f)(x^{\frac{1}{b}})dx\Big) \\
&= lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(\int_0^1 g^\lambda(f(x^{\frac{1}{a}}))dx \int_0^1 g^\lambda(f(x^{\frac{1}{b}}))dx\Big) \\
&= \Big(lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(\int_0^1 g^\lambda(f(x^{\frac{1}{a}}))dx\Big)\Big) \\
&\qquad \times \Big(lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(\int_0^1 g^\lambda(f(x^{\frac{1}{b}}))dx\Big)\Big) \\
&= \Big(lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} f(x^{\frac{1}{a}})dm\Big)\Big(lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} f(x^{\frac{1}{b}})dm\Big) \\
&= \Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{a}})dm\Big)\Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{b}})dm\Big).
\end{aligned}$$

The proof now is complete. ■

EXAMPLE 3.2. Suppose that $g^\lambda(x) = x^{-\lambda}$. Then

$$x \oplus y = (x^{-\lambda} + y^{-\lambda})^{-\frac{1}{\lambda}}$$

and

$$x \odot y = xy.$$

Therefore, the Stolarsky type inequality from Theorem 3.1 reduces to

$$sup(f(x^{\frac{1}{a+b}}) + \psi(x) \geq sup(f(x^{\frac{1}{a}}) + \psi(x))sup(f(x^{\frac{1}{b}}) + \psi(x)),$$

where ψ is from Theorem 2.4.

Theorem 3.3. *Let $a, b > 0$, $f : [0,1] \to [0,1]$ be a continuous and strictly decreasing function and let m be the same as in Theorem 2.4. If $\odot$ is represented by an increasing multiplicative generator g, then the inequality*

$$\int_{[0,1]}^{sup} f(x^{\frac{1}{a+b}})dm \geq \Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{a}})dm\Big)\Big(\int_{[0,1]}^{sup} f(x^{\frac{1}{b}})dm\Big),$$

holds.

P r o o f. The proof is similar with Theorem 3.1.

Example 3.4. Let $g^{\lambda}(x) = e^{\lambda x}$. Then

$$x \oplus y = lim_{\lambda\to\infty}\frac{1}{\lambda}\ln(e^{\lambda x} + e^{\lambda y}) = max(x,y),$$

and

$$x \odot y = lim_{\lambda\to\infty}\frac{1}{\lambda}\ln(e^{\lambda x}e^{\lambda y}) = x + y.$$

Therefore, the Stolarsky type inequality from Theorem 3.3 reduces to

$$sup(f(x^{\frac{1}{a+b}}) + \psi(x)) \geq sup(f(x^{\frac{1}{a}}) + \psi(x)) + sup(f(x^{\frac{1}{b}}) + \psi(x)),$$

where ψ is from Theorem 2.4.

Note that the second important case $\oplus = max$ and $\odot = min$ has been studied in [5] and the pseudo-integral in such a case yields the *Sugeno integral.*

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