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ON ARRANGEMENTS OF REAL ROOTS OF A REAL POLYNOMIAL AND ITS DERIVATIVES

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Abstract. We prove that all arrangements (consistent with the Rolle theorem and some other natural restrictions) of the real roots of a real polynomial and of its s-th derivative are realized by real polynomials.

In the present paper we consider a real polynomial of one real variable $P(x,a) = x^n + a_1x^{n-2} + \ldots + a_{n-1}$. We are interested in the question what *arrangements* between the real roots of P and $P^{(s)}$ are possible $(1 \leq s \leq n-1)$. To define an arrangement means to write down the roots of P and $P^{(s)}$ in a chain in which every two consecutive roots are connected either by an equality or by an inequality $<$. The arrangement α is said to belong to the closure of

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the arrangement β if it is obtained from β by replacing some inequalities by equalitites. The results are the first step towards the study of real discriminant sets $\{a \in \mathbf{R}^{n-1} | \mathrm{Res}(P, P^{(s)}) = 0\}$.

In an earlier paper [3] it is shown that if P is *hyperbolic*, i.e. with n real roots, then *the standard Rolle restrictions* are necessary and sufficient conditions for a root arrangement to be realizable (see Theorems 2 and 4.4 in [3]). Namely, denote by $x_1 \leq \ldots \leq x_n$ the roots of P and by $\xi_1 \leq \ldots \leq \xi_{n-s}$ the ones of $P^{(s)}$ (which is also hyperbolic). Then one has

$$x_l \leq \xi_l \leq x_{l+s} \tag{1}$$

for $l = 1, \ldots, n-s$ and every arrangement of the roots of P and $P^{(s)}$ which is consistent with (1) is realizable. One presumes also that the following conditions hold:

A) *If a root of P of multiplicity $d > s$ coincides with a root of $P^{(s)}$ of multiplicity g, then $g = d - s$ (self-evident).*

B) *If a root ξ of $P^{(s)}$ coincides with a root of P of multiplicity $\kappa \leq s$, then ξ is a simple root of $P^{(s)}$ (see* [3]*, Lemma* 4.2*) and one has $\kappa \leq s-1$.*

C) *If $x_l = \xi_l$ or $x_{l+s} = \xi_l$, then $x_l = x_{l+1} = \ldots = x_{l+s} = \xi_l$ (self-evident for $s = 1$ and easy to prove by induction on s for $s > 1$).*

Example 1. If $n = 2$, $s = 1$, then there are two possible arrangements (i.e. consistent with (1), A) B) and C)) : $x_1 < \xi_1 < x_2$ and $x_1 = \xi_1 = x_2$. They are both realizable by hyperbolic polynomials.

In the present paper we treat the case when P is arbitrary (not necessarily hyperbolic). (Notice that $P^{(s)}$ can be hyperbolic even if P is not.)

Definition 2. *Suppose that P has m conjugate couples of complex roots and $n - 2m$ real roots. Then a priori $P^{(s)}$ has at least $n - 2m - s$ real roots counted with the multiplicities. Indeed, a real root of $P^{(i)}$ of multiplicity $l \geq 1$ is a root of $P^{(i+1)}$ of multiplicity $l - 1$ and between every two real roots of $P^{(i)}$ there is a root of $P^{(i+1)}$. Iterating this rule s times one obtains the existence of $n - 2m - s$ real roots of $P^{(s)}$ (we call them* Rolle *roots) which together with the real roots of P satisfy conditions (1), A) and B). A Rolle root is multiple only if it coincides with a root of P of multiplicity $> s$. Eventually, $P^{(s)}$ can have $\leq 2m$*

*other (*non-Rolle*) real roots counted with the multiplicities some (or all) of which can coincide with Rolle ones. Which real roots of $P^{(s)}$ should be chosen as Rolle and which as non-Rolle ones is not always uniquely defined and when it is not we assume that a choice is made.*

Example 3. The polynomial $x^6 - x^2 = x^2(x^2 - 1)(x^2 + 1)$ has real roots $x_1 = -1$, $x_2 = x_3 = 0$, $x_4 = 1$ (and complex roots $\pm i$). One has $P' = 6x^5 - 2x = 2x(\sqrt{3}x^2 - 1)(\sqrt{3}x^2 + 1)$, i.e. P' has three Rolle roots (and no non-Rolle ones) – 0 and $\pm 1/3^{1/4}$ where 0 is a common root for P and P', see A). It has also two complex roots $\pm i/3^{1/4}$. One has $P'' = 30x^4 - 2$, i.e. P'' has two Rolle roots $\pm 1/15^{1/4}$, no non-Rolle ones and two complex roots $\pm i/15^{1/4}$. One has $P''' = 120x^3$, i. e. P''' has a triple real root at 0 and no complex roots. One copy of this real root should be considered as a Rolle one and the other two as non-Rolle ones.

Proposition 4. *Suppose that a real root of P of multiplicity d coincides with a real root of $P^{(s)}$ of multiplicity g. Then*

1) *if $d > s$, then one has $g = d - s$; in this case this is a Rolle root of $P^{(s)}$ of multiplicity $d - s$;*

2) *if $0 \leq d \leq s$, then one has $g \leq 2m + 1$ (and if $g \geq 1$, then $d < s$).*

Observe that in the above example one has $m = 1$ and for $s = 3$ the estimation $2m + 1$ is attained by the multiplicity of 0 as a root of P'''. The proposition generalizes conditions A) and B) in the case of arbitrary m.

P r o o f. Part 1) is self-evident. Prove part 2). If the root is non-Rolle and does not coincide with a Rolle one, then its multiplicity is $\leq 2m$. If the root is Rolle and does not coincide with a non-Rolle one, then either it coincides with a root of P of multiplicity $> s$ and we are in case 1) or it is a simple root. Finally, if the root is Rolle and coincides with a non-Rolle one, then the Rolle root must be simple (otherwise there will be a contradiction with part 1)) and the sum of their two multiplicities is $\leq 2m + 1$. □

Definition 5. *An arrangement of the real roots of P and $P^{(s)}$ is called* a priori admissible *if there exist $n - 2m - s$ Rolle roots of $P^{(s)}$ in the sense of Definition* 2 *and if conditions* 1) *and* 2) *of Proposition* 4 *hold.*

Theorem 6. *All a priori admissible root arrangements are realizable by real polynomials of degree n.*

P r o o f. 1^0. We explain first in 1^0 – 7^0 why all a priori admissible arrangements in which the derivative $P^{(s)}$ is hyperbolic and which are the *least generic* are realizable. "Least generic" means that all non-Rolle roots of $P^{(s)}$ coincide with Rolle ones or with roots of P . The general case is treated in 8^0 – 11^0.

To realize an a priori admissible arrangement with $P^{(s)}$ hyperbolic and with the necessary multiplicities of the real roots of P consider the family of polynomials

$$P(x,w,g,t) = \prod_{j=1}^{q}(x-w_j)^{m_j}\prod_{j=1}^{m}((x-g_j)^2+t_j^2) \tag{2}$$

where w_j, $j=1,\ldots,q$, are the real roots of P, of multiplicities m_j ($w_0 = 0 \leq w_1 \leq \ldots \leq w_q \leq 1 = w_{q+1}$), and $g_j \pm it_j$ are its complex roots (not necessarily distinct), $t_j \geq 0$, $0 \leq g_j \leq 1$. We allow here equalities between the roots w_j for convenience; it will be shown that the necessary arrangement is realized for roots with strict inequalities between them.

Denote by $\xi_1 \leq \ldots \leq \xi_{n-s}$ the real parts of the roots of $P^{(s)}$ ($n-2m-s$ of them are just Rolle roots) and by $\theta_1 \leq \ldots \leq \theta_m$ the biggest nonnegative imaginary parts of the roots of $P^{(s)}$ (recall that for a least generic arrangement one has $\theta_j = 0$). Set $\xi_0 = 0$, $\xi_{n-s+1} = 1$. (Notice that $P^{(s)}$ has not more conjugate couples of complex roots than P, i.e. not more than m.) The functions ξ_i, θ_j are continuous in (w,g,t).

2^0. Suppose that for the desired arrangement of the real roots of P and $P^{(s)}$ the Rolle and non-Rolle roots of $P^{(s)}$ are fixed. Denote the non-Rolle roots by $u_1 \leq \ldots \leq u_{2m}$. Impose additional requirements upon the numbers g_j as follows: if the non-Rolle roots with odd indices $u_{2p-1}, u_{2p+1}, \ldots, u_{2p+2p'-1}$ belong to the interval $[w_j, w_{j+1})$, $j < q$, or to $[w_q, w_{q+1}]$, then we require that $w_j \leq g_p \leq \ldots \leq g_{p+p'} \leq w_{j+1}$. Define the variables $h_1 \leq \ldots \leq h_{q+m}$ as the union of the variables w_j ($j = 1,\ldots,q$) and g_i ($i = 1,\ldots,m$) with the order defined above. Hence, they belong to the unit simplex Σ_{q+m}.

3^0. In what follows we assume that the variables t_j belong to some interval $[0, N]$ where $N > 1$. We define with the help of the variables h_j, t_i continuous functions η_j, ζ_i such that $(\eta_1, \ldots, \eta_{q+m}) \in \Sigma_{q+m}$, $\zeta_i \in [0, N]$. The set $\mathcal{S} = \Sigma_{q+m} \times [0, N]^m$ is homeomorphic to Σ_{q+2m}. By the Brouwer fixed point theorem (see [1], p. 57), there exists a fixed point of the mapping $\tau : \mathcal{S} \to \mathcal{S}$, $\tau : (h, t) \mapsto (\eta, \zeta)$, i.e. a point where one has $\eta_j = h_j$, $\zeta_i = t_i$. The functions η_j, ζ_i are defined such that the arrangement of the real roots of P and $P^{(s)}$ at the fixed point is the required one.

4^0. Define the functions η_j by the following rules:

1) We want to achieve the additional conditions (at the fixed point) $g_p = u_{2p-1}$, …, $g_{p+p'} = u_{2p+2p'-1}$ for all appropriate indices, see 2^0; therefore we set $\eta_{i_1} = \xi_{i_2}$ whenever h_{i_1} is a variable g_{p+l} and ξ_{i_2} is the corresponding function $u_{2p+2l-1}$;

2) If a variable h_j, which is a root w_i of multiplicity $< s+1$, must coincide with a simple root ξ_k of $P^{(s)}$ or, more generally, with the roots $\xi_k = \xi_{k+1} = \ldots = \xi_{k+l}$, then we set $\eta_j = \xi_k$;

3) If the variables $h_r < h_{r+1} < \ldots < h_{r+l}$ (which are all consecutive roots w_j and among which there might be roots w_j of multiplicity $\geq s+1$) lie between the Rolle roots ξ_k and ξ_{k+v} of $P^{(s)}$ and all roots among the roots ξ_{k+1}, …, ξ_{k+v-1} (if $v > 1$) coincide with roots w_j ($r \leq j \leq r+l$) of multiplicity $\geq s+1$, then we set

$$\eta_{r+j} = \xi_k + (j+1)(\xi_{k+v} - \xi_k)/(l+2),\ j = 0, 1, \ldots, l.$$

Remark 7. It follows from rules 1) – 3) that there are $q+m$ functions η_j – as many as the variables h_j.

Recall that the arrangement is least generic, i.e. for every non-Rolle root ξ_i of $P^{(s)}$ one has either $\xi_i = \xi_{i_1}$ where ξ_{i_1} is a Rolle one or $\xi_i = w_{i_2} = h_j$ for some i_2, j. Denote by l_1, …, l_{2m} the absolute values $|\xi_i - \xi_{i_1}|$ and $|\xi_i - w_{i_2}|$ for all i, i_1 and i_2 as above. Set $\Phi = l_1 + \ldots + l_{2m}$ and

$$\zeta_i = \left| t_i - \frac{1}{3m}\sum_{j=1}^{m}\theta_j - \frac{t_i}{3(N+1)^m}|t_1 t_2 \ldots t_m - 1| - \frac{t_i}{12m}\Phi \right| \tag{3}$$

5^0. Denote by t_{i_0} the greatest variable t_i at the fixed point (see 3^0). Observe first that one can assume that $t_{i_0} > 0$. Indeed, if $t_{i_0} = 0$, then $t_i = 0$ for all i, P is hyperbolic and the roots of P and $P^{(s)}$ define an arrangement α from the closure of the desired least generic one β.

Lemma 8. *For $t_{i_0} = 0$ there exists a real-analytic deformation of P into a real polynomial which together with its s-th derivative defines the arrangement β.*

The lemma is proved after the theorem. It allows one to consider only the case $t_{i_0} > 0$.

6^0. One has

$$\zeta_{i_0} = t_{i_0} - \frac{1}{3m}\sum_{j=1}^{m}\theta_j - \frac{t_{i_0}}{3(N+1)^m}|t_1t_2\dots t_m - 1| - \frac{t_{i_0}}{12m}\Phi \ .$$

Indeed, all roots of $P^{(s)}$ lie within the convex hull of all roots of P (see [4], p. 108). Hence, one has $\theta_j \leq t_{i_0}$, $j = 1, \dots, m$. One has also $|t_1t_2\dots t_m - 1| \leq t_1t_2\dots t_m + 1 < (N+1)^m$ and $\Phi \leq 4m$ (because for each term l_j one has $l_j \leq 2$). Thus

$$\frac{1}{3m}\sum_{j=1}^{m}\theta_j + \frac{t_{i_0}}{3(N+1)^m}|t_1t_2\dots t_m - 1| + \frac{t_{i_0}}{12m}\Phi < mt_{i_0}/3m + t_{i_0}/3 + 4mt_{i_0}/12m = t_{i_0} \quad (4)$$

and for $i = i_0$ one can delete the absolute value sign in the right hand-side of (3). But then to have $\zeta_{i_0} = t_{i_0}$ one must have $\theta_j = 0$ for $j = 1, \dots, m$, $t_1t_2\dots t_m - 1 = 0$ and $l_1 = \dots = l_{2m} = 0$. This means that $t_j \neq 0$, i.e. no root $g_j + it_j$ of P will be real, that $P^{(s)}$ will indeed be hyperbolic ($\theta_j = 0$) and that all non-Rolle roots of $P^{(s)}$ equal either roots w_j of P or Rolle roots of $P^{(s)}$.

Remark 9. The condition $N > 1$ makes possible the choice of the values of the variables t_i so that $t_1t_2\dots t_m - 1 = 0$. One can prove by analogy with (4) that $|\zeta_i| < N$, i.e. the mapping τ is indeed from $\mathcal{S}$ into itself.

7^0. A priori the fixed point assures the existence of an arrangement only from the closure of the necessary one. The fact that at the fixed point no inequality between roots of P is replaced by equality is proved by analogy with 4^0 – 7^0 of the proof of Theorem 4.4 from [3] where the case of P hyperbolic is considered. The proof there shows that equalities replacing inequalities between roots of P imply that a root of P of multiplicity $m \geq s+1$ is a root of $P^{(s)}$ of multiplicity $\geq m-s+1$ which contradicts part 1) of Proposition 4. In the general case (P not necessarily hyperbolic) the proof is essentially the same, the presence of eventual non-Rolle roots can only increase the multiplicity of the root as a root of $P^{(s)}$.

Hence, the fixed point provides the necessary arrangement.

8^0. To obtain (in 8^0 – 9^0) all arrangements in which $P^{(s)}$ is hyperbolic but which are not necessarily least generic we use the same construction but with another function Φ. Namely, consider a family of such functions Φ depending on a parameter $b \in (\mathbf{R}_+, 0)$ defined as follows: if instead of $\xi_i - \xi_{i_1} = 0$, see 4^0, one must have $\xi_i - \xi_{i_1} > 0$ or $\xi_i - \xi_{i_1} < 0$ (and no root ξ_j or w_j lies between ξ_i and ξ_{i_1}), then in Φ we replace the absolute value $l_\nu = |\xi_i - \xi_{i_1}|$ by $|\xi_i - \xi_{i_1} - b|$ (resp. by $|\xi_i - \xi_{i_1} + b|$); in the same way for $\xi_i - w_{i_2}$, see 4^0. In a sense, we obtain the not least generic arrangements by deforming least generic ones the deformation parameter being b.

9^0. Denote by $F(b)$ the set of fixed points of the mapping τ from 3^0. For b small enough one has $(\eta, \zeta) \in \mathcal{S}$. The set $F(0)$ contains all limit points of the family of sets $F(b)$ when $b \to 0$ and there exists at least one such limit point because all sets $F(b)$ (for b small enough) are non-empty and belong to $\mathcal{S}$ which is compact. Hence, one can choose $b > 0$ small enough and a fixed point of $F(b)$ at which there is an inequality between two roots in the arrangement if there is an inequality in the arrangement for $b = 0$, and the equalities $\xi_i - \xi_{i_1} = 0$ or $\xi_i - w_{i_2} = 0$ where this is necessary are replaced by the desired inequalities.

10^0. Obtain all arrangements in which $P^{(s)}$ is not hyperbolic and which are least generic. Suppose that $P^{(s)}$ must have exactly m' conjugate couples of complex roots. In this case we assume that m' of the couples of roots $g_j \pm it_j$ are replaced by a couple $\pm iv$ where $v > 0$ is "large", i.e. much bigger than N. Hence, $P^{(s)}$ also has exactly m' couples of conjugate complex roots with "large" imaginary parts. One has

$$Q := P/v^{2m'} = (1 + x^2/v^2)^{m'} \prod_{j=1}^{q} (x - w_j)^{m_j} \prod_{j=1}^{m-m'} ((x - g_j)^2 + t_j^2) \quad ,$$

i.e. the family Q is a one-parameter deformation of a family of polynomials like (2) (the role of the small parameter is played by $1/v^2$) and the existence of the necessary arrangements can be deduced by analogy with 1^0 – 7^0 (see 9^0 for the role of the small parameter; however, the function Φ is the one from 1^0 – 7^0).

11^0. To obtain the existence of all arrangements (which are not necessarily least generic and with $P^{(s)}$ not necessarily hyperbolic) one has to combine 8^0, 9^0 and 10^0. The theorem is proved. □

P r o o f o f L e m m a 8. 1^0. We assume that P has the same number of distinct real roots as in the desired arrangement β, otherwise one can deform P within the class of hyperbolic polynomials to obtain this condition while remaining in the closure of β. See [2] for such deformations. We begin with two observations:

1) for $a > 0$, $\mu \in \mathbf{N} \cup \{0\}$ and ν even the polynomial $Q = x^\mu(x^\nu + a)$ has a μ-fold root for $x = 0$ and its s-th derivative for $s > \mu$ has a $(\mu + \nu - s)$-fold one; Q has also $\nu/2$ couples of conjugate complex roots;

2) with a, μ and ν as above, the polynomial $Q_1 = x^\mu(x^\nu + a + aQ_2(x, a))$ where Q_2 is a polynomial in x of degree $\leq \nu - 1$, $Q_2(0, a) \equiv 0$, has ν complex zeros for a small enough and a real μ-fold root at 0; to see this set $a = c^\nu$, $x = cy$; one has $Q_1(cy, c^\nu) = c^{\mu+\nu}y^\mu(y^\nu + 1 + Q_2(cy, c^\nu))$; the last polynomial has a μ-fold root at 0 and ν roots which for c small enough are close to the roots of $y^\nu + 1$, hence, are complex.

2^0. Suppose that the polynomial P of degree n realizing with $P^{(s)}$ the arrangement α has a real root of multiplicity $\mu+\nu$ (with ν even) which (in order to obtain the arrangement β) must split into $\nu/2$ couples of conjugate complex roots and into a real root of multiplicity μ. (If several roots of P must split, we make them split one by one.) Suppose in addition that in the deformed polynomial (denoted by R) the real root of multiplicity μ must coincide with a root of $R^{(s)}$ of multiplicity $\mu + \nu - s$. Assume that the bifurcating root is at 0 and that

$$P = x^{\mu+\nu}(1 + h(x)) \quad , \quad h(0) = 0 \tag{5}$$

(P is not necessarily monic). Construct the necessary deformation of P in the form

$$R(x, a) = x^\mu(x^\nu + a + b_{s-\mu}x^{s-\mu} + \ldots + b_{\nu-1}x^{\nu-1})(1 + g(x, a)) \tag{6}$$

where $a \in (\mathbf{R}, 0)$ and $b_i = b_i(a)$ and $g(x,a)$ ($g(0,a) \equiv 0$) are defined such that all equalities of the form $x_i = \xi_j$ defining the arrangement β will be preserved.

3^0. Suppose first that in (6) one has $g(x,a) \equiv h(x)$. The condition

$$(A) \quad : \quad R^{(s)} \text{ has a } (\mu+\nu-s)\text{-fold root at } 0$$

is a triangular linear non-homogeneous system with unknown variables b_i; the system defines unique functions $b_i = b_i^* a$, $b_i^* \in \mathbf{R}$. This can be checked directly.

Suppose that in (6) one has $g = h(x) + \sum_{j=1}^{l} d_j h_j(x,d)$ where $d = (d_1, \ldots, d_l) \in (\mathbf{R}^l, 0)$ and h_j depend smoothly on d. Then condition (A) defines unique functions $b_i(a,d) = b_i^* a + a \sum_{j=1}^{l} d_j \tilde{b}_{i,j}(d)$ where $b_i^* \in \mathbf{R}$ and $\tilde{b}_{i,j}$ are smooth in d. This can also be checked directly.

4^0. For each root $w_j \neq 0$ of P of multiplicity $< s$ which must be equal to a root ξ_i of $P^{(s)}$ denote by d_j the deviation from its position in a deformation of P. Admitting such deviations means that in (5) the function h should be replaced by $h(x) + \sum_{j=1}^{l} d_j h_j(x,d)$.

Denote by (B) the system of all conditions $w_j = \xi_i$ for all such equalities with $w_j \neq 0$ characterizing the arrangement β.

5^0. For any deformation $R^*(x,a,d) = x^\mu(x^\nu + a + b_{s-\mu}x^{s-\mu} + \ldots + b_{\nu-1}x^{\nu-1})(1 + g(x,d))$ of P (where b_k are considered as small parameters) one can find d depending smoothly on a and b_k such that for all a small enough all equalities from (B) hold. This follows from Propositions 11 and 13 from [2] where it is shown that the linearizations of the conditions (B) w.r.t. d are linearly independent. (In [2] their linear independence is proved only when P is hyperbolic; this independence is an "open" property, so it holds for all nearby polynomials as well.)

6^0. The independence of these linearizations implies that for a small enough the system of conditions (B) applied to the deformation

$$\tilde{R}(x,a,d) = x^\mu(x^\nu + a + b_{s-\mu}(a,d)x^{s-\mu} + \ldots + b_{\nu-1}(a,d)x^{\nu-1})(1 + h(x) + \sum_{j=1}^{l} d_j h_j(x,d))$$

(with $b_i(a,d)$ defined as in 3^0) defines unique $d_j = d_j(a)$ smooth in a. Indeed, the linearizations w.r.t. d of the system of conditions (B) from 6^0 and from 5^0 are the same.

On the other hand, b_i were defined such that condition (A) holds. Hence, for $d = d(a)$ and $b_i = b_i(a, d(a))$ (where $a > 0$ is small enough) the $(\mu+\nu)$-fold root of P at 0 splits into a real μ-fold root at 0 and ν complex roots close to 0 (see observation 2) from 1^0) and $P^{(s)}$ has a $(\mu + \nu - s)$-fold root at 0. The arrangement of the other real roots of P and $P^{(s)}$ remains the same. □

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REFERENCES

[1] A. Dold. Lectures on algebraic topology. Classics in Mathematics, Springer, 1980.

[2] Vl. P. Kostov. On arrangements of the roots of a hyperbolic polynomial and of one of its derivatives. Electronic preprint math.AG/0211132.

[3] Vl. P. Kostov, B. Z. Shapiro. On arrangements of roots for a real hyperbolic polynomial and its derivatives. *Bull. Sci. Math.* **126** (2002), 45–60.

[4] G. Polya, G. Szegö. Problems and Theorems in Analysis, vol. 1. Springer-Verlag, 1972.

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