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APPLICATIONS OF SUBORDINATION PRINCIPLE TO LOG-HARMONIC MAPPINGS

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Dedicated to Professor Gheorghe Oros
on the occasion of his 60th birthday

Abstract

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z| \quad |z| < 1\}$. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation $\overline{f_{\overline{z}}} = w(z) f_z \left(\frac{\overline{f}}{f}\right)$, where $w(z) \in H(\mathbb{D})$ is the second dilatation of f such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. It has been shown that if f is non-vanishing log-harmonic mapping, then f can be expressed as $f(z) = h(z)\overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0, g(0) = 1$. If f vanishes at $z = 0$ but it is not identically zero, then f admits the representation $f(z) = z|z|^{2\beta} h(z)\overline{g(z)}$, where $Re\beta > -1/2$ and $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0, g(0) = 1$([1], [2], [4]). The class of all log-harmonic mappings is denoted by $\mathcal{S}_{lh}$. We say that f is a starlike log-harmonic mapping of complex order b ($b \neq 0$ and complex) if $Re\left[1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right)\right] > 0$, the class of all starlike log-harmonic mapping of complex order is denoted by $\mathcal{S}^*_{lh}(1-b)$.

The aim of this paper is to give some applications of subordination principle to log-harmonic mappings.

MSC 2010: 30C45, 30C55

Key Words and Phrases: starlike functions, starlike log-harmonic mappings, two point distortion theorem, starlike log-harmonic mapping of complex order

1. Introduction

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the unit disc $\mathbb{D}$. A log-harmonic mapping, (i.e. $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 > 0$) is the solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{1}$$

where $w(z)$ is the second dilation function of f and $w(z) \in H(\mathbb{D})$, $|w(z)| < 1$ for every $z \in \mathbb{D}$. It has been shown ([2]) that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z)\overline{g(z)} \tag{2}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0, g(0) = 1$. On the other hand, if f vanishes at $z = 0$, but it is not identically zero then f admits the following representation

$$f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}, \tag{3}$$

where $Re\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0, g(0) = 1$. We note that the univalent log-harmonic mappings have been studied extensively ([1], [2], [3], [4]), and the class of all univalent log-harmonic functions is denoted by $\mathcal{S}_{lh}$.

Let $f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping of complex order if

$$Re\left(1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right)\right) > 0 \tag{4}$$

for all $z \in \mathbb{D}$. The class of all starlike log-harmonic mappings of complex order is denoted by $\mathcal{S}^*_{lh}(1-b)$.

If we give specific values to b we obtain the following subclasses of starlike log-harmonic functions of complex order:

- If $b = 1$, then $\mathcal{S}^*_{lh}(1-1) = \mathcal{S}^*_{lh}(0)$ is the class of starlike log-harmonic functions,
- If $b = 1-\alpha,\ 0 \leq \alpha < 1$, then $\mathcal{S}^*_{lh}(1-(1-\alpha)) = \mathcal{S}^*_{lh}(\alpha)$ is the class of starlike log-harmonic functions of order α,
- If $b = e^{-i\lambda},\ |\lambda| < \frac{\pi}{2}$, then $\mathcal{S}^*_{lh}(1-e^{-i\lambda})$ is the class of $\lambda-$ spirallike log-harmonic functions,
- If $b = (1-\alpha)e^{-i\lambda},\ 0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}$, then $\mathcal{S}^*_{lh}(1-(1-\alpha)e^{-i\lambda})$ is the class of $\lambda-$ spirallike log-harmonic functions of order α.

Finally, Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0, |\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $S_1(z) = z + a_2z^2 + a_3z^3 + \cdots$, $S_2(z) = z + b_2z^2 + b_3z^3 + \cdots$ be the analytic functions in $\mathbb{D}$. We say that $S_1(z)$ is subordinate to $S_2(z)$ if there exists $\phi(z) \in \Omega$ such that $S_1(z) = S_2(\phi(z))$ and denote $S_1(z) \prec S_2(z)$([5]).

Let $s(z)$ be analytic function in $\mathbb{D}$ with the normalization $s(0) = 0$, $s'(0) = 1$. If $s(z)$ satisfies the condition

$$Re\left(1 + \frac{1}{b}\left(z\frac{s'(z)}{s(z)} - 1\right)\right) > 0 \tag{5}$$

for every $z \in \mathbb{D}$, then $s(z)$ is called starlike function of complex order. The class of all starlike functions of complex order is denoted by $\mathcal{S}^*(1-b)$ ([6]). Also we note that in our proofs we will need the following theorems.

Theorem 1.1. ([6]) *A necessary and sufficient condition for* $s_1(z) \in \mathcal{S}^*(1-b)$ *is that for each member* $s_2(z) \in \mathcal{S}^*(0) = \mathcal{S}^*$ *the equation*

$$s_2(z) = z\left(\frac{s_1(z)}{z}\right)^{1/b} \Leftrightarrow z\left(\frac{s_2(z)}{z}\right)^{b} = s_1(z)$$

must be satisfied, where $\left(\frac{s_1(z)}{z}\right)^{1/b} = 1$ *at* $z = 0$.

Theorem 1.2. ([2]) *Let* $f(z) = zh(z)\overline{g(z)}$ *be univalent log-harmonic mapping. Then*

$$f \in \mathcal{S}^*_{lh} \Leftrightarrow \left(z\frac{h(z)}{g(z)}\right) \in \mathcal{S}^*.$$

2. Main results

LEMMA 2.1. *Let* $f \in \mathcal{S}^*_{lh} \Leftrightarrow s(z) = z\left(\frac{h(z)}{g(z)}\right)^b \in \mathcal{S}^*(1-b)$.

P r o o f.

$$s(z) = z\left(\frac{h(z)}{g(z)}\right)^b \Rightarrow \log s(z) = \log\left[z\left(\frac{h(z)}{g(z)}\right)^b\right] \Rightarrow$$

$$Re\left[\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right] = Re\left[1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right] = Re\left[1 + \frac{1}{b}\left(z\frac{s'(z)}{s(z)} - 1\right)\right]$$

This shows that the lemma is true. ■

LEMMA 2.2. *Let* $f = zh(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{lh}$, *then*

$$\frac{\frac{\phi'(z)}{\phi(z)}}{\frac{f_z}{f}} \prec 1 - z, \quad \frac{\frac{\overline{f_{\overline{z}}}}{\overline{f}}}{\frac{\phi'(z)}{\phi(z)}} \prec \frac{z}{1-z},$$

where $\phi(z) = z\frac{h(z)}{g(z)}$.

P r o o f. Let $\phi(z) = z\frac{h(z)}{g(z)}$, then we have

$$z\frac{\phi'(z)}{\phi(z)} = 1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}. \tag{6}$$

On the other hand, since $f = zh\overline{g}$ is the solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w(z)f_z\left(\frac{\overline{f}}{f}\right),$$

then we have $w(z)$, that is the second dilatation of f:

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}}\frac{f}{f_z}.$$

Using $w(0) = 0$, we can write

$$w(z) = \frac{\frac{\overline{f_{\overline{z}}}}{\overline{f}}}{\frac{f_z}{f}} = \frac{z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)}}.$$

This shows that the second dilatation satisfies the condition of the Schwarz lemma and we get these equalities:

$$1-w(z)=\frac{\frac{\phi'(z)}{\phi(z)}}{\frac{f_z}{f}},\qquad \frac{w(z)}{1-w(z)}=\frac{\frac{\overline{f_{\overline{z}}}}{f}}{\frac{\phi'(z)}{\phi(z)}}. \tag{7}$$

Using the subordination principle the equalities (7) can be written in the following forms

$$\frac{\frac{\phi'(z)}{\phi(z)}}{\frac{f_z}{f}}\prec 1-z$$

and

$$\frac{\frac{\overline{f_{\overline{z}}}}{f}}{\frac{\phi'(z)}{\phi(z)}}\prec \frac{z}{1-z}.$$

■

THEOREM 2.3. *Let* $f(z)=zh(z)\overline{g(z)}\in\mathcal{S}^*_{lh}(1-b)$, *then*

$$\left|\frac{zf_z}{f}\right|\le\frac{(1+|1-b|)+[2|b|-1-|1-b|]r}{|b|(1-r)^2}, \tag{8}$$

$$\left|\frac{z\overline{f_{\overline{z}}}}{\overline{f}}\right|\le\frac{r[(1+|1-b|)+(2|b|-|1-b|-1)r]}{|b|(1-r)^2}. \tag{9}$$

P r o o f. Using Lemma 2.1 and Lemma 2.2,

$$1-w(z)=\frac{1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}}{1+z\frac{h'(z)}{h(z)}}=\frac{1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)}{\frac{zf_z}{f}}, \tag{10}$$

$$\frac{w(z)}{1-w(z)}=\frac{z\frac{g'(z)}{g(z)}}{1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}}=\frac{\frac{z\overline{f_{\overline{z}}}}{\overline{f}}}{1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)}. \tag{11}$$

On the other hand, since the transformations $w_1(z)=1-z$ and $w_2=\frac{z}{1-z}$ map $|z|=r$ onto the discs with the centers $C_1(r)=(1,0)$, $C_2(r)=\left(\frac{r^2}{1-r^2},0\right)$, and radii $\rho_1(r)=r$, $\rho_2=\frac{r}{1-r^2}$ respectively. Using Lemma 2.2 and the subordination principle, we can write

$$\left|\frac{1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)}{\frac{zf_z}{f}}-1\right|<r,\qquad \left|\frac{\frac{z\overline{f_{\overline{z}}}}{\overline{f}}}{1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)}-\frac{r^2}{1-r^2}\right|\le\frac{r}{1-r^2}. \tag{12}$$

After the simple calculations from (12) we get

$$\frac{\left|1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right|}{1+r}\leq\left|\frac{zf_z}{f}\right|\leq\frac{\left|1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right|}{1-r}, \tag{13}$$

$$\frac{-r\left|1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right|}{1+r}\leq\left|\frac{\overline{z\overline{f_{\overline{z}}}}}{\overline{f}}\right|\leq\frac{r\left|1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right|}{1-r}. \tag{14}$$

On the other hand, we have

$$\begin{gathered}\left|1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right|=\left|1+\frac{1}{b}z\frac{s'(z)}{s(z)}-\frac{1}{b}\right|=\left|\frac{1}{b}z\frac{s'(z)}{s(z)}-\left(\frac{1}{b}-1\right)\right|,\\ \left|\frac{1}{b}z\frac{s'(z)}{s(z)}\right|-\left|\frac{1}{b}-1\right|\leq\left|1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right|\leq\left|\frac{1}{b}z\frac{s'(z)}{s(z)}\right|+\left|\frac{1}{b}-1\right|.\end{gathered} \tag{15}$$

Since $Re\left[1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right]>0$, then using subordination principle we can write

$$\left|\left[1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right]-\frac{1+r^2}{1-r^2}\right|\leq\frac{2r}{1-r^2}. \tag{16}$$

The inequality (16) can be written in the following form

$$\frac{-1-(2|b|+1)r}{|b|(1+r)}\leq\left|\frac{1}{b}\left(z\frac{s'(z)}{s(z)}\right)\right|\leq\frac{1+(2|b|-1)r}{|b|(1-r)}. \tag{17}$$

Applying (13), (14), (15) to the inequalities (11) and (12), we get (8) and (9). ■

Lemma 2.4. *Let $f=zh(z)\overline{g(z)}$ be starlike log-harmonic mapping of complex order b. Then we have the following distortion*

$$\frac{1}{(1+r)^2}\leq\left|\frac{h(z)}{g(z)}\right|\leq\frac{1}{(1-r)^2}. \tag{18}$$

P r o o f. From the inequality (16) we can write

$$\left|\left[1+\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)\right]-\frac{1+r^2}{1-r^2}\right|\leq\frac{2r}{1-r^2},$$

since f is a starlike log-harmonic mapping of complex order b, we have

$$\left|\left[1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right]-\frac{1+r^2}{1-r^2}\right|\leq\frac{2r}{1-r^2}. \tag{19}$$

Using the following equality in (19)

$$Re\left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) = r\frac{\partial}{\partial r}\left[\log|h(z)| - \log|g(z)|\right],$$

we obtain that

$$-\frac{2r}{1+r} \leq \frac{\partial}{\partial r}\left[\log|h(z)| - \log|g(z)|\right] \leq \frac{2r}{1+r}. \tag{20}$$

Integrating both sides 0 to r inequality (20), we get the inequality (18).

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