Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 5 (2010) ISSN 1311-0454

APPLICATION OF SALAGEAN AND RUSCHEWEYH OPERATORS ON UNIVALENT HOLOMORPHIC FUNCTIONS WITH FINITELY MANY COEFFICIENTS

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Abstract

The purpose of the present paper is to introduce a new subclass of holomorphic univalent functions with negative and fixed finitely coefficient based on Salagean and Ruscheweyh differential operators. The various results investigated in this paper include coefficient estimates, extreme points and Radii properties.

MSC 2010: 30C45, 30C50

Key Words and Phrases: univalent functions, coefficient inequality, radii of starlikeness and convexity, Salagean operator, Ruscheweyh operator

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=t+1}^{\infty} a_k z^k$$

which are holomorphic in the unit disk Δ. We denote by $\mathcal{N}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are holomorphic, univalent in Δ and are of the form (see [2])

$$f(z) = z - \sum_{k=t+1}^{\infty} a_k z^k \quad , \quad a_k \geq 0 \,. \tag{1}$$

Definition 1.1. Let $n \in \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$. By $\Omega_\lambda^n f$ we denote the operator $\Omega_\lambda^n : \mathcal{N} \longrightarrow \mathcal{N}$ defined by

$$\Omega_\lambda^n f(z) = (1-\lambda)S^n f(z) + \lambda R^n f(z), \quad z \in \Delta, \tag{2}$$

where $S^n f$ is the Salagean differential operator ([6]) and $R^n f$ is the Ruscheweyh differential operator ([5]).

For $f(z) \in \mathcal{N}$ given by (1) we have respectively

$$S^n f(z) = z - \sum_{k=t+1}^{\infty} k^n a_k z^k, \tag{3}$$

and

$$R^n f(z) = z - \sum_{k=t+1}^{\infty} B_k(n) a_k z^k, \tag{4}$$

where

$$B_k(n) = \binom{k+n-1}{n} = \frac{(n+1)(n+2)...(n+k-1)}{(k-1)!}. \tag{5}$$

Further by replacing (3) and (4) in (2), we obtain

$$\Omega_\lambda^n f(z) = z - \sum_{k=t+1}^{\infty} [k^n(1-\lambda) + \lambda B_k(n)] a_k z^k. \tag{6}$$

It is observed that

$$\Omega_\lambda^0 f(z) = (1-\lambda)S^0 f(z) + \lambda R^0 f(z) = f(z) = S^0 f(z) = R^0 f(z).$$

Definition 1.2. A function $f(z) \in \mathcal{N}$ is said to belong to the class $\Psi_\lambda^n(\alpha, \beta)$ if and only if

$$Re\left\{\frac{\alpha z[\Omega_\lambda^n f(z)]' + z^2[\Omega_\lambda^n f(z)]''}{\alpha z[\Omega_\lambda^n f(z)]' + (1-\alpha)\Omega_\lambda^n f(z)}\right\} > \beta, \tag{7}$$

where $0 \leq \beta < 1, \quad 0 \leq \alpha \leq 1$ and $\alpha > \beta$.

Now we introduce the class $\Psi_\lambda^n(\alpha, \beta, c_m)$, the subclass of $\Psi_\lambda^n(\alpha, \beta)$ consisting of functions with negative and fixed finitely many coefficients of the form

$$f(z) = z - \sum_{m=2}^{t} \frac{(\alpha-\beta)c_m}{[m(m-\alpha\beta) - (1-\alpha)(m+\beta)][m^n(1-\lambda) + \lambda B_m(n)]} - \sum_{k=t+1}^{\infty} a_k z^k. \tag{8}$$

Such type of studies were recently carried out by many authors. See for example [1], [4].

To prove our main results, we need the following lemma.

LEMMA 1.1. *A function $f(z)$ given by (1) is in the class $\Psi_\lambda^n(\alpha,\beta)$ if and only if*

$$\sum_{k=t+1}^{\infty}[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)]a_k \leq \alpha-\beta,$$

where $B_k(n)$ is defined by (5). The result is sharp.

P r o o f. See [3]. ∎

2. Main Results

We begin by proving a necessary and sufficient conditions for a function belonging to the class $\Psi_\lambda^n(\alpha,\beta,c_m)$.

THEOREM 2.1. *Let $f(z)$ defined by (1), then $f(z)\in\Psi_\lambda^n(\alpha,\beta,c_m)$ if and only if*

$$\sum_{k=t+1}^{\infty}\frac{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)]}{\alpha-\beta}<1-\sum_{m=2}^{t}c_m.$$

P r o o f. By using Lemma 1.1 and relation (8) we get the result. ∎

Now we obtain the radii of starlikeness and convexity for the elements of the class $\Psi_\lambda^n(\alpha,\beta,c_m)$.

THEOREM 2.2. *Let the function $f(z)$ defined by (8) be in the class $\Psi_\lambda^n(\alpha,\beta,c_m)$, then $f(z)$ is starlike of order γ $(0\leq\gamma<1$ in $|z|<R_1$, where R_1 is the largest value such that*

$$\sum_{m=2}^{t}\frac{c_m}{[m(m-\alpha\beta)-(1-\alpha)(m+\beta)][m^n(1-\lambda)+\lambda B_m(n)]}R_1^{m-1}+$$

$$\frac{1-\sum_{m=2}^{\infty}c_m}{[k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)]}R_1^{k-1}<\frac{1}{\alpha-\beta}$$

for $k\geq t+1$.

By using the fact that "$f(z)$ is convex if and only if $zf'(z)$ is starlike", we conclude the following corollary.

COROLLARY 2.3. *Let $f(z)\in\Psi_\lambda^n(\alpha,\beta,c_m)$, then $f(z)$ is convex of order γ $(0\le\gamma<1)$ in $|z|<R_2$, where R_2 is the largest value such that*

$$\sum_{m=2}^{t}\frac{mc_m}{[m(m-\alpha\beta)-(1-\alpha)(m+\beta)][m^n(1-\lambda)+\lambda B_m(n)]}R_2^{m-1}$$

$$+\frac{k(1-\sum_{m=2}^{t}c_m)}{k(k-\alpha\beta)-(1-\alpha)(k+\beta)][k^n(1-\lambda)+\lambda B_k(n)]}R_2^{k-1}<\frac{1}{\alpha-\beta}.$$

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Received: GFTA, August 27-31, 2010