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NOTE ON RADIUS PROBLEMS FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS

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We dedicate this paper to the 60th anniversary of Professor Y. Polatoglu

Abstract

For analytic functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$, the subclass $\mathcal{A}(\alpha, \beta; \lambda)$ of functions $f(z)$ which satisfy

$$\left| \alpha \left(\frac{z}{f(z)} \right)'' + \beta z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leqq \lambda \quad (z \in \mathbb{U})$$

for some complex numbers α and β and for some real $\lambda > 0$ is introduced. The object of the present paper is to discuss some radius properties for $S^*(\gamma)$ such that $\frac{1}{\delta} f(\delta z) \in \mathcal{A}(\alpha, \beta; \lambda)$.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a function $f(z) \in \mathcal{A}$, we say that $f(z)$ belongs to the class $\mathcal{A}(\alpha, \beta; \lambda)$ if it satisfies $\dfrac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$ and

$$\left| \alpha \left(\frac{z}{f(z)} \right)'' + \beta z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' \right| \leqq \lambda \quad (z \in \mathbb{U}) \tag{1.2}$$

for some complex numbers α and β and for some real $\lambda > 0$.

Let us consider a function $f_k(z)$ given by

$$f_k(z) = \frac{z}{(1-z)^k} \qquad (k \in \mathbb{R}).$$

Then $f_k(z)$ satisfies $\dfrac{f_k(z)}{z} = \dfrac{1}{(1-z)^k} \neq 0 \quad (z \in \mathbb{U})$ and

$$\left(\frac{z}{f_k(z)} \right)'' = k(k-1)(1-z)^{k-2}.$$

Further, if we write that

$$f_k(z) = \frac{z}{1 + \sum_{n=1}^{\infty} a_n z^n}$$

with

$$a_n = (-1)^n \binom{k}{n},$$

then we see that

$$z^2 \left(\frac{1}{f_k(z)} - \frac{1}{z} \right)' = \sum_{n=1}^{\infty} (n-1) a_n z^n.$$

Therefore, we have that

$$\left| \alpha \left(\frac{z}{f_k(z)} \right)'' + \beta z^2 \left(\frac{1}{f_k(z)} - \frac{1}{z} \right)' \right| = \left| \alpha k(k-1)(1-z)^{k-2} + \beta \sum_{n=1}^{\infty} (n-1) a_n z^n \right|$$

$$< |\alpha| k(k-1) 2^{k-2} + |\beta| \sum_{n=1}^{\infty} (n-1) |a_n|$$

for $k \geqq 2$. This means that $f_k(z) \in \mathcal{A}(\alpha, \beta; \lambda)$ for $\lambda \geqq 2|\alpha| + |\beta|$ if $k = 2$, and $f_k(z) \in \mathcal{A}(\alpha, \beta; \lambda)$ for $\lambda \geqq 12|\alpha| + 5|\beta|$ if $k = 3$.

The classes $\mathcal{A}(1, 0; \lambda)$ and $\mathcal{A}(0, 1; \lambda)$ were introduced by Obradović and Ponnusamy [2], the generalized classes of $\mathcal{A}(1, 0; \lambda)$ and $\mathcal{A}(0, 1; \lambda)$ were considered by Shimoda, Hayami, Hashidume and Owa [4] and Kobashi, Kuroki, Shiraishi and Owa [1].

Let $\mathcal{S}^*(\gamma)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real γ $(0 \leqq \gamma < 1)$. A function $f(z) \in \mathcal{S}^*(\lambda)$ is said to be starlike of order γ in $\mathbb{U}$ (cf. Robertson [3]). We also write that $\mathcal{S}^*(0) = \mathcal{S}^*$. For $f(z) \in \mathcal{A}$ given by (1.1), we write

$$\frac{z}{f(z)} = \frac{1}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

Then we know that $b_1 = -a_2$, $b_2 = a_2^2 - a_3$ and $b_3 = 2a_2a_3 - a_4 - a_3^2$.

2. Radius problems

To discuss our radius problems for $f(z) \in \mathcal{A}(\alpha, \beta; \lambda)$, we need the following lemmas.

LEMMA 2.1. *Let $f(z) \in \mathcal{A}$ be given by* (1.4) *with* $\dfrac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *If $f(z)$ satisfies*

$$\sum_{n=2}^{\infty} (n-1)\left(|\alpha| n + |\beta|\right)|b_n| \leqq \lambda \tag{2.1}$$

for some complex numbers α and β and for some real $\lambda > 0$, then $f(z) \in \mathcal{A}(\alpha, \beta; \lambda)$.

P r o o f. It follows that

$$\left|\alpha\left(\frac{z}{f(z)}\right)'' + \beta z^2\left(\frac{1}{f(z)} - \frac{1}{z}\right)'\right| = \left|\alpha \sum_{n=2}^{\infty} n(n-1) b_n z^{n-2} + \beta \sum_{n=2}^{\infty} (n-1) b_n z^n\right|$$

$$< |\alpha|\sum_{n=2}^{\infty} n(n-1)|b_n| + |\beta|\sum_{n=2}^{\infty}(n-1)|b_n|$$
$$= \sum_{n=2}^{\infty}(n-1)\left(|\alpha|n+|\beta|\right)|b_n|.$$

Therefore, if the coefficient inequality (2.1) holds true, then we say that $f(z) \in \mathcal{A}(\alpha,\beta;\lambda)$. ∎

LEMMA 2.2. *Let* $f(z) \in \mathcal{A}$ *be given by* (1.4) *with* $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *Further let* $b_n = |b_n|e^{in\theta}$ $(\theta \in \mathbb{R})$. *If* $f(z) \in \mathcal{S}^*(\gamma)$, *then*

$$\sum_{n=1}^{\infty}(n+\gamma-1)|b_n| \leqq 1-\gamma. \tag{2.2}$$

P r o o f. Note that $f(z) \in \mathcal{S}^*(\gamma)$ implies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1-\sum\limits_{n=1}^{\infty}(n-1)b_n z^n}{1+\sum\limits_{n=1}^{\infty}b_n z^n}\right)$$
$$= \operatorname{Re}\left(\frac{1-\sum\limits_{n=1}^{\infty}(n-1)|b_n|e^{in\theta}z^n}{1+\sum\limits_{n\ 1}^{\infty}|b_n|e^{in\theta}z^n}\right) > \gamma \qquad (z \in \mathbb{U}).$$

If we consider z such that $z = |z|e^{-i\theta}$, then we obtain that

$$\frac{1-\sum\limits_{n=1}^{\infty}(n-1)|b_n||z|^n}{1+\sum\limits_{n=1}^{\infty}|b_n||z|^n} > \gamma \qquad (|z|<1).$$

Now, letting $|z| \to 1^-$, we have that

$$\sum_{n=1}^{\infty}(n+\gamma-1)|b_n| \leqq 1-\gamma,$$

∎

REMARK 2.1. In view of the coefficient inequality (2.2), we know that

$$\sum_{n=2}^{\infty}(n+\gamma-1)|b_n| \leqq 1-\gamma-\gamma|b_1|$$

which shows that

$$|b_n| \leqq \frac{1-\gamma-\gamma|b_1|}{n+\gamma-1} < 1 \qquad (n = 2, 3, 4, \cdots).$$

This implies that

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leqq 1-\gamma-\gamma|b_1|. \tag{2.3}$$

Now, we derive the following

THEOREM 2.1. *Let $f(z) \in \mathcal{A}$ be given by* (1.4) *with* $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *Further let $b_n = |b_n|e^{in\theta}$ $(\theta \in \mathbb{R})$ and $\delta \in \mathbb{C}$ $(|\delta| < 1)$. If $f(z) \in \mathcal{S}^*(\gamma)$ with $0 \leqq \gamma < \frac{1}{1+|b_1|}$, then $\frac{1}{\delta}f(\delta z)$ belongs to the class $\mathcal{A}(\alpha, \beta; \lambda)$ for $0 < |\delta| \leqq |\delta_0(\lambda)|$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation*

$$|\delta|^2\left(|\alpha|\sqrt{2(2+|\delta|^2)}+|\beta|(1-|\delta|^2)\right)\sqrt{1-\gamma-\gamma|b_1|} = \lambda(1-|\delta|^2)^2. \tag{2.4}$$

P r o o f. By means of (1.4), we have that

$$\frac{z}{\frac{1}{\delta}f(\delta z)} = 1+\sum_{n=1}^{\infty}b_n\delta^n z^n.$$

Thus, we have to prove that

$$\sum_{n=2}^{\infty}(n-1)\left(|\alpha|n+|\beta|\right)|b_n||\delta|^n \leqq \lambda$$

by Lemma 2.1. Applying the Cauchy-Schwarz inequality, we see that

$$\sum_{n=2}^{\infty}(n-1)\left(|\alpha|n+|\beta|\right)|b_n||\delta|^n = |\alpha|\sum_{n=2}^{\infty}n(n-1)|b_n||\delta|^n+|\beta|\sum_{n=2}^{\infty}(n-1)|b_n||\delta|^n$$

$$\leqq |\alpha|\left(\sum_{n=2}^{\infty}n^2(n-1)|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} +|\beta|\left(\sum_{n=2}^{\infty}(n-1)|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}.$$

We note that

$$\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \leqq \sqrt{1-\gamma-\gamma|b_1|}$$

from (2.3) of Remark 2.1. If we put $|\delta|^2 = x$, then

$$\sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n} = x^2 \sum_{n=2}^{\infty} n^2(n-1)x^{n-2}$$
$$= x^2 \left(\sum_{n=2}^{\infty} nz^n\right)'' = x^2 \left(\frac{x}{(1-x)^2} - x\right)'' = \frac{2x^2(2+x)}{(1-x)^4}$$

and

$$\sum_{n=2}^{\infty}(n-1)|\delta|^{2n} = x^2 \sum_{n=2}^{\infty}(n-1)x^{n-2}$$
$$= x^2 \left(\sum_{n=2}^{\infty} x^{n-1}\right)' = x^2 \left(\frac{x}{1-x}\right)' = \frac{x^2}{(1-x)^2}.$$

Therefore, we obtain that

$$\sum_{n=2}^{\infty}(n-1)\left(|\alpha|n + |\beta|\right)|b_n||\delta|^n$$
$$\leqq \left(\frac{|\alpha||\delta|^2\sqrt{2(2+|\delta|^2)}}{(1-|\delta|^2)^2} + \frac{|\beta||\delta|^2}{1-|\delta|^2}\right)\sqrt{1-\gamma-\gamma|b_1|}.$$

Considering $\lambda > 0$ such that

$$\left(\frac{|\alpha||\delta|^2\sqrt{2(2+|\delta|^2)}}{(1-|\delta|^2)^2} + \frac{|\beta||\delta|^2}{1-|\delta|^2}\right)\sqrt{1-\gamma-\gamma|b_1|} = \lambda$$

which is equivalent to (2.4), we define $h(|\delta|)$ by

$$h(|\delta|) = |\delta|^2\left(|\alpha|\sqrt{2(2+|\delta|^2)} + |\beta|(1-|\delta|^2)\right)\sqrt{1-\gamma-\gamma|b_1|} - \lambda(1-|\delta|^2)^2.$$

Then we have that $h(0) = -\lambda < 0$ and $h(1) = \sqrt{6}|\alpha|\sqrt{1-\gamma-\gamma|b_1|}$. This implies that $h(|\delta|) = 0$ has a positive root $|\delta_0(\lambda)|$ for $0 < |\delta| < 1$. This completes the proof of the theorem. ■

Taking $\gamma = 0$ in Theorem 2.1, we have the following

Corollary 2.1. *Let $f(z) \in \mathcal{A}$ be given by* (1.4) *with* $\dfrac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *Further, let $b_n = |b_n|e^{in\theta}$ $(\theta \in \mathbb{R})$ and $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. If $f(z) \in \mathcal{S}^*$, then $\dfrac{1}{\delta}f(\delta z)$ belongs to the class $\mathcal{A}(\alpha, \beta; \lambda)$ for $0 < |\delta| \leqq |\delta_0(\lambda)|$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation*

$$|\delta|^2\left(|\alpha|\sqrt{2(2+|\delta|^2)} + |\beta|(1-|\delta|^2)\right) = \lambda(1-|\delta|^2)^2. \tag{2.5}$$

REMARK 2.2. If we take $\delta = \frac{1}{2}e^{i\theta}$ in (2.5), then we have

$$\lambda = \frac{2\sqrt{2}}{3}|\alpha| + \frac{1}{3}|\beta|.$$

If we consider $|\alpha| = |\beta| = \lambda = 1$, then

$$|\delta|^2\sqrt{2(2+|\delta|^2)} + |\delta|^2(1-|\delta|^2) - (1-|\delta|^2)2 = 0.$$

Therefore, we see that $0.4623 < |\delta_0(1)| < 0.4625$.

Taking $\alpha = 1$ and $\beta = 0$ in Theorem 2.1, we have

COROLLARY 2.2. *Let* $f(z) \in \mathcal{A}$ *be given by* (1.4) *with* $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *Further, let* $b_n = |b_n|e^{in\theta}$ $(\theta \in \mathbb{R})$ *and* $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. *If* $f(z) \in \mathcal{S}^*(\gamma)$ *with* $0 \leqq \gamma < \frac{1}{1+|b_1|}$, *then* $\frac{1}{\delta}f(\delta z)$ *belongs to the class* $\mathcal{A}(1,0;\lambda)$ *for* $0 < |\delta| \leqq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$\frac{|\delta|^2\sqrt{2(2+|\delta|^2)}}{(1-|\delta|^2)^2}\sqrt{1-\gamma-\gamma|b_1|} = \lambda.$$

If we take $\alpha = 0$ and $\beta = 1$ in Theorem 2.1, then we have

COROLLARY 2.3. *Let* $f(z) \in \mathcal{A}$ *be given by* (1.4) *with* $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *Further let* $b_n = |b_n|e^{in\theta}$ $(\theta \in \mathbb{R})$ *and* $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. *If* $f(z) \in \mathcal{S}^*(\gamma)$ *with* $0 \leqq \gamma < \frac{1}{1+|b_1|}$, *then* $\frac{1}{\delta}f(\delta z)$ *belongs to the class* $\mathcal{A}(0,1;\lambda)$ *for* $0 < |\delta| \leqq |\delta_0(\lambda)|$, *where*

$$|\delta_0(\lambda)| = \left(\frac{\lambda}{\lambda + \sqrt{1-\gamma-\gamma|b_1|}}\right)^{\frac{1}{2}}.$$

Finally, since $b_1 = 0$ implies that $a_2 = 0$, we have

COROLLARY 2.4. *Let* $f(z) \in \mathcal{A}$ *be given by* (1.4) *with* $a_2 = 0$ *and* $\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. *Further, let* $b_n = |b_n|e^{in\theta}$ $(\theta \in \mathbb{R})$ *and* $\delta \in \mathbb{C}$ $(|\delta| < 1)$. *If* $f(z) \in \mathcal{S}^*(\gamma)$ *with* $0 \leqq \gamma < 1$, *then* $\frac{1}{\delta}f(\delta z)$ *belongs to the class* $\mathcal{A}(\alpha,\beta;\lambda)$ *for* $0 < |\delta| \leqq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$|\delta|^2\left(|\alpha|\sqrt{2(2+|\delta|^2)} + |\beta|(1-|\delta|^2)\right)\sqrt{1-\gamma} = \lambda(1-|\delta|^2)^2.$$

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