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SYLOW p-SUBGROUPS OF ABELIAN GROUP RINGS

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Abstract. Let PG be the abelian modular group ring of the abelian group G over the abelian ring P with 1 and prime char $P = p$. In the present article, the p-primary components $U_p(PG)$ and $S(PG)$ of the groups of units $U(PG)$ and $V(PG)$ are classified for some major classes of abelian groups.

Suppose K is a first kind field with respect to p in char $K \neq p$ and A is an abelian p-group. In the present work, the p-primary components $U_p(KA)$ and $S(KA)$ of the group of units $U(KA)$ and $V(KA)$ in the semisimple abelian group ring KA are studied when they belong to some central classes of abelian groups. The established criteria extend results obtained by us in Compt. rend. Acad. bulg. Sci. (1993). Moreover, the question for the isomorphic type of the basic subgroup of $S(KA)$ is also settled. As a final result, it is proved that if A is a direct sum of cyclics, the group of all normed p-units $S(KA)$ modulo A, that is, $S(KA)/A$, is a direct sum of cyclics too. Thus A is a direct factor of $S(KA)$ with a direct sum of cyclics complementary factor provided $s_p(K) \supseteq \mathbb{N}$. This generalizes a result due to T. Mollov in Pliska Stud. Math. Bulgar. (1986).

1. Introduction. Let G be an arbitrary abelian group and A be an abelian p-group both written multiplicatively. Throughout this paper P denotes

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an arbitrary commutative ring in prime char $P = p$ and with 1, L designates a field in char $L = p$ and K a field of the first kind with respect to p of char $K \neq p$ such that its spectrum contains all naturals. As usual, let $U(P)$ and $U_p(P)$ be the unit group and its p-component in P, and let $U(K)$ and $U_p(K)$ be the multiplicative group and the p-multiplicative group (i.e. the p-unit group) of K. Besides, define $N(P)$ as the nilradical of P.

For G a group, G_p will denote the p-component of the maximal torsion subgroup (= torsion part) tG of G. For A a p-group, $A^1 \stackrel{def}{=} \bigcap_{n=1}^{\infty} A^{p^n}$ will designate the group of all elements of infinite height (i.e. in other words, the first Ulm subgroup) of A, and B_A the basic subgroup of A.

Suppose that PG and KA are group rings, $U(PG)$ and $U(KA)$ are their unit groups with subgroups of normalized (i.e. of augmentation 1 — the coefficients sum equal to 1) units $V(PG)$ and $V(KA)$, and $U_p(PG)$ and $S(PG) = V_p(PG)$, respective $U_p(KA)$ and $S(KA) = V_p(KA)$, are their Sylow p-subgroups. For C a subgroup of G and an arbitrary ring R, the symbol $I(RG; C)$ denotes the relative augmentation(fundamental) ideal of RG with respect to C. All other notations and the terminology are in agreement with the excellent classical monographs of L. Fuchs [10], G. Karpilovsky [11] and D. Passman [14].

In [15], S. Berman and G. Rossa classified $S(KA)$ when A is countable.

In 1986, T. Mollov has described in [12] up to an isomorphism the structure of $S(KA)$ provided A is a direct sum of cyclics. Moreover, he decomposed in [12] the group $S(KA)$ into a divisible part and a separable reduced part. The divisible subgroup was classified in [12], and the reduced subgroup was characterized via the Ulm-Kaplansky functions in [13]. Nevertheless, his results are no complete and some new considerations in this aspect are needed.

In [16, 17, 18] Z. Chatzidakis and P. Pappas have given the isomorphic type of $V(KA)$ when A is a direct sum of countables and they considered certain variants of the Splitting Problem and the Direct Factor Problem as well. Moreover they showed that the classification problem for $S(KA)$ is difficult provided A is not countable.

Comparing our results stated and argued below with those independently obtained by Berman-Rossa, Chatzidakis-Pappas, and Mollov, we can say that the new major moments in this paper are the study of the Direct Factor Question and the characterization of the isomorphic type of the basic subgroup both in a semisimple aspect. The methods used by us are not identical to these in the cited above research articles.

The main purpose that motivates this manuscript is to investigate some characteristic properties of the above defined groups in the modular and semisimple directory by developing of the Mollov's assertions. More especially, we examine the Direct Factor Conjecture and the problem for isomorphism structure of the basic subgroup. Thus our results strengthen those obtained by Mollov. The work is organized in two sections as follows:

2. Modular case. The basic facts selected in this point concern the decomposition into direct sums of cyclic groups and the quasicompleteness of p-torsion components of commutative modular group rings. Foremost, for a convenience of the reader, we recall some facts needed for our presentation.

Theorem [4]**.** *Let H be a pure p-subgroup of G. The group $V(PG;H)$ is a direct sum of cyclics if and only if H is a direct sum of cyclics. In particular, when $H=G$, $V(PG)$ is a direct sum of cyclics if and only if G is.*

The purity condition on H being a pure subgroup in G is not sufficient. This may be demonstrated by the following.

Proposition 1. *Let $H \leq A$ such that A/H be bounded. The group $V(PA;H)$ is a direct sum of cyclics if and only if H is a direct sum of cyclics. Moreover, if H is a direct sum of cyclic groups, then $V(PA;H)/H$ is a direct sum of cyclics and H is a direct factor of $V(PA;H)$.*

P r o o f 1. We treat the sufficiency. Because H is a direct sum of cyclic groups, the same holds for A by virtue of [10]. From the above theorem, $V(PA)$ is a direct sum of cyclics. Thus, [10] ensures that $V(PA;H) \subseteq V(PA)$ is one also.

Besides, $V(PA)/A$ is a direct sum of cyclics invoking to [3]. Therefore $V(PA;H)/H \cong V(PA;H)A/A \subseteq V(PA)/A$ is with the same property, and it is a routine matter to see that H is pure in $V(PA;H)$. So, the direct factor claim holds true. This finishes the proof. □

P r o o f 2. Write $H = \bigcup_{n=1}^{\infty} H_n = \bigcup_{k=t}^{\infty} H_k$; $H_k \subseteq H_{k+1}$ for this $t \in \mathbb{N}$ such that $A^{p^t} \subseteq H$. Clearly $V(PA;H) = \bigcup_{k=t}^{\infty} V(PA;H_k)$. By applications of lemmas from [4], we conclude that, $V(PA;H_k) \cap V^{p^{s_k+t}}(PA;H) = V(PA;H_k) \cap V(P^{p^{s_k+t}}A^{p^{s_k+t}};H^{p^{s_k+t}}) \subseteq V(PA;H_k) \cap V(P^{p^{s_k}}H^{p^{s_k}};H^{p^{s_k}}) = V(P^{p^{s_k}}H^{p^{s_k}};H^{p^{s_k}} \cap H_k) = 1$ because $H_k \cap H^{p^{s_k}} = 1$ for some $s_k \in \mathbb{N}$ and any $k \in \mathbb{N}$. Finally, $V(PA;H)$ is a direct sum of cyclic groups owing to the well-known and documented Kulikov's criterion in [10]. Further, the same conclusions are valid and for $V(PA;H)/H$. The proof is finished. □

Proposition 2. *Suppose P is weakly perfect, that is, $P^{p^i} = P^{p^{i+1}}$ for some natural i. Then $U_p(PG)$ is a direct sum of cyclic groups if and only if the* *maximal perfect subring of P has no nilpotents and G_p is a direct sum of cyclic groups.*

P r o o f. If $U_p(PG)$ is a direct sum of cyclics, as subgroups $G_p \subseteq U_p(PG)$ is a direct sum of cyclic groups and $U_p(P^{p^i})$ is reduced. But $U_p(P^{p^i})$ is divisible since P^{p^i} is perfect and hence $U_p(P^{p^i}) = 1$, i.e. $N(P^{p^i}) = 0$.

We presume now that $N(P^{p^i}) = 0$ and that G_p is a direct sum of cyclics. By using of the first formulated theorem, $S(P^{p^i}G^{p^i}) = U_p(P^{p^i}G^{p^i}) = U_p^{p^i}(PG)$ is a direct sum of cyclics whence so is $U_p(PG)$ applying a theorem due to Sasiada-Mostowski and Fuchs [10]. The claim is proved. □

We begin now with the quasi and torsion completeness in modular group rings.

Proposition 3. *Assume that P is weakly perfect. Then $U_p(PG)$ is quasi complete (in particular torsion complete) if and only if the maximal perfect subring of P is without nilpotent elements and G_p is bounded.*

P r o o f. First, $U_p(PG)$ is assumed to be quasi complete. Therefore for some $i \in \mathbb{N}$, the group $U_p(P^{p^i})$ is reduced and divisible, i.e. $U_p(P^{p^i}) = 1$ which is equivalent to $N(P^{p^i}) = 0$. On the other hand, $U_p^{p^i}(PG)$ must be also quasi complete. Because $U_p(P^{p^i}G^{p^i}) = S(P^{p^i}G^{p^i})$, according to [8] we can deduce that $(G^{p^i})_p = (G_p)^{p^i}$ is bounded, i.e. the same holds and for G_p.

Conversely, G_p bounded means that $(G^{p^i})_p$ is so. The last property together with [2] lead us to $U_p(P^{p^i}G^{p^i}) = U_p^{p^i}(PG)$ is bounded. So, $U_p(PG)$ is bounded hence quasi complete [10]. The proof is over. □

Next, we start with the other main statements concerning

3. Semisimple case.

First, we proceed by proving the following criteria (see [2, 3, 5, 6, 7] as well).

Theorem 4. *The following are fulfilled:*

(0) *$S(KA)$ is cyclic $\iff$ $A = 1$.*

(1) *$S(KA)$ is a direct sum of cocyclics $\iff$ A/A^1 is a direct sum of cyclics.*

(2) *$S(KA)$ is finitely generated (i.e. finite) $\iff$ A is finite.*

(3) *$S(KA)$ is finitely cogenerated (i.e. finite) $\iff$ A is finite.*

(4) *$S(KA)$ is elementary $\iff$ A is elementary, $p = 2$ and $K \neq K(\varepsilon_2)$.*

(5) *$S(KA)$ is reduced homogeneous no elementary, i.e. is a direct sum of cyclics of the same order p^t for $t \geq 2$ $\iff$ A is a direct sum of cyclics, $p \neq 2$ or $p = 2$ but $K = K(\varepsilon_2)$, $A^{p^i} = 1$ and $i < t = \mathrm{const}_p(K)$.*

(6) $S(KA)$ *is a direct sum of* σ*-summables* $\iff$ A *is a direct sum of cyclics.*

(7) $S(KA)$ *is a* p^α*-projective for some* $\alpha \geq \omega$ $\iff$ A *is a direct sum of cyclics.*

(8) $S(KA)$ *is divisible* $\iff$ A *is divisible.*

Proof. Utilizing [12], we detect

$$S(KA) \cong S^1(KA) \times S(K(A/A^1)),$$

where $S^1(KA)$ is the divisible part of $S(KA)$.

(0) Follows in virtue of the decomposition formula in [12] as well as (4) and (5) below considered.

(1) Certainly, $S(KA)$ is a direct sum of cocyclics if and only if $S(K(A/A^1))$ is a direct sum of cyclics. Thus A/A^1 is a direct sum of cyclics again complying with [12].

(2) and (3). If $S(KA)$ is finite, then A is also finite as a subgroup. Conversely, let us assume that A is finite. Following the proof of [12, Lemma 5], we obtain that $S(KA)$ must be finite.

(4) and (5). By making use of [12], we may write $S(KA) \cong \coprod_{|A|}(p)$ when $p = 2$ and $K \neq K(\varepsilon_2)$; or $S(KA) \cong \coprod_{|A|}(p^t)$ with $\exp(A) = i < t = \operatorname{const}_p(K)$ when $p \neq 2$ or $p = 2$ but $K = K(\varepsilon_2)$.

(6) and (7). Evidently $S(KA)$ must be reduced whence separable and so $S(KA)$ is a direct sum of cyclics, i.e. the same is A.

(8) If A is divisible, then $A = A^1$ and hence $S(KA) \cong S^1(KA)$. Thus $S(KA)$ is divisible. Conversely, if $S(KA)$ is divisible, then $S(K(A/A^1))$ is divisible and separable (whence reduced), and therefore $S(K(A/A^1)) = 1$, i.e. $A/A^1 = 1$. Finally $A = A^1$, i.e. $A = A^p$. Furthermore, A is divisible. So, the assertion is verified. $\square$

Next, we are in position to state

Theorem 5. *The following are valid:*

($0'$) $U_p(KA)$ *is cyclic* $\iff$ $A = 1$.

($1'$) $U_p(KA)$ *is a direct sum of cocyclics* $\iff$ A/A^1 *is a direct sum of cyclics.*

($2'$) $U_p(KA)$ *is finitely generated* $\iff$ G *and* $U_p(K)$ *are both finite.*

($3'$) $U_p(KA)$ *is finite* $\iff$ G *and* $U_p(K)$ *are finite.*

($4'$) $U_p(KA)$ *is elementary* $\iff$ A *and* $U_p(K)$ *are elementary,* $p = 2$ *and* $K \neq K(\varepsilon_2)$.

($5'$) $U_p(KA)$ *is a direct sum of cyclics of the same order* p^t *for* $t \geq 2$ $\iff$ A *is a direct sum of cyclics,* $U_p(K)$ *is a direct sum of cyclics of the same order* p^t *for* $t \geq 2$, $p \neq 2$ *or* $p = 2$ *but* $K = K(\varepsilon_2)$, $A^{p^i} = 1$ *and* $i < t = \operatorname{const}_p(K)$.

(6′) *$U_p(KA)$ is a direct sum of σ-summables $\iff$ A is a direct sum of cyclics.*

(7′) *$U_p(KA)$ is a p^α-projective for some $\alpha \geq \omega$ $\iff$ A is a direct sum of cyclics.*

(8′) *$U_p(KA)$ is divisible $\iff$ A is divisible and $U_p(K) = 1$.*

(9) *$U_p(KA)$ is simply presented $\iff$ A/A^1 is a direct sum of cyclic groups.*

(10) *$U_p(KA)$ is totally projective $\iff$ A is a direct sum of cyclic groups.*

(11) *$U_p(KA)$ is a direct sum of countable groups $\iff$ A/A^1 is a direct sum of cyclic groups.*

(12) *$U_p(KA)$ is algebraically compact $\iff$ A/A^1 is bounded.*

(13) *$U_p(KA)$ is summable $\iff$ A is a direct sum of cyclic groups.*

Proof. Since $U_p(KA) = S(KA) \times U_p(K)$ and $U_p(K)$ is cyclic, following step by step the method for proof of the preceding Theorem 4 and [2], we conclude obviously that the listed dependences are true. The theorem is proved. □

The following affirmation is crucial for our further investigation.

Lemma [9]**.** *The subgroup A is pure in $S(KA)$.*

Thus if A is torsion-complete, then A is a direct factor of $S(KA)$, hence of $V(KA)$ by [10]. The structure of the complementary factor $S(KA)/A$ is unknown yet. Probably it is a direct sum of cyclic groups.

Lemma 6. *Let $G = B \times C$ be an abelian group and let R be any commutative unitary ring. Then $S(RG) = S(RB) \times [S(RG) \cap (1 + I(RG; C))]$.*

Proof. For $x \in S(RG)$ we write $x = \sum_{c\in C} f_{bc}c = \sum_{c\in C} f_{bc} + \sum_{c\in C} f_{bc}(c-1)$, whenever $f_{bc} \in RB$. Moreover, there is a positive integer t with $x^{p^t} = 1$. Henceforth, $1 = (\sum_{c\in C} f_{bc})^{p^t} + (\sum_{c\in C} f_{bc}(c-1))^{p^t} + \cdots$ and from [3], $1 - (\sum_{c\in C} f_{bc})^{p^t} \in RB \cap I(RG; C) = I(RB; B \cap C) = 0$. Thus, $(\sum_{c\in C} f_{bc})^{p^t} = 1$ and $\sum_{c\in C} f_{bc} \in S(RB)$. Finally, we obtain $x = \sum_{c\in C} f_{bc}(1+(\sum_{c\in C} f_{bc})^{-1}(\sum_{c\in C} f_{bc}(c-1))) \in S(RB)(1+I(RG; C))$. The fact that the intersection of this production is equal to 1 follows by ideas from [3]. This verifies the proof of the assertion. □

Well, we come now to one of the significant attainments.

Theorem 7 (Direct Factor)**.** *Suppose that A is a direct sum of p-primary cyclic groups. Then $S(KA)/A$ is a direct sum of cyclic groups and thus A is a direct factor of $S(KA)$ with a direct sum of cyclics complement. More generally, A is a direct factor of $V(KA)$.*

Proof. Write $A = \coprod_{n=1}^{\infty} A_n$, where $A_n = \coprod_{\alpha_n} \langle p^n \rangle$ is homogeneous of order p^n. Set $H_n = A_1 \times \cdots \times A_n$. It is clear that $A = \bigcup_{n=1}^{\infty} H_n$, $H_n \subseteq H_{n+1}$ and $H_n \cap A^{p^n} = 1 = H_n^{p^n}$. That is why $S(KA) = \bigcup_{n=1}^{\infty} S(KH_n) = \bigcup_{n=f}^{\infty} S(KH_n)$, where $S(KH_n) \subseteq S(KH_{n+1})$, $S^{p^n}(KH_n) = 1$ and $f = \mathrm{const}_p(K)$. Apparently $S(KA)/A = \bigcup_{n=f}^{\infty} [S(KH_n)A/A]$. Since we observe that $A = H_n \times M$ for some group M, consulting with Lemma 6 we have $S(KA) = S(KH_n) \times T$, where $T = S(KA) \cap [1 + I(KA; M)]$. Consequently $S(KA)/A = S(KH_n)A/A \times TA/A$. Really, it is enough to show that $[S(KH_n)A] \cap [TA] = A$, or owing to the modular law in [10] the relation is equivalent to $[S(KH_n)A] \cap T \subseteq A$. In fact, $[S(KH_n)A] \cap [S(KA) \cap (1 + I(KA; M))] = [S(KH_n)A] \cap (1 + I(KA; M)) \subseteq A$ adapting the technique described in [3]. Therefore $S(KH_n)A/A$ must be pure in $S(KA)/A$ and in conclusion, $[S(KH_n)A/A] \cap [S(KA)/A]^{p^n} = [S(KH_n)A/A]^{p^n} = S^{p^n}(KH_n)A/A = 1$. By virtue of the important criterion due to L. Kulikov [10], we establish that $S(KA)/A$ is a direct sum of cyclics. Finally, complying with the purity Lemma along with an other classical theorem of L. Kulikov, argued in [10], we derive that A is a direct factor of $S(KA)$ and the complement is isomorphic to $S(KA)/A$. The final part follows via [17, Proposition 1.6] since $S(KA)$ is a direct factor of $V(KA)$. The proof is completed. □

The following is a direct consequence of the above central theorem.

Corollary 8. *Given that A is a direct sum of p-torsion cyclics. Then $U_p(KA)/A$ is a direct sum of cyclics and so A is a direct factor of $U_p(KA)$ with a direct sum of cyclics complementary factor. Thus A is a direct factor of $U(KA)$.*

Proof. Since $U_p(KA) = S(KA) \times U_p(K)$ where $U_p(K)$ is cyclic and according to the previous assertion, we can deduce that $U_p(KA) = A \times S(KA)/A \times U_p(K)$, as desired. This fulfilles the proof. □

Next, we concentrate on the problem for the basic subgroup in commutative semisimple group algebras. First we start with one key

Proposition 9. *Suppose $H \leq A$ is a subgroup of the separable group A. Then $S(KH) \subseteq B_{S(KA)}$ if and only if $H \subseteq B_A$.*

Proof. "necessity". Consuming a result of L. Kovacs documented in [10], we may write $S(KH) = \bigcup_{n=1}^{\infty} S_n$ so that $S_n \subseteq S_{n+1}$ and $S_n \cap S^{p^n}(KA) = 1$.

Furthermore $H = \bigcup_{n=1}^{\infty} (S_n \cap H)$, where $S_n \cap H \subseteq S_{n+1} \cap H$ and $S_n \cap H \cap A^{p^n} \subseteq S_n \cap S^{p^n}(KA) = 1$, as required.

"sufficiency". We write $H = \bigcup_{n=1}^{\infty} H_n$, $H_n \subseteq H_{n+1}$ and $H_n \cap A^{p^n} = 1$. Therefore $S(KH) = \bigcup_{n=1}^{\infty} S(KH_n) = \bigcup_{n=f}^{\infty} S(KH_n)$, where $f = \mathrm{const}_p(K)$. Clearly $S(KH_n) \subseteq S(KH_{n+1})$. Besides, we calculate that $S(KH_n) \cap S^{p^n}(KA) = 1$. In fact, choose x to belongs to the last intersection. Thus $x \in S(KH_n)$ whence $x \in S(KF)$, where $F \subseteq H_n$ is a finite direct factor of A (for instance, cf. [10]). Consequently as a direct factor $S(KF)$ is pure in $S(KA)$ and following the proof of Proposition 11 in [12], we detect that $S^{p^n}(KF) = 1$. Finally, the above mentioned Kovacs criterion completes the proof. □

As an immediate consequence, we extract a nontrivial relation, namely (announced in [1, Theorem 12]).

Corollary 10. $S(KB_A) \subseteq B_{S(KA)}$ *provided* A *is separable.*

Proof. Follows automatically from the last proposition at setting $H = B_A$. □

Further, we formulate one attainment announced as [1, Theorem 12].

Proposition 11. *The following isomorphism holds*

$$B_{S(KA)} \cong S(KB_A).$$

Proof. If A is finite, then $A = B_A$. In conjunction with (2-3), $S(KA)$ is finite hence $B_{S(KA)} = S(KA) = S(KB_A)$.

Let us now A be infinite. First assume that A is separable. Hence $S(KA)$ is the same by [12, 13], and we elementary see that [10] is applicable to obtain that $S(KA)$ and $B_{S(KA)}$ have equal Ulm-Kaplansky invariants. On the other hand, employing [13], we derive that $S(KB_A)$ and $S(KA)$ also have equal Ulm-Kaplansky functions. Finally from [10] we find that $B_{S(KA)}$ and $S(KB_A)$ must be isomorphic, proving the first half.

For the general part, taking into account that $S(KA) \cong S^1(KA) \times S(K(A/A^1))$ (see cf. [12]) where $S^1(KA)$ is divisible and consulting with [10, p.185, Exercise 8], we obtain $B_{S(KA)} \cong B_{S(K(A/A^1))}$. But by what we have just shown, $B_{S(K(A/A^1))} \cong S(KB_{A/A^1}) = S(K(B_A A^1/A^1)) \cong S(K(B_A/B_A \cap A^1)) \cong S(K(B_A/B_A^1)) \cong S(KB_A)$, as claimed. The proof is verified. □

In the abelian group theory, it is well-known that (by Khabbaz – see the bibliography of [10]), the infinite abelian p-group A is said to be starred if $|A| = |B_A|$. Evidently all finite groups are starred, but the divisible groups are not from this group class.

As a valuable consequence to the last affirmation, we may deduce

Proposition 12. *Given A is an abelian p-torsion group. Then $S(KA)$ is starred if and only if A is starred, and $U_p(KA)$ is starred if and only if so is A.*

Proof. We shall consider only the infinite case since (2) completes the situation when A is finite. Therefore B_A is infinite because otherwise point (2) does imply that $S(KB_A)$ is finite whence Proposition 11 together with the suppositions yield that $S(KA)$ is finite that is a contradiction. As we have seen, Proposition 11 guarantees that $|B_{S(KA)}| = |S(KB_A)|$. Because of this that $|S(KB_A)| = |B_A|$ and $|S(KA)| = |A|$ (see [12]), we establish $|B_{S(KA)}| = |S(KA)|$ only when $|B_A| = |A|$. This verifies the first half.

For the second part, we take into account that $U_p(KA) = S(KA) \times U_p(K)$, where $U_p(K)$ is cyclic. Foremost, if A is finite, (3') will imply that so is $U_p(KA)$ whence starred. Otherwise, $|U_p(KA)| = |S(KA)| > |U_p(K)|$ and the further proof is trivial. The proposition is shown. □

Claim 13. *Suppose $H \leq A$. If $S(KH)$ is a basic subgroup of $S(KA)$, then H is so in A.*

Proof. The subgroup $H \subseteq S(KH)$ is a direct sum of cyclics utilizing [10].

Besides, by virtue of the purity Lemma, H is pure in $S(KH)$ hence in $S(KA)$ and so from [10], H is pure in A.

After this, since $S(KA)/S(KH)$ is divisible, the same holds and for the epimorphic image $S(KA)/\ker\varphi$, where $S(KH) \subseteq \ker\varphi$ and $\varphi\colon S(KA) \to S(K(A/H))$ is an epimorphism (cf. [12]). Consequently $S(KA)/\ker\varphi \cong S(K(A/H))$ is divisible and thus (8) is applicable to get that A/H is divisible, as required. This completes the proof. □

Remark 14. Oppositely to the last assertion, if H is basic in the separable p-group A, then $S(KH)$ is pure in $S(KA)$ and $S(KH)$ is a direct sum of cyclics (see cf. [12]). But whether or not $S(KA)/S(KH)$ is divisible is unknown yet. If yes, $S(KH)$ would be a basic subgroup of $S(KA)$.

The Generalized Direct Factor Problem in the commutative modular group aspect asks does $S(LA)/A$ is totally projective whenever A is reduced and L is perfect. By a reason of symmetry, in the commutative semisimple case, we can state

Problem 15. *Whether or not $S(KA)/A$ is totally projective or simply presented.*

However, the following sheds some light in this direction, namely:

Proposition 16. (*a*) *If A is separable, the quotient group $S(KA)/A$ is separable or equivalently A is nice in $S(KA)$;*

(*b*) *$S(KA)/A$ is totally projective if and only if $S(KA)/A$ is a direct sum of cyclics.*

Proof. (*a*) Given $x \in S(KA)/A$, hence $x \in S(KF)A/A \cong S(KF)/F$ for some finite subgroup F of A. Therefore, consuming (2), $S(KF)/F$ is finite whence separable. On the other hand F should be a direct factor of A. Therefore, as we previously have seen above, $S(KF)A/A$ must be a direct factor of $S(KA)/A$, thus it is its pure subgroup. That is why, for arbitrary $y \in (S(KF)A/A) \cap (S(KA)/A)^1 \cong (S(KF)/F)^1 = 1$ we have $y = 1$, so we are done.

(*b*) By making use of the purity Lemma, $S^1(KA)/A^1 \cong S^1(KA)A/A \subseteq S(KA)/A$ is indebted to be divisible and reduced, owing also and to the above cited Mollov's result in [12]. Therefore $S^1(KA) = A^1$. But then $A^1 = 1$. Really, if not, there exists $1 \neq a \in A^1$. Let us now e_1 and e_2 be minimal ortogonal idempotents for some finite subgroup $F \leq A$ such that $F \cap \langle a \rangle = 1$ (F may be choosen to be a finite direct factor of A whence $F \cap A^1 = 1$), i.e. ${e_1}^2 = e_1$, ${e_2}^2 = e_2$ plus $e_1 + e_2 = 1$ and $e_1 e_2 = 0$ in KF for such a group F. Thus, the element $e_1 + e_2 a$ clearly belongs to $S^1(KA)$. That is why $e_1 + e_2 a$ lies in A^1, but this is impossible because it is an element in canonical form. The obtained contradiction extract our claim.

By what we have just argued $S(KA)/A$ reduced yields that A is separable. And so, since by (*a*) the factor group $S^1(KA)A/A$ is equal to $(S(KA)/A)^1$, i.e. $S(KA)/A$ is separable, consulting with [10] we are done. □

Furthermore, in the remaining case when A is not separable and A^1 is not divisible, we have doubts about the validity of the Direct Factor Conjecture. In more precise words, A is not however a direct factor of $S(KA)$.

So, we reformulate Problem 15 in the following way.

Problem 17. *If A is separable, does it follow that $S(KA)/A$ is a direct sum of cyclics?*

In the case of separable p-groups the Mollov's formula is trivially satisfied, hence another idea to solve the stated question in negative or in affirmative is necessary. For direct sums of p-cyclics, it was done via Theorem 7. Conforming with the representation theorem for separable abelian p-groups (see [10], p.24, Corollary 68.2 – L. Kulikov), we observe that it is enough to show that $S(KA)/A$ is a direct sum of cyclic groups provided A is torsion complete only. In this direction, if there exists a commutative unitary ring E of prime characteristic p such that $K \subset E$, we will be done since we have proved that $S(EA)/A$ is a direct sum of cyclics provided A is separable, so the same will be valid and for its subgroup $S(KA)/A$.

Well, we conjecture that Problem 17 holds true in a positive way, and so as we have seen $S(KA) \cong A \times S(KA)/A$ where $S(KA)/A$ is a direct sum of cyclics. Thus, for the full description of $S(KA)$, the Ulm-Kaplansky functions of $S(KA)/A$ must be computed (although Mollov has claimed that the calculated in [13] Ulm-Kaplansky invariants of $S(KA)$ are sufficient that is, of course, wrong). However, this is a work of some other research study.

4. Concluding remarks and problems. From the begining we mention the nice fact due to N. Nachev (for example, see [2]) that $U_p(QA) = U_p(RA)$ and $S(QA) = S(RA)$ whenever $R = Z[\frac{1}{p}]$ is the ring of all rational numbers so that their denominators are a power of the prime number p. Thus all proved statements concerning the group ring QA may be replaced with such similar claims for RA.

After this, we list once again (see [2]) the conjecture that $U_p(KA)$, respective $S(KA)$, is quasi complete (in particular torsion complete) if and only if A is bounded. In the case for torsion completeness, the reader can see (cf. [9]). An interesting problem is also what is the criterion (i.e. the necessary and sufficient condition) illustrated $S(KA)$ to be quasi pure injective (q.p.i.) and quasi pure projective (q.p.p.)? For other interesting group classes, we refer the reader also to the papers [3, 5, 6, 7].

And as a final discussion, a global problem is the theme for finding of the basic subgroup of $S(KA)$ which, as we have seen, must be isomorphic to $S(KB_A)$, but probably they are not equal.

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