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A CHARACTERIZATION OF WEAKLY LINDELÖF DETERMINED BANACH SPACES*

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Abstract. We prove that a Banach space X is weakly Lindelöf determined if (and only if) each non-separable Banach space isomorphic to a complemented subspace of X has a projectional resolution of the identity. This answers a question posed by S. Mercourakis and S. Negrepontis and yields a converse of Amir-Lindenstrauss' theorem. We also prove that a Banach space of the form $C(K)$ where K is a continuous image of a Valdivia compactum is weakly Lindelöf determined if (and only if) each non-separable Banach space isometric to a subspace of $C(K)$ has a projectional resolution of the identity.

1. Introduction. Projectional resolutions of the identity (or, shortly, PRI's) are a powerful tool in studying the structure of non-separable Banach

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spaces. For applications see e. g. [29] or [6, Section 6.2]. First projectional resolutions were constructed by J. Lindenstrauss [16], [17]. Their importance became obvious after the famous paper of D. Amir and J. Lindenstrauss [1] where it is proved that every nonseparable weakly compactly generated Banach space admits a PRI. Let us remark that WCG spaces are stable to isomorphisms and to taking complemented subspaces. Hence it follows from [1] that if X is WCG, then each nonseparable Banach space isomorphic to a complemented subspace of X admits a PRI.

The result of [1] was later extended to several larger classes of Banach spaces. In [28] it was extended to weakly countably determined spaces (called also Vašák spaces by some authors). This class contains all subspaces of WCG spaces, and even all weakly K-analytic Banach spaces introduced by M. Talagrand [23], [24]. A further generalization was proved by M. Valdivia [25] who proved the existence of a PRI in weakly Lindelöf determined (WLD) spaces (see [3]). Again, each of these larger classes is stable to isomorphisms and to taking (even arbitrary, not necessarily complemented) subspaces.

Another yet larger class of Banach spaces admitting a PRI is formed by 1-*Plichko spaces*. It was proved in various settings and in different degrees of generality in [22], [26], [27]; our terminology follows [14] and [15]. However, this class is stable neither to isomorphisms (see [9], [7] and [13]) nor to taking (complemented) subspaces (see [9] and [12]).

In view of these results it is natural to ask 'converse questions'. What is the largest class of Banach spaces admitting a PRI? It was observed in [7] that a Banach space of density $\aleph_1$ is 1-Plichko whenever it admits a PRI. However, for larger densities the analogous statement is not valid; for some partial characterization of such spaces see [15, Theorem 4.14]. While the just mentioned result is rather technical and not easy to formulate, if we search for the largest class of Banach spaces admitting a PRI, which has some nice stability properties, the answer could be more clear. In [18, § 4, p. 517] the following question was asked.

Problem. *Let X be a Banach space such that each nonseparable Banach space isomorphic to a complemented subspace of X admits a PRI. Is then X weakly Lindelöf determined?*

It follows from the results of [13] that a Banach space of density $\aleph_1$ is WLD whenever it has a PRI in every equivalent norm. In particular, this yields the positive answer to the above problem within spaces of density $\aleph_1$. In the present paper we prove the positive answer for spaces of an arbitrary density. Let us remark that there are spaces which are not WLD but still have a PRI in every equivalent norm (see [13]).

Having answered the above problem, it is natural to ask the following question.

Question. *Let X be a Banach space such that each nonseparable Banach space isometric to a (complemented) subspace of X admits a PRI. Is then X weakly Lindelöf determined?*

We also give a partial positive answer. However, this question in full generality seems to be open.

We start by the basic definitions. Let us begin with PRI's. For technical reasons we give its definition in two steps.

Definition 1. *Let $(X, \|\cdot\|)$ be a Banach space. By a* long sequence of projections *on X we mean an indexed family $(P_\alpha : \omega \le \alpha \le \mu)$, where μ is a limit ordinal, of projections on X satisfying the following conditions.*

(*i*) $P_\omega = 0$, $\|P_\alpha\| = 1$ *for* $\omega < \alpha \le \mu$*;*

(*ii*) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ *whenever* $\omega \le \alpha \le \beta \le \mu$*;*

(*iii*) $P_\lambda X = \overline{\bigcup_{\omega \le \alpha < \lambda} P_\alpha X}$ *whenever* $\lambda \in (\omega, \mu]$ *is limit.*

Definition 2. *Let $(X, \|\cdot\|)$ be a nonseparable Banach space with density character μ. By a* projectional resolution of the identity (PRI) *of X we mean a long sequence of projections $(P_\alpha : \omega \le \alpha \le \mu)$ on X satisfying moreover the following conditions.*

(*iv*) $P_\mu = \mathrm{Id}_X$*;*

(*v*) $\operatorname{dens} P_\alpha X \le \operatorname{card} \alpha$ *for* $\omega < \alpha \le \mu$.

This is the classical notion of a PRI. It turns out that this concept is not the best one. In some applications it is too strong, in other situations it seems to be too weak. So let us introduce the following two notions.

Definition 3. *Let $(X, \|\cdot\|)$ be a nonseparable Banach space with density character μ.*

(1) *By a* weak projectional resolution of the identity (weak PRI) *of X we mean a long sequence of projections $(P_\alpha : \omega \le \alpha \le \mu)$ on X satisfying, moreover, the condition (iv) and the following one.*

  (v') $\operatorname{dens} P_\alpha X < \mu$ *for* $\omega < \alpha < \mu$.

(2) *By a* strong projectional resolution of the identity (strong PRI) *of X we mean a long sequence of projections $(P_\alpha : \omega \le \alpha \le \mu)$ on X satisfying, moreover, the condition (iv) and the following one.*

 (v'') $\operatorname{dens} P_\alpha X = \operatorname{card} \alpha$ *for* $\omega < \alpha \le \mu$.

It can be proved that each Banach space from one of the classes mentioned in the introductory paragraphs admits even a strong PRI (cf. [15, Section 4.1]). On the other hand, to be able to use transfinite induction (which is probably the main application of PRI's) it would be sufficient to use a weak PRI.

Now we are going to give the definitions of some classes of spaces closely related to PRI's. These are Corson and Valdivia compacta and associated Banach spaces.

Definition 4.

(1) *For any set Γ we put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\}$ is countable$\}$.*

(2) *A compact space K is called* Corson *if it is homeomorphic to a subset of some $\Sigma(\Gamma)$.*

(3) *A compact space K is called* Valdivia *if it is homeomorphic to a subset $K' \subset \mathbb{R}^\Gamma$ such that $K' \cap \Sigma(\Gamma)$ is dense in K'.*

In Banach spaces we need a counterpart of the notion of a dense set. Therefore we recall definitions of the following two notions.

Definition 5. *Let X be a Banach space. A subspace S of X^* is called* norming (1-norming) *if the norm defined by $|x| = \sup\{|\langle \xi, x\rangle| : \xi \in S, \|\xi\| \le 1\}$ is equivalent (equal, respectively) to the original norm on X.*

Definition 6.

(1) *Let X be a Banach space. We say that $S \subset X^*$ is a* Σ-subspace *of X^* if there is a linear one-to-one weak* continuous mapping $T : X^* \to \mathbb{R}^\Gamma$ such that $S = T^{-1}(\Sigma(\Gamma))$.*

(2) *A Banach space X is called* weakly Lindelöf determined (WLD) *if X^* is a Σ-subspace of itself.*

(3) *A Banach space X is called* Plichko (1-Plichko) *if X^* has a norming (1-norming, respectively) Σ-subspace.*

Plichko and 1-Plichko spaces were studied, using various definitions, for example in [22], [27], [20], [7], [13], [12], [14] and [15]. The above definition and terminology follows the last two named papers.

The class of Corson compact spaces is closely related to the class of WLD Banach spaces. Namely, a Banach space is WLD if and only if the dual unit ball is Corson in it weak* topology ([21, Proposition 4.1], see [7] for a different proof). The relationship between properties of K and $C(K)$ is more complicated – the space $C(K)$ is WLD if and only if K is a Corson compact space with the property (M) defined below (see [4, Theorem 3.5]).

Definition 7. *A compact space is said to have* the property (M) *if each Radon probability measure on it has separable support.*

2. Main results.

In this section we present our main results. The first one is the following answer to the Problem formulated in the introduction.

Theorem 1. *Let X be a Banach space. Then the following assertions are equivalent.*

- (a) *X is weakly Lindelöf determined.*
- (b) *Each nonseparable Banach space which is isomorphic to a subspace of X admits a strong PRI.*
- (c) *Each nonseparable Banach space which is isomorphic to a complemented subspace of X admits a weak PRI.*

The proof of the new implication $(c) \Rightarrow (b)$ of Theorem 1 is based on the results of [13]. Namely, we prove that any space X satisfying (c) is 1-Plichko in any equivalent norm and then use [13] to conclude that X is WLD. This shows that the isometric question is completely different.

We continue by the following partial answer to Question asked in the introduction. Let us remark that by $C(K)$ we mean the space of all real-valued continuous functions on the compact space K equipped with the supremum norm.

Theorem 2. *Let K be a compact Hausdorff space which is a continuous image of a Valdivia compactum. Then the following assertions are equivalent.*

- (a) *$C(K)$ is weakly Lindelöf determined.*
- (a') *K is a Corson compactum with property (M).*

(b) *Each nonseparable Banach space which is isometric to a subspace of* $C(K)$ *has a strong PRI.*

(b') *Each nonseparable Banach space which is isometric to a subspace of* $C(K)$ *has a weak PRI.*

The same idea can be used to characterize Corson compact spaces.

Theorem 3. *Let* K *be a compact Hausdorff space which is a continuous image of a Valdivia compactum. Then the following assertions are equivalent.*

(a) K *is a Corson compactum.*

(b) *For each* L*, a (non-metrizable) continuous image of* K*, the space* $C(L)$ *has a strong PRI.*

(b') *For each* L*, a (non-metrizable) continuous image of* K*, the space* $C(L)$ *has a weak PRI.*

Let us remark that in Theorems 2 and 3 we make the assumptions on *all* subspaces, not only on the complemented ones. We do not know whether it suffices to consider only complemented subspaces. However, in some special cases it is enough, as expressed in the following remark.

Remark. If K is a Valdivia compactum with a dense set of G_δ points, then the assertions of Theorem 2 are also equivalent to the following one.

(c) Each nonseparable Banach space which is isometric to a complemented subspace of $C(K)$ has a weak PRI.

Under the same hypothesis the assertions of Theorem 3 are also equivalent to the following one.

(c') For each L, a (non-metrizable) continuous image of K such that the canonical copy of $C(L)$ is complemented in $C(K)$, the space $C(L)$ has a weak PRI.

The proof of this remark is included in the proofs of Theorems 2 and 3.

3. Proofs. To prove Theorem 1 we need several lemmas. First two of them are trivial and we omit the obvious proofs but we state them for the convenience of the reader.

Lemma 1. *Let X be a Banach space and P be a bounded projection on X with $Y = PX$. Then the mapping $R : P^*X^* \to Y^*$ defined by $R(\xi) = \xi \restriction Y$ is an isomorphism of P^*X^* onto Y^*. Moreover, if $\|P\| = 1$, then R is an isometry.*

Lemma 2. *Let X be a Banach space and $(P_\alpha : \omega \le \alpha \le \mu)$ be a long sequence of projections on X. If $\omega \le \alpha < \beta \le \mu$, $\xi \in P_\alpha^* X^*$ and $x \in (P_\beta - P_\alpha)X$, then $\langle \xi, x\rangle = 0$.*

Lemma 3. *Let X be a Banach space and $(P_\alpha : \omega \le \alpha \le \mu)$ be a long sequence of projections on X. Then $S = \bigcup_{\omega\le\alpha<\mu} P_\alpha^* X^*$ is a 1-norming subspace of $P_\mu^* X^* = (P_\mu X)^*$ (using the identification due to Lemma 1).*

Proof. Choose $x \in P_\mu X$ and $\varepsilon > 0$. By the condition (iii) there is $\alpha < \mu$ and $y \in P_\alpha X$ such that $\|x - y\| < \frac{\varepsilon}{2}$. Pick $\eta \in X^*$ with $\|\eta\| = 1$ such that $\langle \eta, y\rangle = \|y\|$. Put $\xi = P_\alpha^* \eta$. Then clearly $\xi \in S$ and $\|\xi\| \le 1$. Moreover,

$$\begin{aligned}\langle \xi, x\rangle &= \langle \xi, y\rangle + \langle \xi, x - y\rangle \ge \langle P_\alpha^*\eta, y\rangle - \|\xi\| \cdot \|x - y\| \ge \langle \eta, P_\alpha y\rangle - \frac{\varepsilon}{2} \\ &= \langle \eta, y\rangle - \frac{\varepsilon}{2} = \|y\| - \frac{\varepsilon}{2} \ge \|x\| - \|x - y\| - \frac{\varepsilon}{2} \ge \|x\| - \varepsilon\end{aligned}$$

Hence, S is 1-norming. □

Lemma 4. *Let X be WLD and $S \subset X^*$ be a norming Σ-subspace. Then $S = X^*$.*

Proof. If S is norming, then there is an equivalent norm on X such that S is 1-norming with respect to this new norm. As WLD spaces are stable to renormings, we can without loss of generality suppose that S is 1-norming. It follows from the Hahn-Banach separation theorem that $S \cap B_{X^*}$ is weak* dense in B_{X^*}. Further, by the definition of a Σ-subspace we have that S is *countably closed* in X^* (i. e., $\overline{C} \subset S$ whenever $C \subset S$ is countable). Finally, B_{X^*} is a Corson compactum, and hence it is angelic [19, Theorem 2.1]. Therefore $B_{X^*} \subset S$, so $S = X^*$. □

Lemma 5. *Let X be a Banach space and S be a subspace of X^*. Then S is a Σ-subspace of X^* if and only if there is a linearly dense set $M \subset X$ such that $S = \{\xi \in X^* : \{m \in M : \langle \xi, m\rangle \ne 0\}$ is countable$\}$.*

Proof. This follows easily from the well-known fact that weak* continuous linear functionals on X^* belong to X [8, Theorem 55]. □

Now we are ready to prove Theorem 1.

P r o o f o f T h e o r e m 1. $(a) \Rightarrow (b)$ This follows from the fact that WLD spaces have strong PRI and are stable to taking subspaces (by [2, Corollary IV.3.15]). (The assertion can be proved in a more direct and elementary way, using [6, Proposition 8.3.1] together with an idea of the proof of [6, Proposition 6.1.10].)

$(b) \Rightarrow (c)$ This is trivial.

$(c) \Rightarrow (a)$ We will proceed by transfinite induction on density character of X. As separable spaces are clearly WLD, the assertion holds if $\operatorname{dens} X \leq \aleph_0$. Suppose that κ is an uncountable cardinal such that the assertion is valid for every X with $\operatorname{dens} X < \kappa$.

Let X be a Banach space with $\operatorname{dens} X = \kappa$ satisfying the condition (c). Fix $|\cdot|$ an arbitrary equivalent norm on X. By the assumptions there is a weak PRI $(P_\alpha : \omega \leq \alpha \leq \kappa)$ on $(X, |\cdot|)$. By the induction hypothesis $(P_{\alpha+1} - P_\alpha)X$ is a WLD space for each $\alpha \in [\omega, \kappa)$. So there is, by Lemma 5 (using the identification from Lemma 1), a linearly dense set $M_\alpha \subset (P_{\alpha+1} - P_\alpha)X$ such that for every $\xi \in (P^*_{\alpha+1} - P^*_\alpha)X^*$ the set $\{m \in M_\alpha : \langle \xi, m\rangle \neq 0\}$ is countable. Put $M = \bigcup_{\omega\leq\alpha<\kappa} M_\alpha$. Then M is linearly dense in X. We will show by transfinite induction on α that

$$\forall \alpha \in [\omega, \kappa)\ \forall \xi \in P^*_\alpha X^* \text{ the set } \{m \in M : \langle \xi, m\rangle \neq 0\} \text{ is countable.}$$

For $\alpha = \omega$ it is trivially satisfied.

Suppose it holds for some $\alpha < \kappa$. Let us prove it for $\alpha + 1$. Choose $\xi \in P^*_{\alpha+1}X^*$ and denote by A the set of all $m \in M$ such that $\langle \xi, m\rangle \neq 0$. By Lemma 2 we have $A \cap M_\beta = \emptyset$ for $\beta > \alpha$. If $m \in A \cap M_\alpha$, then

$$0 \neq \langle \xi, m\rangle = \langle \xi, (P_{\alpha+1} - P_\alpha)m\rangle = \langle (P^*_{\alpha+1} - P^*_\alpha)\xi, m\rangle,$$

and hence $A\cap M_\alpha$ is countable by the choice of M_α. Further, if $m \in A \cap \bigcup_{\omega\leq\beta<\alpha} M_\beta$, then $m \in P_\alpha X$, hence

$$0 \neq \langle \xi, m\rangle = \langle \xi, P_\alpha m\rangle = \langle P^*_\alpha \xi, m\rangle,$$

and thus $A \cap \bigcup_{\omega\leq\beta<\alpha} M_\beta$ is countable by induction hypothesis. Indeed, we already know that the set $\{m' \in M : \langle P^*_\alpha \xi, m'\rangle \neq 0\}$ is countable. Therefore A is countable.

Suppose that $\alpha \in (\omega, \kappa)$ is limit and that we have proved our claim for all $\beta < \alpha$. Put

$$\tilde{M} = M \cap P_\alpha X = \bigcup_{\omega\leq\beta<\alpha} M_\beta.$$

Then M is linearly dense in $P_\alpha X$. Further, put

$$\tilde{S} = \{\xi \in P_\alpha^* X^* : \{m \in \tilde{M} : \langle \xi, m\rangle \neq 0\} \text{ is countable}\}.$$

By Lemma 5 (using Lemma 1) it is a Σ-subspace of $P_\alpha^* X^*$. By the induction hypothesis it contains $\bigcup_{\omega\le\beta<\alpha} P_\beta^* X^*$, and hence it is 1-norming by Lemma 3. Moreover, the space $P_\alpha X$ is WLD (as $\operatorname{dens} P_\alpha X < \kappa$), and thus $\tilde{S} = P_\alpha^* X^*$ by Lemma 4. Finally, by Lemma 2 our claim holds also for α.

To complete the argument, put

$$S = \{\xi \in X^* : \{m \in M : \langle \xi, m\rangle \neq 0\} \text{ is countable}\}.$$

By Lemma 5 it is a Σ-subspace of X^*. By the just proved claim it contains $\bigcup_{\omega\le\alpha<\kappa} P_\alpha^* X^*$, and hence it is 1-norming by Lemma 3. It follows that $(X, |\cdot|)$ is 1-Plichko. As $|\cdot|$ was an arbitrary equivalent norm, X is WLD by [13, Theorem 1]. □

To prove Theorems 2 and 3 we need three more lemmas.

Lemma 6. *Let K be a compact Hausdorff space, κ be an infinite cardinal and $H \subset K$ be a closed set of cardinality at most κ. Then there is L, a continuous image of K of weight at most κ, such that L contains a homeomorphic copy of H.*

Proof. It is well-known that we can suppose that $K \subset [0,1]^\Gamma$ for some Γ. If h, h' are two distinct points of H, there is some $\gamma_{h,h'} \in \Gamma$ such that $h(\gamma_{h,h'}) \neq h'(\gamma_{h,h'})$. Put $\Gamma' = \{\gamma_{h,h'} : h, h' \in H, h \neq h'\}$. Then clearly $\operatorname{card}\Gamma' \le \kappa$. Let $R : [0,1]^\Gamma \to [0,1]^{\Gamma'}$ be the canonical restriction mapping. Put $L = R(K)$. Then clearly L is a continuous image of K and the weight of L is at most κ. Moreover, it follows from the choice of Γ' that $R \restriction H$ is one-to-one, so $R(H)$ is a homeomorphic copy of H contained in L. □

Lemma 7. *Let K be a non-Corson Valdivia compactum. Then there is a retract L of K such that the weight of L is $\aleph_1$ and L is a non-Corson Valdivia compactum. If, moreover, K has a dense set of G_δ points, then L can be chosen to have a dense set of G_δ points as well.*

Proof. Suppose that $K \subset [0,1]^\Gamma$ such that $A = K \cap \Sigma(\Gamma)$ is dense in K. For $J \subset \Gamma$ let us denote by R_J the canonical restriction mapping. It follows from [5, Claim on p. 254] that we have

$$(*) \qquad I \subset \Gamma, \operatorname{card} I \le \aleph_1 \Rightarrow \exists J \supset I : \ \operatorname{card} J \le \aleph_1 \ \& \ R_J(K) \subset K.$$

As K is not Corson, there is by [10, Proposition 2. 7] a homeomorphic injection $h : [0, \omega_1] \to K$ with $h([0, \omega_1)) \subset A$. Put $I = \bigcup\{\operatorname{supp} h(\alpha) : \alpha \in [0, \omega_1)\}$. Then clearly $\operatorname{card} I \leq \aleph_1$. Let J be the set from (*). Then $R_J \restriction K$ is a retraction on K, the weight of $R_J(K)$ is at most $\aleph_1$, $R_J(K) \cap A$ is dense in $R_J(K)$ (as it contains $R_J(A)$), so $R_J(K)$ is a Valdivia compactum, and finally $R_J(K)$ contains a copy of $[0, \omega_1]$, and hence it is not Corson. This completes the proof of the first statement.

Now let us suppose that K has a dense set of G_δ points. By [10, Proposition 2.2(3)] all G_δ points of K are contained in A. As A is a *Fréchet-Urysohn space* (i. e., whenever $M \subset A$, $x \in A$, $x \in \overline{M}$, then there is a sequence $x_n \in M$, $x_n \to x$; see [19, Theorem 2.1] or [10, Proposition 2.2(1)]), we have the following.

$$(**) \qquad \forall x \in A \, \exists g_n(x) \in A, n \in \mathbb{N} : g_n(x) \text{ is a } G_\delta \text{ point of } K \ \& \ g_n(x) \to x$$

Let us construct by induction sets $M_k, D_k \subset A$, $J_k \subset \Gamma$ such that

(α) $D_1 = h([0, \omega_1))$;

(β) $M_k = D_k \cup \{g_n(x) : x \in D_k, n \in \mathbb{N}\}$;

(γ) $J_k \supset \bigcup\{\operatorname{supp} x : x \in M_k\} \cup \bigcup\{J_l : l < k\}$, $\operatorname{card} J_k \leq \aleph_1$, $R_{J_k}(K) \subset K$;

(δ) D_{k+1} is a dense subset of $R_{J_k}(K) \cap A$, $\operatorname{card} D_{k+1} \leq \aleph_1$, $D_{k+1} \supset M_k$.

Put $J = \bigcup\limits_{k \in \mathbb{N}} J_k$. Then clearly $\operatorname{card} J \leq \aleph_1$ and $R_J(K) \subset K$. Hence $R_J(K)$ is a retraction of K of weight at most $\aleph_1$. Further, $R_J(K) \cap A$ is dense in $R_J(K)$ and $R_J(K)$ contains a copy of $[0, \omega_1]$. Thus $R_J(K)$ is a non-Corson Valdivia compactum. It remains to prove that $R_J(K)$ has a dense set of G_δ points. Let G denote the set of all G_δ points of $R_J(K)$. By the construction we have $D_k \subset \overline{G}$ for all k. Hence $R_{J_k}(K) \subset \overline{G}$ for all k. To finish the proof it suffices to observe that $\bigcup\limits_{k \in \mathbb{N}} R_{J_k}(K)$ is dense in $R_J(K)$. □

Lemma 8. *Let K be a Corson compact space without property (M). Then there is a retract L of K such that the weight of L is $\aleph_1$ and L is a Corson compact space without property (M).*

Proof. Suppose that $K \subset \Sigma(\Gamma) \cap [0, 1]^\Gamma$ and that μ is a Radon probability measure on K with nonseparable support H. Construct by transfinite induction $h_\alpha \in H$ for $\alpha < \omega_1$ such that $h_\alpha \notin \overline{\{h_\beta : \beta < \alpha\}}$ for each $\alpha < \omega_1$. Put $I = \bigcup\{\operatorname{supp} h_\alpha : \alpha < \omega_1\}$. Then $\operatorname{card} I \leq \aleph_1$, so there is (by (*) in the proof of the previous proposition) $J \supset I$ with $\operatorname{card} J \leq \aleph_1$ and such that $R_J(K) \subset K$. Let

$\nu = R_J(\mu)$ (i. e., ν is the image of μ by the mapping R_J). Then the support of ν is equal to $R_J(H)$. If $R_J(H)$ were separable, it would be metrizable (as a separable Corson compactum), and so $\{h_\alpha : \alpha < \omega_1\} \subset R_J(H)$ would be separable as well, which contradicts the construction. □

P r o o f of T h e o r e m 3. $(a) \Rightarrow (b)$ This follows from the fact that Corson compact spaces are stable to continuous images (see e. g. [2, Corollary IV.3.15]) and that $C(K)$ has a strong PRI whenever K is Corson (see e. g. [26]).

$(b) \Rightarrow (b')$ This is trivial.

$(b') \Rightarrow (a)$ Let K be a continuous image of a Valdivia compactum which is not Corson. By [11, Theorem 1] the space K contains a copy of $[0, \omega_1]$. By Lemma 6 there is K', a continuous image of K of weight $\aleph_1$ such that K' contains a copy of $[0, \omega_1]$. Then K' is a non-Corson continuous image of a Valdivia compactum of weight $\aleph_1$. By [12, Theorem 2] there is a continuous image L of K such that $C(L)$ has no PRI. As $C(L)$ has density $\aleph_1$, it has no weak PRI, as these two notions coincide for spaces of density $\aleph_1$.

$(b') \Rightarrow (c')$ This is trivial.

$(c') \Rightarrow (a)$ Let K be a non-Corson Valdivia compactum with a dense set of G_δ points. By Lemma 7 there is K', a retract of K, such that K' is a non-Corson Valdivia compactum of weight $\aleph_1$ with a dense set of G_δ points. Clearly $C(K')$ is complemented in $C(K)$. By [12, Proposition 1] there are $a, b \in K'$ such that $B_{C(L)^*}$ is not Valdivia if L is the quotient space made from K' by identification of a and b. Then $C(L)$ has no PRI by [7, Lemma 2]. Further, $C(L)$ form a hyperplane in $C(K')$, so it is complemented in $C(K')$. Finally it is clear that $C(L)$ is complemented in $C(K)$. □

P r o o f of T h e o r e m 2. $(a) \Leftrightarrow (a')$ This is proved in [4, Theorem 3.5].

$(a) \Rightarrow (b)$ This follows from Theorem 1.

$(b) \Rightarrow (b')$ This is trivial.

$(b') \Rightarrow (a')$ If K is not Corson, then this assertion follows from Theorem 3. If K is Corson without property M, there is, by Lemma 8, a retract L of K such that L has weight at most $\aleph_1$ and does not have the property (M). By [12, Proposition 3] there is a hyperplane $Y \subset C(L)$ such that B_{Y^*} is not Valdivia. As $\operatorname{dens} Y = \aleph_1$, the space Y has no PRI by [7, Lemma 2]. Let us remark that in this case Y is complemented in $C(K)$.

$(b') \Rightarrow (c)$ This is trivial

$(c) \Rightarrow (a')$ If K is not Corson, the assertion follows from Theorem 3. If K is Corson without property (M), then the proof is the same as that of the implication $(b') \Rightarrow (a')$ above. □

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