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# THE OPERATORS $A_\gamma = \gamma A + \overline{\gamma} A^*$ FOR A CLASS OF NONDISSIPATIVE OPERATORS $A$ WITH A LIMIT OF THE CORRESPONDING CORRELATION FUNCTION*

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Abstract. In this work we present the operators $A_\gamma = \gamma A + \overline{\gamma} A^*$ with $\mathrm{Re}\,\gamma = 1/2$ in the case of an operator $A$ from the class of nondissipative operators generating nonselfadjoint curves, whose correlation functions have a limit as $t \to \pm\infty$. The asympthotics of the stationary curves $e^{itA_\gamma} f$ as $t \to \pm\infty$ onto the absolutely continuous subspace of $A_\gamma$ are obtained. These asymptotics and the obtained asymptotics in [9] of the nondissipative curves $e^{itA} f$ allow to construct the scattering theory for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$. We consider the basic terms from the scattering theory - wave operators, a scattering operator and the question of a similarity of $A$ and $A_\gamma$. We obtain explicitly the wave operators, the scattering operator and the similarity of $A$ and $A_\gamma$.

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*Key words:* operator colligation, nondissipative curve, correlation function, wave operator, scattering operator.

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**1. Introduction.** This paper is a continuation and an application of the results in [2, 9] concerning the asymptotics of nondissipative curves $e^{itA}f$ as $t \to \pm\infty$, generated by the class $\widetilde{\Omega}_{\mathbb{R}}$ of nondissipative operators $A$ with a limit of the corresponding correlation function $V(t+\tau, s+\tau) = (e^{i(t+\tau)A}f, e^{i(s+\tau)A}f)$ as $\tau \to \pm\infty$, and the abstract scattering theory for the couple $(A^*, A)$.

The purpose of the present paper is to obtain the asymptotics of the curves $e^{itA_\gamma}f$ onto the absolutely continuous subspace of the operator $A_\gamma = \gamma A + \overline{\gamma} A^*$ with $\operatorname{Re}\gamma = 1/2$ and to construct the scattering theory for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$.

The development of the study of nonselfadjoint operators began with the works of M. S. Livšic and his associates in 1950's ([6, 5]) and later that of Sz.-Nagy, Foias, de Branges and Rownyak ([14, 3, 4]), M. S. Livšic, A. A. Yantsevich, V. Vinnikov et al. ([13, 12]), K. Kirchev and V. Zolotarev ([10, 11]). The theory, created from M. S. Livšic, consider mostly operators $A : H \longrightarrow H$ in a Hilbert space $H$ with a nonhermitian part $A - A^*$ from finite rank $(\dim(A - A^*)H < +\infty)$ or trace class. It is based on the connection between the theory of nonselfadjoint operators and the theory of bounded analytic functions on the upper half-plane.

Let us denote for the operator $A : H \longrightarrow H$ with $\dim(A - A^*) < +\infty$

$$E = (A - A^*)H, \quad \Phi = P_E, \quad L = \frac{1}{i}(A - A^*)|_E,$$

where $P_E$ is the orthogonal projector of $H$ onto $E$. Then the matrix function (called a characteristic function of $A$)

$$W(\lambda) = I_E - iP_E(A - \lambda I)^{-1}P_E L$$

is defined and analytic in the resolvent set of $A$, analytic in a neighbourhood $|\lambda| > a$ of $\lambda = \infty$, $W(\infty) = I$ and $W(\lambda)$ possesses the metric properties

$$W^*(\lambda)LW(\lambda) \geq L \quad (\operatorname{Im}\lambda > 0),$$

$$W^*(\lambda)LW(\lambda) = L \quad (\operatorname{Im}\lambda = 0),$$

$$W^*(\lambda)LW(\lambda) \leq L \quad (\operatorname{Im}\lambda < 0)$$

for a regular point $\lambda$ of the operator $A$. In other words to every bounded operator $A$ in a Hilbert space with a finite imaginary part there corresponds a matrix valued function which characterizes these operators up to an unitary equivalence on the principal subspace of $A$. This relation is the essence of the theory of M. S. Livšic and it allows to make a classification of the considered operators.

More generally the characteristic function of $A : H \longrightarrow H$ can be introduced in the form

$$W(\lambda) = I - i\Phi(A - \lambda I)^{-1}\Phi^* L$$

by the so called operator colligation

$$X = (A; H, \Phi, E; L),$$

where $E$ is a Hilbert space, $\Phi : H \longrightarrow E$ and $L : E \longrightarrow E$ are bounded linear operators with $L^* = L$ and $(A - A^*)/i = \Phi^* L \Phi$.

The main point in this investigations is the relation between the invariant subspaces of the operator $A$ and the factorizations of the characteristic function $W(\lambda)$ (given by Potapov's factorization theorem).

An arbitrary finite matrix can be presented in a triangular form by a corresponding unitary mapping. Analogous problem can be solved for classes of nonselfadjoint operators – the operators from these classes are presented in the so called triangular models using unitary mappings.

One of the applications of this theory is the study of nonstationary random processes and more generally continuous curve $g(t)$ in a Hilbert space $H$:

$$g(t) = e^{itA} f \quad (f \in H).$$

The obtained asymptotics of a nonstationary curves for classes of nonselfadjoint operators allow us to construct a scattering theory for a couple $(A^*, A)$, where $A$ is an operator from a given class.

From the class of all nonselfadjoint operators in a Hilbert space with a finite nonhermitian rank we consider the operators $A$ presented as a coupling of a dissipative operator and an antidissipative one. For these operators $A$ we consider the operators

$$A_\gamma = \gamma A + \overline{\gamma} A^* \quad \text{with} \quad \operatorname{Re}\gamma = 1/2.$$

The reason for our interest in the operators $A_\gamma$ with $\operatorname{Re}\gamma = 1/2$ is the connection between the complete characteristic matrix function $W(\lambda) = I - iL\|((A - \lambda I)^{-1} g_\alpha, g_\beta)\|$ of the nondissipative operator $A$ and the matrix function $V_\gamma(\lambda) = |\gamma|.\|((A_\gamma - \lambda I)^{-1} g_\alpha, g_\beta)\|$ where $\{g_\alpha\}_1^m$ are the channel elements of $A$. $W(\lambda)$ and $V_\gamma(\lambda)$ determine completely each other and the consideration of the matrix function $V_\gamma(\lambda)$ allows to describe the characteristic matrix functions of a class of operators (see, for example, [6] in the case of $\gamma = 1/2$). In [6] M. S. Livšic and M. S. Brodskii have considered the operators $A_\gamma$ with $\gamma = 1/2$ for operators $A$ with a finite imaginary part. In [16] L. A. Sakhnovich has presented the operators $A_\gamma$ with $\operatorname{Re}\gamma = 1/2$ for dissipative operators $A$ with a trace class imaginary parts.

The main purpose of this article is to obtain the asymptotics of the curves $e^{itA_\gamma}f$ onto the absolutely continuous subspace of the selfadjoint operator $A_\gamma$ and to construct a scattering theory for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$. We obtain explicitly the wave operators $W_\pm(A, A_\gamma)$, $W_\pm(A_\gamma, A)$, the scattering operator and the similarity of $A$ and $A_\gamma$. In this paper we essentially use the asymptotics of the nondissipative curves $e^{itA}f$ for $A \in \widetilde{\Omega}_{\mathbb{R}}$ and other results obtained in [9].

**2. Wave operators for the class $\widetilde{\Omega}_{\mathbb{R}}$ of nondissipative operators.** We consider the class $\widetilde{\Omega}_{\mathbb{R}}$ of all nonselfadjoint linear bounded operators in a Hilbert space with a finite nonhermitian rank, real spectrum and presented as a coupling of a dissipative operator and an antidissipative one. This class has been considered in [9, 2]. Let the operator $A$ belong to the class $\widetilde{\Omega}_{\mathbb{R}}$. We may assume without loss of generality that the operator $A$ has the form

$$\begin{gathered} Af(x) = \alpha(x)f(x) - i\int_0^x f(\xi)\Pi(\xi)S^*\Pi^*(x)d\xi + \\ +i\int_x^l f(\xi)\Pi(\xi)S\Pi^*(x)d\xi + i\int_0^x f(\xi)\Pi(\xi)L\Pi^*(x)d\xi \end{gathered} \tag{1}$$

in the Hilbert space $\mathbf{L}^2(0,l;\mathbb{C}^n)=\{f(x)=(f_1(x),\dots,f_n(x)) : [0,l] \to \mathbb{C}^n : f_k(x)\in \mathbf{L}^2(0,l), k=1,2,\dots,n\}$ with an inner product $(f(x),g(x))=\int_0^l f(x)g^*(x)dx$. Here $\alpha(x)$ is a bounded non-decreasing function on a finite interval $[0;l]$ which is continuous at 0 and continuous from the left on $(0;l]$, $\Pi(x)$ is a measurable $n \times m$ $(1 \le n \le m)$ matrix function on $[0;l]$, whose rows are linearly independent at each point of a set of positive measure, and satisfying the condition

$$tr\Pi^*(x)\Pi(x) = 1,$$

the selfadjoint operator $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ with $\det L \neq 0$ has the representation

$$L = J_1 - J_2 + S + S^*,$$

where $J_1$, $J_2$, $S$, $S^* : \mathbb{C}^m \longrightarrow \mathbb{C}^m$,

$$J_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}, \; S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix},$$

$I_k$ is the identity matrix in $\mathbb{C}^k$ $(k = r, m-r)$, $\widehat{S}$ is a $(m-r)\times r$ matrix, $r$ is the number of the positive eigenvalues and $m-r$ is the number of the negative eigenvalues of the operator $L$, the matrix function $B(x) = \Pi^*(x)\Pi(x)$ satisfies the condition

$$B(x)J_1 = J_1 B(x) \tag{2}$$

for almost all $x \in [0;l]$. The model (1) is a suitable form of the triangular model of M. S. Livšic [13, 12] which generates a class of nondissipatives curves having a limit of the corresponding correlation function ([2, 9]).

Let us suppose that $\Pi(x)$ has a linearly independent rows for almost all $x \in [0;l]$. If we consider a measurable $m \times n$ matrix function $Q(x)$ on $[0;l]$ satisfying the condition $\Pi(x)Q(x) = I$ where $I$ is the identity matrix in $\mathbb{C}^n$, we can present the operator $A$ from (1) as a coupling of a dissipative operator and an antidissipative one

$$A = P_1AP_1 + P_2AP_2 + P_1AP_2, \tag{3}$$

where the orthogonal projectors $P_1$ and $P_2$ have the form

$$P_1 f(x) = f(x)\Pi(x)J_1Q(x), \quad P_2 f(x) = f(x)\Pi(x)J_2Q(x) \tag{4}$$

(see [9]), $P_1AP_1$ is dissipative and $P_2AP_2$ is antidissipative.

In this paper we shall be considering only operators from the class $\widetilde{\Omega}_{\mathbb{R}}$ with an absolutely continuous spectrum, i. e. the inverse function $\sigma(u)$ of $\alpha(x)$ is absolutely continuous on $[a;b]$, where $a = \alpha(0)$, $b = \alpha(l)$.

In this paper we will denote by $\| \ \|$ the norm of a matrix function in $\mathbb{C}^n$ and by $\| \ \|_{L^2}$ - the norm in $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

For the simplification of writing we will also assume that the matrix function $B(x) \in C_{\alpha_1}[0;l]$ (i. e. $\|B(x_1) - B(x_2)\| \le C|x_1 - x_2|^{\alpha_1}$, $0 < \alpha_1 \le 1$, $\forall x_1, x_2 \in [0;l]$) and the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies the conditions:

**(i)** $\alpha(x)$ is continuous strictly increasing on $[0;l]$ :

**(ii)** the inverse function $\sigma(u)$ of $\alpha(x)$ is absolutely continuous on $[a;b]$;

**(iii)** $\sigma'(u)$ is continuous and satisfies the condition

$$|\sigma'(u_1) - \sigma'(u_2)| \le C|u_1 - u_2|^{\alpha_2} \ \ (0 < \alpha_2 \le 1)$$

for all $u_1,\ u_2 \in [a;b]$ and for some constant $C > 0$.

For our further applications let us denote the next operators for a non-negative (non-positive) matrix function $T(x) \in C_{\alpha_1}[a;b]$ $(\alpha_1 > 0)$ (following the denotations in [9]):

$$F_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \overset{u}{\underset{w}{\overrightarrow{\int}}} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv}, \tag{5}$$

$$P_w(x,u) = F_w^+(x,u) - F_w^-(x,u) \tag{6}$$

for all $w, u, x$ such that $a \le w < u \le b$, $a \le x \le b$ and

$$F_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \overset{\to}{\int_w^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \overset{\to}{\int_{x+\delta}^{u}} e^{\frac{-iT(v)}{v-x}dv}, \tag{7}$$

$$\begin{aligned} R_w^{\pm 1}(x) &= (F_w^{\pm}(x,u)(F_w^{\pm}(x,u))^*)^{\frac{1}{2}} = \\ &= s - \lim_{\delta\to 0} \overset{\to}{\int_w^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \left( \overset{\to}{\int_w^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} \right)^{-1}, \end{aligned} \tag{8}$$

$$U_w(x,u) = R_w^{\mp 1}(x) F_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \overset{\to}{\int_w^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} \overset{\to}{\int_{x+\delta}^{u}} e^{\frac{-iT(v)}{v-x}dv}, \tag{9}$$

$$U_{1w}(x,u) = \lim_{\delta\to 0} \overset{\to}{\int_w^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} e^{-i\int_{x+\delta}^{u} \frac{T(x)}{v-x}dv}, \tag{10}$$

$$U_{2w}(x) = \lim_{\delta\to 0} \overset{\to}{\int_w^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} e^{i\int_{w}^{x-\delta} \frac{T(x)}{v-x}dv}, \tag{11}$$

$$P_{2w}(x,u) = (R_w(x) - R_w^{-1}(x)) U_{2w}(x) e^{-iT(x)\ln\frac{u-x}{x-w}}, \tag{12}$$

$$Q_w(x,u) = P_{2w}(x,u) e^{iT(x)\ln(u-x)} e^{-iT(u)\ln(u-x)}, \tag{13}$$

$$Q_w^{\pm}(x) = R_w^{\pm 1}(x) U_{2w}(x) e^{iT(x)\ln(x-w)}, \tag{14}$$

$$Q_w(x) = Q_w^+(x) - Q_w^-(x) \tag{15}$$

for all $w$, $u$, $x$ such that $a \le w < x < u \le b$. The existence of these limits follows from the formula about the limit values for multiplicative integrals, obtained by L. A. Sakhnovich in [16]

$$s - \lim_{\delta\to 0} \overset{\to}{\int_a^{b}} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv} = s - \lim_{\delta\to 0} \overset{\to}{\int_a^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \overset{\to}{\int_{x+\delta}^{b}} e^{\frac{-iT(v)}{v-x}dv}.$$

Using the introduced notations for $T(x) \in C_\alpha[a;b]$ $(0 < \alpha \le 1)$ we shall recall several inequalities obtained by L. A. Sakhnovich in ([16]) which we will use (see, for example, [9]):

$$\|U_w(x,u) - U_{1w}(x,u)\| \le \int_x^u \frac{\|T(x) - T(v)\|}{|x-v|} dv \tag{16}$$

for all $w$, $u$, $x$ such that $a \le w \le x \le u \le b$,

$$\|U_{2a}(x_1) - U_{2a}(x_2)\| \le C \left( \frac{x_2 - x_1}{x_1 - a} \right)^{\alpha'}, \tag{17}$$

$$\|R_a(x_1) - R_a(x_2)\| \le C \left( \frac{x_2 - x_1}{x_1 - a} \right)^{\alpha'}, \tag{18}$$

$$\|F_a^{\pm}(x_1, b) - F_a^{\pm}(x_2, b)\| \le C \left( \left( \frac{x_2 - x_1}{x_1 - a} \right)^{\alpha'} + \left( \frac{x_2 - x_1}{b - x_2} \right)^{\alpha'} \right), \tag{19}$$

for all $x_1, x_2 \ : \ a \le x_1 < x_2 \le b$ where $C > 0$ is a suitable constant and $\alpha' = \alpha/(1+\alpha)$.

We will also use the next inequalities obtained in [9]

$$\|F_w^+(x, u) - Q_w^+(x, u)\| \le C(u - x)^{\alpha'}, \tag{20}$$

$$\|Q_w^+(x) - Q_w^+(u)\| \le C \left( \frac{u - x}{x - w} \right)^{\alpha'} \tag{21}$$

for all $w, u, x : a < w \le x \le u \le b$,

$$\left\| \overset{\rightarrow}{\int_w^u} e^{\frac{-iT(v)}{v-x} dv} - U_{2w}(x) e^{-iT(x) \ln \frac{x-u}{x-w}} \right\| \le C(x - u)^{\alpha} \tag{22}$$

for all $w, u, x : a \le w \le u < x \le b$,

$$\|e^{iT(x) \ln(x-w)} - e^{iT(u) \ln(u-w)}\| \le C \left( \frac{x - u}{x - w} \right)^{\alpha'} \tag{23}$$

for all $x, u, w : w < u \le x$,

$$\|e^{-iT(x) \ln(x-u)} - e^{-iT(u) \ln(x-u)}\| \le C(x - u)^{\alpha'} \tag{24}$$

for all $x, u : a \le u < x \le b$,

$$\left\| \overset{\rightarrow}{\int_w^b} e^{\frac{-iT(x)}{v-x} dv} - \overset{\rightarrow}{\int_w^b} e^{\frac{-iT(x)}{v-u} dv} \right\| \le C \left( \left( \frac{u - x}{x - a} \right)^{\alpha'} + \left( \frac{u - x}{w - x} \right)^{\alpha'} \right) \tag{25}$$

when $a < x \le u < w \le b$, where $C > 0$ is a suitable constant and $\alpha' = \alpha/(1+\alpha)$.

For the simplification of writing suppose that the initial function

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x)) \in \mathbf{L}^2(0,l;\mathbb{C}^n)$$

is chosen from the dense set $\widetilde{H}_0$ in $\mathbf{L}^2(0,l;\mathbb{C}^n)$ such that there exist $f_k'(x) \in \mathbf{L}^2(0,l)$ $(k = 1,2,\dots,n)$ and $f(0) = f(l) = (0,0,\dots,0)$.

For our further applications we shall denote the matrix functions defined by (5) (or (7)), (6), (8), (9), (10), (11), (12), (13), (15), (14) with $F_w^{\pm}(x,u)$, $P_w(x,u)$, $R_w^{\pm 1}(x)$, $U_w(x,u)$, $U_{1w}(x,u)$, $U_{2w}(x)$, $P_{2w}(x,u)$, $Q_w(x,u)$, $Q_w(x)$, $Q_w^{\pm}(x)$ respectively for the nonnegative matrix function $T(x) = J_1B(\sigma(x))J_1\sigma'(x)$ on $[a;b]$ and with $\widetilde{F}_w^{\pm}(x,u)$, $\widetilde{P}_w(x,u)$, $\widetilde{R}_w^{\pm 1}(x)$, $\widetilde{U}_w(x,u)$, $\widetilde{U}_{1w}(x,u)$, $\widetilde{U}_{2w}(x)$, $\widetilde{P}_{2w}(x,u)$, $\widetilde{Q}_w(x,u)$, $\widetilde{Q}_w(x)$, $\widetilde{Q}_w^{\pm}(x)$ respectively for the nonpositive matrix function $T(x) = -J_2B(\sigma(x))J_2\sigma'(x)$ on $[a;b]$.

In [9] the asymptotics of nondissipative curve $e^{itA}f$ with basic operators from the class $\widetilde{\Omega}_{\mathbb{R}}$ have been obtained and the existence of the limits of the corresponding correlation function $V(t+\tau, s+\tau) = (e^{i(t+\tau)A}f, e^{i(s+\tau)A}f)$ as $\tau \to \pm\infty$ has been proved. In other words (see [9] Theorem 4, Theorem 5) let for the model $A \in \widetilde{\Omega}_R$, defined by (1), next conditions hold:

1) the function $\alpha \ : \ [0;l] \longrightarrow \mathbb{R}$ satisfies (i), (ii), (iii);

2) $Q^*(x)$ is a smooth matrix function on $[0;l]$ (i. e. $Q^*(x)$ is differentiable and $Q^{*\prime}(x)$ is continuous on $[0;l]$ by norm in $\mathbb{C}^n$)) ;

3) $B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \leq 1)$.

Then the nondissipative curve $e^{itA}f$ for each $f \in \widetilde{H}_0$ after the change of the variable $x = \sigma(u)$ has the asymptotics

$$\left\| e^{itA} f(\sigma(u)) - e^{itu} S_{\pm} f(\sigma(u)) \right\|_{L^2} \to 0 \tag{26}$$

as $t \to \pm\infty$ and there exist the limits of the corresponding correlation function $V(t+\tau, s+\tau)$ as $\tau \to \pm\infty$ for $e^{itA}f$ in $\mathbf{L}^2(0,l;\mathbb{C}^n)$ and after the change of the variable $x = \sigma(u)$ these limits have the form

$$\begin{aligned} \lim_{\tau\to\pm\infty} V(t+\tau, s+\tau) &= \int_a^b e^{i(t-s)} \widetilde{S}_{\pm}(f(\sigma(u)))(\widetilde{S}_{\pm}(f(\sigma(u))))^* \sigma'(u)du = \\ &= (e^{itu}\widetilde{S}_{\pm} f(\sigma(u)), e^{isu}\widetilde{S}_{\pm} f(\sigma(u))) \end{aligned} \tag{27}$$

for all $t, s \in \mathbb{R}$. The operators $S_{\pm}$ and $\widetilde{S}_{\pm}$ are bounded linear operators defined in the subspace $\widetilde{H}_0$ by the equalities (after the change of the variable $x = \sigma(u)$):

$$S_{\pm} f(\sigma(u)) = (\widehat{S}_{\pm} f(\sigma(u)))T_{\pm} Z(t,u), \tag{28}$$

$$\widetilde{S}_{\pm} f(\sigma(u)) = T_{\pm} \widehat{S}_{\pm} f(\sigma(u)), \tag{29}$$

where

$$\widehat{S}_\pm f(\sigma(u)) =$$

$$= \int_a^u \widetilde{f}'(w) \int_a^{\overleftarrow{w}} e^{\frac{i\widetilde{B}_1(v)}{v-u}dv} dw J_1 + \int_a^u \widetilde{f}'(w) \int_a^{\overleftarrow{w}} e^{\frac{-i\widetilde{B}_2(v)}{v-u}dv} dw J_2 - \int_a^b \widetilde{f}'(w)\widetilde{F}_w^{\mp}(u,b) dw S,$$

$$\begin{aligned} T_\pm h = h \Big( & J_1 U_{2a}(u)(u-a)^{i\widetilde{B}_1(u)} e^{\mp\frac{\pi}{2}\widetilde{B}_1(u)} \mathbf{\Gamma}^{-1}(I + i\widetilde{B}_1(u))J_1 + \\ & + J_2 \widetilde{U}_{2a}(u)(u-a)^{-i\widetilde{B}_2(u)} e^{\pm\frac{\pi}{2}\widetilde{B}_2(u)} \mathbf{\Gamma}^{-1}(I - i\widetilde{B}_2(u))J_2 \Big) \Pi^*(\sigma(u)) \end{aligned} \tag{30}$$

(for all $h \in \mathbb{C}^m$) where $\mathbf{\Gamma}(\varepsilon I + i\widetilde{B}_1(u)) = \int\limits_0^{+\infty} e^{-x} e^{((\varepsilon-1)I - i\widetilde{B}_1(u))\ln u} dx$ $(\varepsilon > 0)$ is the analogue in $\mathbb{C}^m$ of the classical gamma-function (considered in [9]),

$$Z(t,u) = \Pi(\sigma(u))(J_1|t|^{i\widetilde{B}_1(u)} J_1 + J_2|t|^{-i\widetilde{B}_2(u)} J_2) Q(\sigma(u)), \tag{31}$$

$$\widetilde{B}_1(u) = J_1 B(\sigma(u)) J_1 \sigma'(u), \quad \widetilde{B}_2(u) = J_2 B(\sigma(u)) J_2 \sigma'(u), \tag{32}$$

$$\widetilde{f}(w) = f(\sigma(w)) Q^*(\sigma(w)). \tag{33}$$

We can embed the operator $A$ from (1) in a colligation

$$X = (A; \mathbf{L}^2(0,l;\mathbb{C}^n), \Phi, \mathbb{C}^m, L), \tag{34}$$

where the operator $\Phi : \mathbf{L}^2(0,l;\mathbb{C}^n) \longrightarrow \mathbb{C}^m$ is defined by

$$\Phi(x) = \int_0^l f(x)\Pi(x) dx, \tag{35}$$

$$(A - A^*)/i = \Phi^* L \Phi \quad \text{and} \quad \Phi^* h = h\Pi^*(x). \tag{36}$$

In [9] the construction of a scattering theory for the couples $(A^*, A)$ with $A$ from $\widetilde{\Omega}_{\mathbb{R}}$ is presented and the wave operators for $(A^*, A)$ as weak limits are obtained. The proved similarity of $A$ and the operator $D$ of a multiplying by an independent variable in [9] allows us to prove the existence of the wave operators as strong limits.

**Theorem 1.** *Let for the model $A \in \widetilde{\Omega}_{\mathbb{R}}$, defined by* (1), *next conditions hold:*

1) *the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies* (i), (ii), (iii);
2) *$Q^*(x)$ is a smooth matrix function on* $[0;l]$;

3) $B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \le 1)$.
*Then there exist the strong limits*

$$s - \lim_{t\to\pm\infty} e^{itA^*}e^{-itA}$$

*on* $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

P r o o f. Let us consider the operator function

$$W(t) = e^{itA^*}e^{-itA}$$

in $\mathbf{L}^2(0,l;\mathbb{C}^n)$. Then $\frac{dW(t)}{dt} = e^{itA^*}\frac{A-A^*}{i}e^{-itA}$ and consequently

$$W(t)f = f - i\int_0^t e^{i\tau A^*}(A-A^*)e^{-i\tau A}f\,d\tau \tag{37}$$

for $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$. But $\frac{A-A^*}{i}e^{-itA}f = \sum\limits_{\alpha,\beta=1}^{m}(e^{-itA}f,g_\alpha)(Le_\alpha,e_\beta)g_\beta$, where $\{e_\alpha\}_1^m$ is an orthonormal basis in $\mathbb{C}^m$, $g_\alpha(x) = \Phi^*e_\alpha = e_\alpha\Pi^*(x)$ $(x \in [0;l])$, $\alpha = 1,2,\ldots,m$, are channel elements of $A$. Then

$$\begin{aligned}\left\|\tfrac{A-A^*}{i}e^{-itA}f\right\|_{\mathbf{L}^2} &\le \left\|\sum_{\alpha,\beta=1}^{m}(e^{-itA}f,g_\alpha)(Le_\alpha,e_\beta)g_\beta\right\|_{\mathbf{L}^2} \le \\ &\le C\sum_{\alpha=1}^{m}|(\widehat{S}_\pm^{-1}e^{-itD}\widehat{S}_\pm f,g_\alpha)| = C\sum_{\alpha=1}^{m}|(e^{-itD}\widehat{S}_\pm f,\widehat{S}_\pm^{-1^*}g_\alpha)| = \\ &= C\sum_{\alpha=1}^{m}\left|\int_a^b e^{-itu}\widehat{S}_\pm f(\sigma(u))(\widehat{S}_\pm^{-1^*}g_\alpha(\sigma(u)))^*\sigma'(u)du\right|,\end{aligned}$$

because $A = \widehat{S}_\pm^{-1}D\widehat{S}_\pm$ onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$ (see [9]) after the change of the variable $x = \sigma(u)$, where $D$ is the operator of a multiplying by an independent variable, $\widehat{S}_\pm^{-1}$ is the inverse operator of $\widehat{S}_\pm$ and it has the form

$$\widehat{S}_\pm^{-1} = G_{11} + G_{22} + G_{12}^\pm, \tag{38}$$

where

$$G_{11}g(\sigma(u)) = \frac{1}{2\pi}\frac{d}{du}\int_a^u g(\sigma(u))P_a(\tau,u)d\tau J_1Q(\sigma(u))(\sigma'(u))^{-1},$$

$$G_{22}g(\sigma(u)) = \frac{1}{2\pi}\frac{d}{du}\int\limits_a^u g(\sigma(u))\widetilde{P}_a(\tau,u)d\tau J_2 Q(\sigma(u))(\sigma'(u))^{-1},$$

$$G_{12}^{\pm}g(\sigma(u)) = -G_{11}\widetilde{S}_{12}^{\pm}G_{22}g(\sigma(u)),$$

$$\widetilde{S}_{12}^{\pm}f(\sigma(u)) = -\int\limits_a^b \widetilde{f}'(w)\widetilde{F}_w^{\mp}(u,b)dwS,$$

(where $g \in \mathbf{L}^2(0,l;\mathbb{C}^m)$ such that $|g'(x)| \leq C$ in $[0;l]$, $C>0$ is a constant). But for each $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$ there exists $g \in Y = \overline{R(\widehat{S}_\pm)} \subset \mathbf{L}^2(0,l;\mathbb{C}^m)$ such that $f = \widehat{S}_\pm^{-1}g$ and then

$$\begin{aligned}(39)\qquad &\left\|\tfrac{A-A^*}{i}e^{-itA}f\right\|_{\mathbf{L}^2} = \left\|\tfrac{A-A^*}{i}e^{-itA}\widehat{S}_\pm^{-1}g\right\|_{\mathbf{L}^2} \leq \\ &\leq C\sum_{\alpha=1}^m \left|\int\limits_a^b e^{-itu}g(\sigma(u))h_\alpha(u)du\right|,\end{aligned}$$

where $h_\alpha(u) = (\widehat{S}_\pm^{-1^*}g_\alpha(\sigma(u)))^*\sigma'(u) \in \mathbf{L}^2(0,l;\mathbb{C}^m)$, $\alpha = 1,2,\ldots,m$ and $R(\widehat{S}_\pm)$ is the range of the operator $\widehat{S}_\pm$.

On the one hand $W(t)$, $t\in\mathbb{R}$, is an uniformly bounded set of operators (i. e. $\|W(t)\| \leq M$ for all $t \in \mathbb{R}$, where $M>0$ is a suitable constant – see [9]). On the other hand from (37) and (39) we obtain

$$\begin{aligned}\|W(t_2)f - W(t_1)f\|_{\mathbf{L}^2}^2 &= \left\|\int\limits_{t_1}^{t_2} e^{i\tau A^*}\tfrac{A-A^*}{i}e^{-i\tau A}f d\tau\right\|_{\mathbf{L}^2}^2 = \\ = &\left\|\int\limits_{t_1}^{t_2}\sum_{\alpha,\beta=1}^m (e^{-i\tau A}f,g_\alpha)(Le_\alpha,e_\beta)e^{i\tau A^*}g_\beta(x)d\tau\right\|_{\mathbf{L}^2}^2 \leq \\ \leq M_1 &\sum_{\alpha,\beta=1}^m |(Le_\alpha,e_\beta)|^2\int\limits_{t_1}^{t_2}|(\Phi e^{-i\tau A}f,e_\alpha)|^2 d\tau\int\limits_{t_1}^{t_2}\|e^{i\tau A^*}\Phi^* e_\beta\|^2 d\tau\end{aligned}$$

($M_1$ is a suitable constant). But straightforward calculations show that $\|\Phi e^{-itA}\|_{\mathbf{L}^2} \in \mathbf{L}^2(\mathbb{R})$ as a function of $t$. Then $\|e^{itA^*}\Phi^* e_\beta\|_{\mathbf{L}^2} \leq \|e^{itA^*}\Phi^*\|.\|e_\beta\| = \|\Phi e^{-itA}\|_{\mathbf{L}^2}$ and hence the function $\|e^{itA^*}\Phi^* e_\beta\|_{\mathbf{L}^2}$ belongs to $\mathbf{L}^2(\mathbb{R})$ as a function of $t$. The integrability of $\|e^{itA^*}\Phi^* e_\beta\|_{\mathbf{L}^2}^2$ together with $\Phi e^{-itA}f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^m)$ and the last inequalities imply that

$$\|W(t_2)f - W(t_1)f\|_{\mathbf{L}^2} \to 0 \quad \text{as } t \to \pm\infty.$$

Then for the uniformly bounded set $W(t)$, $t \in \mathbb{R}$, there exist the limits

$$\lim_{t\to\pm\infty} W(t)f \quad \text{for all} \ \ f \in \mathbf{L}^2(0,l;\mathbb{C}^n),$$

i. e. there exist the strong limits

$$s - \lim_{t\to\pm\infty} W(t) = s - \lim_{t\to\pm\infty} e^{itA^*}e^{-itA}$$

onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$. The proof is complete. □

From the results in [9] about the form of the wave operators as weak limits and Theorem 1 it follows that

$$s - \lim_{t\to\pm\infty} e^{itA^*}e^{-itA} = W_\pm(A^*,A) = \widetilde{S}^*_\mp \widetilde{S}_\mp,$$

where the operators $\widetilde{S}_\pm$ are defined by (29).

**3. The operators $A_\gamma = \gamma A + \overline{\gamma} A^*$ with $\mathrm{Re}\,\gamma = 1/2$.** Properties of the operators from the form $A_\gamma = \gamma A + \overline{\gamma} A^*$ with $\gamma = 1/2$ for operators $A$ with a finite imaginary part are presented in [6], properties of $A_\gamma$ with $\mathrm{Re}\,\gamma = 1/2$ for dissipative operators $A$ with a trace class imaginary part $(A - A^*)/i$ are presented in [16].

The obtained asymptotics (56) for the nondissipative curves $e^{itA}f$ as $t \to \pm\infty$ with $A \in \widetilde{\Omega}_{\mathbb{R}}$ in [9] allow us to study some properties of the operators

$$A_\gamma = \gamma A + \overline{\gamma} A^* \quad (\mathrm{Re}\,\gamma = 1/2)$$

in the case of the operator $A \in \widetilde{\Omega}_{\mathbb{R}}$, satisfying the conditions introduced in part 2, to obtain the asymptotics of the curve $e^{itA_\gamma}f$ onto the absolutely continuous subspace as $t \to \pm\infty$ and to consider the wave operators and the scattering operators for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$

Let the model $A \in \widetilde{\Omega}_{\mathbb{R}}$ be defined by (1). Let the function $\alpha(x)$ satisfies the conditions (i), (ii), (iii) and $\Pi(x)$, $B(x)$, $Q(x)$, $J_1$, $J_2$, $S$, $L$ are like in part 2 stated. Let $A$ be embedded in the colligation $X$ from (32) with $\Phi$, $\Phi^*$ satisfying (33) and (34). Let $\{g_\alpha\}_1^m$ be the channel elements of the operator $A$:

$$g_\alpha(x) = \Phi^* e_\alpha = e_\alpha \Pi^*(x)\ , \ \ x \in [0;l], \tag{40}$$

where $\{e_\alpha\}_1^m$ is an orthonormal basis in $\mathbb{C}^m$. Then the characteristic operator function $W(\lambda)$ of the colligation $X$ has the representation

$$W(\lambda) = I - iL\|((A - \lambda I)^{-1} g_\alpha, g_\beta)\|. \tag{41}$$

Let us consider the next matrix function

$$V_\gamma(\lambda) = |\gamma|.\|((A_\gamma - \lambda I)^{-1} g_\alpha, g_\beta)\|. \tag{42}$$

But

$$(A - \lambda I)^{-1} - (A_\gamma - \lambda I)^{-1} = -\overline{\gamma}(A_\gamma - \lambda I)^{-1}(A - A^*)(A - \lambda I)^{-1}, \tag{43}$$

$$(A - \lambda I)^{-1} - (A_\gamma - \lambda I)^{-1} = -\overline{\gamma}(A - \lambda I)^{-1}(A - A^*)(A_\gamma - \lambda I)^{-1}. \tag{44}$$

Using (41), (42) and $(A - A^*)/i = \Phi^* L \Phi$ from (43) and (44) it can be obtained

$$I - W(\lambda) = i|\gamma|^{-1}(\gamma I + \overline{\gamma} W(\lambda)) L V_\gamma(\lambda), \tag{45}$$

$$I - W(\lambda) = i|\gamma|^{-1} L V_\gamma(\lambda)(\gamma I + \overline{\gamma} W(\lambda)). \tag{46}$$

Then we have

$$\begin{aligned} V_\gamma(\lambda) = i|\gamma| L^{-1}(\gamma I + \overline{\gamma} W(\lambda))^{-1}(W(\lambda) - I) = \\ = i|\gamma| L^{-1}(W(\lambda) - I)(\gamma I + \overline{\gamma} W(\lambda))^{-1}, \end{aligned} \tag{47}$$

$$\begin{aligned} W(\lambda) = (|\gamma| I - i\gamma L V_\gamma(\lambda))(|\gamma| I + i\overline{\gamma} L V_\gamma(\lambda))^{-1} = \\ = (|\gamma| I + i\overline{\gamma} L V_\gamma(\lambda))^{-1}(|\gamma| I - i\gamma L V_\gamma(\lambda)). \end{aligned} \tag{48}$$

The equalities (45) and (46) show that

$$\begin{aligned} \tfrac{1}{|\gamma|^2 + \gamma^2}(\gamma I + \overline{\gamma} W(\lambda))(\gamma I + i|\gamma| L V_\gamma(\lambda)) = \\ = \tfrac{1}{|\gamma|^2 + \gamma^2}(\gamma I + i|\gamma| L V_\gamma(\lambda))(\gamma I + \overline{\gamma} W(\lambda)) = I \end{aligned} \tag{49}$$

$(\forall \lambda \notin [0; l])$. But $V_\gamma(\lambda)$ has the form

$$V_\gamma(\lambda) h = |\gamma| \Phi (A_\gamma - \lambda I)^{-1} \Phi^* h \quad (h \in \mathbb{C}^m). \tag{50}$$

From the results of Birman and Entina ([1]) it follows that there exist the limits

$$V_\gamma^{\pm}(x) = s - \lim_{\delta \to 0} V_\gamma(x \pm i\delta) \tag{51}$$

for almost all $x \in \mathbb{R}$ for the operators $V_\gamma(\lambda)$ $(\lambda = x \pm i\delta)$ with the form (50).

On the one hand from the form of the characteristic operator function

$$W(\lambda) = \left( -\overset{\overrightarrow{l}}{\underset{0}{\int}} e^{\frac{-iJ_2 B(\theta) J_2}{\lambda - \alpha(\theta)} d\theta} L + L + I \right) \left( \overset{\overrightarrow{l}}{\underset{0}{\int}} e^{\frac{iJ_1 B(\theta) J_1}{\lambda - \alpha(\theta)} d\theta} L - L + I \right) \tag{52}$$

of the colligation $X$, presented in [2] and the analogue for the multiplicative integrals of the well-known Privalov's Theorem, obtained by L. S. Sakhnovich in [16], it follows that there exist the limits

$$W^{\pm}(x) = s - \lim_{\delta \to 0} W(x \pm i\delta)$$

for all $x \in \mathbb{R}$ (using the assumptions for $B(x)$ and $\alpha(x)$).

On the other hand (51) implies that the set of operators $\{V_\gamma(x \pm i\delta)\}$ is uniformly bounded for almost all fixed $x \in \mathbb{R}$ (using for example, [8], Theorem III.1.29). Then from (49) it follows that for almost all $x$ there exist the bounded operators $(\gamma I + \overline{\gamma} W^{\pm}(x))^{-1}$ and from (47) we obtain

$$V_\gamma^{\pm}(x) = i|\gamma| L^{-1}(\gamma I + \overline{\gamma} W^{\pm}(x))^{-1}(W^{\pm}(x) - I).$$

Let us denote the next matrix functions

$$\widetilde{W}(x, \lambda) = I - iL\|((A - \lambda I)^{-1} g_\alpha, \widehat{g}_\beta(x))\|, \tag{53}$$

$$\widetilde{V}_\gamma(x, \lambda) = |\gamma|.\|((A_\gamma - \lambda I)^{-1} g_\alpha, \widehat{g}_\beta(x))\|, \tag{54}$$

where $\widehat{g}_\beta(x) = g_\beta(u)\chi_{[0;x]}(u) = e_\beta \Pi^*(u)\chi_{[0;x]}(u)$, $x \in [0; l]$, $\chi_{[0;x]}(u)$ is the characteristic function of the interval $[0; x]$. Using the notations (53) and (54) analogously to (47) we obtain the next representation of $\widetilde{V}_\gamma(x, \lambda)$

$$\widetilde{V}_\gamma(x, \lambda)h = i|\gamma| h L^{-1}(\gamma I + \overline{\gamma} W(\lambda))^{-1}(\widetilde{W}(x, \lambda) - I) \tag{55}$$

for each $h \in \mathbb{C}^m$. But from the form (40) of $g_\alpha(x)$ it follows that

$$\widetilde{V}_\gamma(x, \lambda)h = |\gamma| \int_0^x h\Pi^*(y)(A_\gamma - \lambda I)^{-1}\Pi(y)dy, \tag{56}$$

$$(\widetilde{W}(x, \lambda) - I)h = -ihL \int_0^x \Pi^*(y)(A - \lambda I)^{-1}\Pi(y)dy, \tag{57}$$

where $h \in \mathbb{C}^m$. Now the representations (55), (56) and (57) imply that

$$\begin{aligned} &\int_0^x h\Pi^*(y)(A_\gamma - \lambda I)^{-1}\Pi(y)dy = \\ &= hL^{-1}(\gamma I + \overline{\gamma} W(\lambda))^{-1} L \int_0^x \Pi^*(y)(A - \lambda I)^{-1}\Pi(y)dy \end{aligned} \tag{58}$$

and by differentiating of the equality (58) it follows that

$$(.)\Pi^*(x)(A_\gamma - \lambda I)^{-1}\Pi(x) = \\ = (.)L^{-1}(\gamma I + \overline{\gamma} W(\lambda))^{-1} L \Pi^*(x)(A - \lambda I)^{-1}\Pi(x). \tag{59}$$

Let us denote

$$G_\gamma(\lambda) = L^{-1}(\gamma I + \overline{\gamma} W(\lambda))^{-1} L. \tag{60}$$

Then from (59) and the form of $Q(x)$ we obtain

$$(A_\gamma - \lambda I)^{-1} h = hQ^*(x) G_\gamma(\lambda)\Pi^*(x)(A - \lambda I)^{-1} \quad (h \in \mathbb{C}^n) \tag{61}$$

But $A$ is a coupling of two operators and from (3) and (4) the resolvent of $A$ has the representation

$$(A - \lambda I)^{-1} = P_1(P_1 A P_1 - \lambda I)^{-1} P_1 + \\ + P_2(P_2 A P_2 - \lambda I)^{-1} P_2 - P_1(P_1 A P_1 - \lambda I)^{-1} P_1 A P_2 (P_2 A P_2 - \lambda I)^{-1} P_2. \tag{62}$$

Straightforward calculations show that for each $f \in \widetilde{H}_0$ after the change of the variable $x = \sigma(u)$ and using the introduced notations, the equalities (62) and (61), $(A_\gamma - \lambda I)^{-1}$ takes the form

$$(A_\gamma - \lambda I)^{-1} f(\sigma(u)) = \tfrac{1}{u-\lambda} \int_a^b \widetilde{g}_\lambda{}'(w)(F_w(\lambda, u) J_1 \chi_{[a;u]}(w) + \\ + \widetilde{F}_w(\lambda, u) J_2 \chi_{[a;u]}(w) - \widetilde{F}_w(\lambda, b) S F_a(\lambda, u) J_1)\Pi^*(\sigma(u)) dw, \tag{63}$$

where $\widetilde{g}_\lambda(w) = \widetilde{f}(w) G_\gamma(\lambda)\Pi^*(\sigma(w)) Q^*(\sigma(w))$ and $\widetilde{f}(w)$ is defined by (33).

Let us denote now

$$F(\lambda) f(\sigma(u)) = (A_\gamma - \lambda I)^{-1} f(\sigma(u)), \tag{64}$$

$$F_\varepsilon(\lambda) f(\sigma(u)) = (u - \lambda)^\varepsilon (A_\gamma - \lambda I)^{-1} f(\sigma(u)) \tag{65}$$

for each $\varepsilon > 0$ sufficiently small and $f \in \widetilde{H}_0$. Let us present the space $\mathbf{L}^2(0, l; \mathbb{C}^n)$ in the form

$$\mathbf{L}^2(0, l; \mathbb{C}^n) = H_{ac}^\gamma \oplus H_s^\gamma$$

where $H_{ac}^\gamma$ is the absolutely continuous subspace of the selfadjoint operator $A_\gamma$ and $H_s^\gamma$ is the singular subspace of $A_\gamma$. Next theorem describes a suitable representation of $F_\varepsilon(\lambda) f(\sigma(u))$ which we will use to obtain the asymptotics for the curve $e^{itA_\gamma} f$ onto $H_{ac}^\gamma$.

From the theory of the selfadjoint operators (using the representation (50)) it follows that $V_\gamma(\lambda)$ has the representation

$$V_\gamma(\lambda) = |\gamma| \int_a^b \frac{Z(t)dt}{t-\lambda}, \tag{66}$$

onto the absolutely continuous subspace $H_{ac}^\gamma$ where the matrix function $Z(t)$ has the form $Z(t) = \omega'(t)$ for almost all $t \in [a;b]$ and $\omega(t)$ is nondecreasing matrix function with $\omega(-\infty) = 0$, $\omega(+\infty) = \|(g_\alpha, g_\beta)\|$.

For the sace of clarity let us suppose that $Z(t)$ satisfies the condition

$$\|Z(t_1) - Z(t_2)\| \le M|t_1 - t_2|^\beta \quad \forall t_1, t_2 \in [a;b], \tag{67}$$

where $M > 0$ is a constant, $0 < \beta \le 1$. The inequality (67) implies that there exist the limits $s - \lim\limits_{\delta\to 0} V_\gamma(x \pm i\delta) = V_\gamma^\pm(x)$ for all $x \in \mathbb{R}$. Hence using the equalities (49) we have

$$s-\lim_{\delta\to 0} G_\gamma(x\pm i\delta) = G_\gamma^\pm(x) = \frac{1}{|\gamma|^2+\gamma^2}(\gamma I + i|\gamma|V_\gamma^\pm(x)L) = L^{-1}(\gamma I + \overline{\gamma}W^\pm(x))^{-1}$$

for all $x \in \mathbb{R}$ and it can be shown that

$$\|G_\gamma^\pm(x)\| \le M_1 + M_1 \max\{1/(x-a)^{\beta_1}; 1/(b-x)^{\beta_1}\} \tag{68}$$

$\forall x \in (a;b)$, $M_1 > 0$ is a suitable constant and $\beta_1 \ : \ 0 < \beta_1 < 1$ is sufficiently small.

**Theorem 2.** *Let for the model $A \in \widetilde{\Omega}_{\mathbb{R}}$, defined by* (1), *next conditions hold:*

1) *the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies* (i), (ii), (iii)*;*
2) *$Q^*(x)$ is a smooth matrix function on $[0;l]$;*
3) *$B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \le 1)$;*
4) *the matrix function $Z(t)$ from the representation* (66) *satisfies the condition* (67).

*Then the operator function $F_\varepsilon(\lambda)$ for each $f \in \widetilde{H}_0 \cap H_{ac}^\gamma$ after the change of the variable $x = \sigma(u)$ has the representation*

$$\begin{aligned} F_\varepsilon(\lambda)f(\sigma(u)) = {} & \tfrac{1}{2\pi i}\int_a^b \tfrac{1}{x-\lambda}\tfrac{1}{(u-x)^{1-\varepsilon}}\int_a^b (\widetilde{g}_x{}'^{+}(w)(F_w^+(x,u)J_1\chi_{[a;u]}(w) + \\ & + \widetilde{F}_w^+(x,u)J_2\chi_{[a;u]}(w) - \widetilde{F}_w^+(x,b)SF_a^+(x,u)J_1) - \\ & - \widetilde{g}_x{}'^{-}(w)(F_w^-(x,u)J_1\chi_{[a;u]}(w) + \widetilde{F}_w^-(x,u)J_2\chi_{[a;u]}(w) - \\ & - \widetilde{F}_w^-(x,b)SF_a^-(x,u)J_1))dw\Pi^*(\sigma(u))dx, \end{aligned} \tag{69}$$

*where*

$$\widetilde{g}_x{}^{\pm}(w) = \widetilde{f}(w) G_\gamma^{\pm}(x) \Pi^*(\sigma(w)) Q^*(\sigma(w)); \quad \widetilde{g}_x{}'^{\pm}(w) = \frac{d}{dw} \widetilde{g}_x^{\pm}(w) \tag{70}$$

*and* $G_\gamma^{\pm}(x) = s - \lim\limits_{\delta \to 0} G_\gamma^{\pm}(x \pm i\delta) = L^{-1}(\gamma I + \overline{\gamma} W^{\pm}(x))^{-1} L.$

Proof. For the proof of the representation (69) we will use the ideas from the proof of Lemma 8 in [9]. Let $\lambda \in \mathbb{C} \setminus [a; b]$. Let $h > 0$ be an arbitrary fixed sufficiently small such that $\lambda$ belongs to the domain $\widehat{G}$ with a boundary $\Gamma = L_R \cup l_1(\delta) \cup l_2(\delta) \cup l_1'(\delta) \cup l_2'(\delta)$, where

$$\begin{aligned} L_R &= \left\{ z = Re^{i\varphi} \ : \ 0 \le \varphi \le 2\pi \right\}, \\ l_1(\delta) &= \left\{ z = x - i\delta : \ a_1 \le x \le b_1 \right\}, \\ l_2(\delta) &= \left\{ z = x + i\delta : \ a_1 \le x \le b_2 \right\}, \\ l_1'(\delta) &= \left\{ z = b_1 + i\tau : \ -\delta \le \tau \le \delta \right\}, \\ l_2'(\delta) &= \left\{ z = a_1 + i\tau : \ -\delta \le \tau \le \delta \right\}, \end{aligned}$$

$a_1 = a - h$, $b_1 = b + h$, for each $R > 0$ sufficiently large and for each $\delta > 0$ sufficiently small. Using the Cauchy integral formula for the operator function $F_\varepsilon(\lambda)$ in the domain $\widehat{G}$ we obtain

$$F_\varepsilon(\lambda) f(\sigma(u)) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{F_\varepsilon(z) f(\sigma(u))}{z - \lambda} dz, \tag{71}$$

where $\lambda \in \widehat{G}$, $f \in \widetilde{H}_0 \cap H_{ac}^\gamma$. But

$$\lim_{R \to +\infty} \int\limits_{L_R} \frac{F_\varepsilon(z) f(\sigma(u))}{z - \lambda} dz = 0 \tag{72}$$

by the Lebesgue convergence theorem because

$$\begin{aligned} &\left\| \frac{Re^{i\varphi}}{Re^{i\varphi} - \lambda} F_\varepsilon(Re^{i\varphi}) f(\sigma(u)) \right\| = \\ &= \left\| \frac{Re^{i\varphi}(u - Re^{i\varphi})^\varepsilon}{Re^{i\varphi} - \lambda} (A_\gamma - Re^{i\varphi} I)^{-1} f(\sigma(u)) \right\| \to 0 \end{aligned} \tag{73}$$

as $R \to +\infty$, $\forall \varphi \in [0; 2\pi]$ and

$$\left\| \frac{Re^{i\varphi}}{Re^{i\varphi} - \lambda} F_\varepsilon(Re^{i\varphi}) f(\sigma(u)) \right\| \le C_1 |f(\sigma(u))| \tag{74}$$

for all $\varphi \in [0; 2\pi]$ and each $R > 0$ sufficiently large, where $C_1 > 0$ is a suitable constant. In (73) and (74) we have used the form of $F_\varepsilon(Re^{i\varphi})$ and the representation of the resolvent of a bounded linear operator.

Now for the integrals we obtain

$$\frac{1}{2\pi i}\int\limits_{l_1'(\delta)} \frac{F_\varepsilon(z)f(\sigma(u))}{z-\lambda}dz + \frac{1}{2\pi i}\int\limits_{l_2'(\delta)} \frac{F_\varepsilon(z)f(\sigma(u))}{z-\lambda}dz =$$
$$= -\frac{1}{2\pi i}\int\limits_{-\delta}^{\delta} \frac{(u-b_1-i\tau)^\varepsilon}{b_1+i\tau-\lambda}F(b_1+i\tau)f(\sigma(u))d\tau - \frac{1}{2\pi i}\int\limits_{-\delta}^{\delta} \frac{(u-a_1-i\tau)^\varepsilon}{a_1+i\tau-\lambda}F(a_1+i\tau)f(\sigma(u))d\tau.$$

Using the connection between $G_\gamma(\lambda)$ and $V_\gamma(\lambda)$:

$$G_\gamma(\lambda) = \frac{1}{|\gamma|^2+\gamma^2}(\gamma I + i|\gamma|V_\gamma(\lambda)L)$$

and the existence of the limits $s - \lim\limits_{\xi\to 0} G_\gamma(a_1 \pm i\xi)$, $s - \lim\limits_{\xi\to 0} G_\gamma(b_1 \pm i\xi)$ it follows that

$$\|G_\gamma(a_1+i\tau)\| \le M_2, \quad \|G_\gamma(b_1+i\tau)\| \le M_2 \tag{75}$$

for all $\tau : |\tau| \le \delta$ and for some constant $M_2 > 0$. The inequalities (75), the form of $\widetilde{g}_{a_1+i\tau}(w)$, $\widetilde{g}_{b_1+i\tau}(w)$, $F(a_1+i\tau)f(\sigma(u))$, $F(b_1+i\tau)f(\sigma(u))$, $F_w(\xi+i\tau,u)$, $\widetilde{F}_w(\xi+i\tau,u)$ and the properties of the multiplicative integrals show that

$$\left\| \frac{(u-b_1-i\tau)^\varepsilon}{b_1+i\tau-\lambda}F(b_1+i\tau)f(\sigma(u)) \right\| \le \frac{1}{|b_1-\lambda|-\delta}\cdot\frac{1}{(h-\delta)^{1-\varepsilon}}M_3,$$
$$\left\| \frac{(u-a_1-i\tau)^\varepsilon}{a_1+i\tau-\lambda}F(a_1+i\tau)f(\sigma(u)) \right\| \le \frac{1}{|a_1-\lambda|-\delta}\cdot\frac{1}{(h-\delta)^{1-\varepsilon}}M_3$$

for all $\tau : |\tau| \le \delta$, $M_3 > 0$ is a suitable constant. Hence

$$\lim_{\delta\to 0}\frac{1}{2\pi i}\int\limits_{l_1'(\delta)} \frac{F_\varepsilon(z)f(\sigma(u))}{z-\lambda}dz = 0\ , \quad \lim_{\delta\to 0}\frac{1}{2\pi i}\int\limits_{l_2'(\delta)} \frac{F_\varepsilon(z)f(\sigma(u))}{z-\lambda}dz = 0. \tag{76}$$

Passing to limits $R \to +\infty$ and $\delta \to 0$ in (71) and applying (72) and (76) we obtain that

$$\begin{aligned} F_\varepsilon(\lambda)f(\sigma(u)) = {} & \frac{1}{2\pi i}\int\limits_{a_1}^{b_1} \frac{1}{x-\lambda}\frac{1}{(u-x)^{1-\varepsilon}}\int\limits_a^b (\widetilde{g}_x{'}^{+}(w)(F_w^+(x,u)J_1\chi_{[a;u]}(w)+ \\ & + \widetilde{F}_w^+(x,u)J_2\chi_{[a;u]}(w) - \widetilde{F}_w^+(x,b)SF_a^+(x,u)J_1)- \\ & - \widetilde{g}_x{'}^{-}(w)(F_w^-(x,u)J_1\chi_{[a;u]}(w) + \widetilde{F}_w^-(x,u)J_2\chi_{[a;u]}(w)- \\ & - \widetilde{F}_w^-(x,b)SF_a^-(x,u)J_1))dw\Pi^*(\sigma(u))dx. \end{aligned} \tag{77}$$

In the equality (77) we have used first the Lebesgue convergence theorem, then the existence of the limits $G_\gamma^\pm(x)$ and $F_w^\pm(x,u)$, $\widetilde{F}_w^\pm(x,u)$ for the multiplicative integrals $F_w(x \pm i\delta, u)$, $\widetilde{F}_w(x \pm i\delta, u)$ as $\delta \to 0$ (when $x \in \mathbb{R}$). Now using the form of the limits $\widetilde{g}_x{}'^{\pm}(w)$, $F_w^\pm(x,u)$, $\widetilde{F}_w^\pm(x,u)$ when $x \in \mathbb{R} \setminus [a;b]$ the equality (77) take the form (69). The theorem is proved. □

From the well-known formula about the representation of $e^{itA_\gamma}$ we have

$$e^{itA_\gamma} = -\frac{1}{2\pi i} \int\limits_\Gamma e^{it\lambda}(A_\gamma - \lambda I)^{-1} d\lambda,$$

where $\Gamma$ is closed contour containing $[\alpha(0);\alpha(l)]$. Then the equality $(A_\gamma - \lambda I)^{-1} f(\sigma(u)) = \lim\limits_{\varepsilon\to 0} F_\varepsilon(\lambda) f(\sigma(u))$ and (69) for $f \in \widetilde{H}_0 \cap H_{ac}^\gamma$ after calculations show that

$$\begin{aligned}
e^{itA_\gamma} f(\sigma(u)) = \lim_{\varepsilon\to 0} \tfrac{1}{2\pi i} \int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} \int\limits_a^b (&\widetilde{g}_x{}'^{+}(w)(F_w^+(x,u)J_1\chi_{[a;u]}(w) + \\
&+ \widetilde{F}_w^+(x,u)J_2\chi_{[a;u]}(w) - \widetilde{F}_w^+(x,b)SF_a^+(x,u)J_1) - \\
&- \widetilde{g}_x{}'^{-}(w)(F_w^-(x,u)J_1\chi_{[a;u]}(w) + \widetilde{F}_w^-(x,u)J_2\chi_{[a;u]}(w) - \\
&- \widetilde{F}_w^-(x,b)SF_a^-(x,u)J_1))dw\Pi^*(\sigma(u))dx.
\end{aligned} \tag{78}$$

We present (78) in the convinience form for the further applications of this formula:

$$\begin{aligned}
e^{itA_\gamma} f(\sigma(u)) = \tfrac{1}{2\pi i} \int\limits_a^u \lim_{\varepsilon\to 0} \Bigg( &\int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} (\widetilde{g}_x{}'^{+}(w) - \widetilde{g}_x{}'^{-}(w)) F_w^+(x,u) dx J_1 + \\
&+ \int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} (\widetilde{g}_x{}'^{+}(w) - \widetilde{g}_x{}'^{-}(w)) \widetilde{F}_w^+(x,u) dx J_2 + \\
&+ \int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) P_w(x,u) dx J_1 + \\
&+ \int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) \widetilde{P}_w(x,u) dx J_2 \Bigg) dw \Pi^*(\sigma(u)) - \\
- \tfrac{1}{2\pi i} \int\limits_a^b \lim_{\varepsilon\to 0} \Bigg( &\int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} (\widetilde{g}_x{}'^{+}(w) - \widetilde{g}_x{}'^{-}(w)) \widetilde{F}_w^+(x,b) S F_a^+(x,u) dx J_1 + \\
&+ \int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) \widetilde{P}_w(x,b) S F_a^+(x,u) dx J_1 + \\
&+ \int\limits_a^b \tfrac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) \widetilde{F}_w^-(x,w) S P_a(x,u) dx J_1 \Bigg) dw \Pi^*(\sigma(u)).
\end{aligned} \tag{79}$$

Next theorem gives the asymptotics of the curve $e^{itA_\gamma}f$ as $t \to \pm\infty$ for $f \in H^\gamma_{ac} \cap \widetilde{H}_0$ after the change of the variable $x = \sigma(u)$. The proof of this asymptotics is based on the ideas of the proofs of the asymptotics (26) for the nondissipative curves $e^{itA}f$ $(A \in \widetilde{\Omega}_{\mathbb{R}})$, obtained in detail in [9].

**Theorem 3.** *Let for the model $A \in \widetilde{\Omega}_{\mathbb{R}}$, defined by (1), next conditions hold:*

1) *the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies* (i), (ii), (iii);
2) *$Q^*(x)$ is a smooth matrix function on* $[0;l]$;
3) $B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \leq 1)$,
4) *the matrix function $Z(t)$ from the representation* (66) *satisfies the condition* (67).

*Then the selfadjoint curve $e^{itA_\gamma}f$ for each $f \in \widetilde{H}_0 \cap H^\gamma_{ac}$ after the change of the variable $x = \sigma(u)$ has the asymptotics*

$$\|e^{itA_\gamma}f(\sigma(u)) - e^{itu}S_{\gamma\pm}f(\sigma(u))\|_{\mathbf{L}^2} \to 0 \tag{80}$$

*as $t \to \pm\infty$, where*

$$S_{\gamma\pm}f(\sigma(u)) =$$

$$= \int_a^u \widetilde{g}_u{}'^{\mp}(w)U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)}dw|t|^{i\widetilde{B}_1(u)}e^{\mp\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))J_1\Pi^*(\sigma(u))+$$
$$+\int_a^u \widetilde{g}_u{}'^{\mp}(w)\widetilde{U}_{2w}(u)(u-w)^{-i\widetilde{B}_2(u)}dw|t|^{-i\widetilde{B}_2(u)}e^{\pm\frac{\pi}{2}\widetilde{B}_2(u)}\mathbf{\Gamma}^{-1}(I-i\widetilde{B}_2(u))J_2\Pi^*(\sigma(u))-$$
$$-\int_a^b \widetilde{g}_u{}'^{\mp}(w)\widetilde{F}^{\mp}_w(u,b)dwSU_{2a}(u)(u-a)^{i\widetilde{B}_1(u)}\cdot$$
$$\cdot|t|^{i\widetilde{B}_1(u)}e^{\mp\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))J_1\Pi^*(\sigma(u)),$$

*$\widetilde{g}_u{}^{\pm}(w)$ is defined by* (70).

P r o o f. We will consider in detail the case when $t \to +\infty$. In the course of proving of (80) we will obtain the asymptotics of the inner integrals in the representation (79) of the curve $e^{itA_\gamma}f$ with $f \in H^\gamma_{ac} \cap \widetilde{H}_0$. Let $\alpha = \min\{\alpha_1, \alpha_2\}$.

From (79) and the form (52) of the complete characteristic operator function $W(\lambda)$ of $A$ we see that

$$\begin{aligned} G^{\pm}_\gamma(x) - G^{\pm}_\gamma(u) = \overline{\gamma}G^{\pm}_\gamma(x)(\widetilde{F}^{\pm}_a(x,b) - \widetilde{F}^{\pm}_a(u,b))(LF^{\pm}_a(u,b) - L + I)- \\ -(-L\widetilde{F}^{\pm}_a(x,b) + L + I)L(F^{\pm}_a(x,b) - F^{\pm}_a(u,b))LG^{\pm}_\gamma(u) \end{aligned} \tag{81}$$

for all $u, x \in [a;b]$. The inequalities (19) for $\widetilde{F}^{\pm}_a(x,b)$, $\widetilde{F}^{\pm}_a(u,b)$, $F^{\pm}_a(x,b)$, $F^{\pm}_a(u,b)$

and (68) show that

$$\lim_{\varepsilon\to 0}\int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}(\widetilde{g}_x{}'^{+}(w)-\widetilde{g}_x{}'^{-}(w))F_w^{+}(x,u)dx \sim \sim (\widetilde{g}_u{}'^{+}(w)-\widetilde{g}_u{}'^{-}(w))\lim_{\varepsilon\to 0}\int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}F_w^{+}(x,u)dx. \tag{82}$$

On the other hand

$$\lim_{\varepsilon\to 0}\int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}F_w^{+}(x,u)dx = \lim_{\varepsilon\to 0}\left(\int_a^w \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\overset{\rightarrow}{\int_w^u} e^{\frac{-i\widetilde{B}_1(v)}{v-x}dv}dx + + \int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}F_w^{+}(x,u)dx + \int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\overset{\rightarrow}{\int_w^u} e^{\frac{-i\widetilde{B}_1(v)}{v-x}dv}dx\right). \tag{83}$$

Using (20) and (21) for the positive matrix function $T(x)=\widetilde{B}_1(x)$, $\alpha'=\alpha/(1+\alpha)$ and applying the Lebesgue convergence theorem and the Lebesgue lemma for the Fourier transform it follows that

$$\lim_{\varepsilon\to 0}\int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}F_w^{+}(x,u)dx \sim \lim_{\varepsilon\to 0}\int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}Q_w^{+}(x)e^{-i\widetilde{B}_1(u)\ln(u-x)}dx \sim \sim Q_w^{+}(u)\lim_{\varepsilon\to 0}\int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(u)\ln(u-x)}dx \tag{84}$$

as $t\to+\infty$. In [9] it has been obtained the asymptotics

$$\lim_{\varepsilon\to 0}\int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(u)\ln(u-x)}dx \sim$$

$$\sim \pi i e^{itu}t^{i\widetilde{B}_1(u)}e^{-\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))(\sinh(\pi\widetilde{B}_1(u)))^{-1}$$

as $t\to+\infty$ which together with (84) implies that

$$\lim_{\varepsilon\to 0}\int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\widetilde{F}_w^{+}(x,u)dx \sim \sim \pi i Q_w^{+}(u)e^{itu}t^{i\widetilde{B}_1(u)}e^{-\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))(\sinh(\pi\widetilde{B}_1(u)))^{-1} \tag{85}$$

as $t\to+\infty$. Now from the inequality (22) for $T(x)=\widetilde{B}_1(x)$ and $a\le w\le u\le x\le b$ and (17) for $U_{2w}(x)$ and $U_{2w}(u)$ applying (23) and (24) we obtain

(by the Lebesgue convergence theorem and the Lebesgue lemma for the Fourier transform)

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\overset{\rightarrow}{\int_w^u} e^{\frac{-i\widetilde{B}_1(v)}{v-x}dv}dx \sim \lim_{\varepsilon\to 0}\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}U_{2w}(x)e^{-i\widetilde{B}_1(x)\ln\frac{x-u}{x-w}}dx \sim \\
(86)\qquad &\sim \lim_{\varepsilon\to 0} U_{2w}(u)\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(x)\ln\frac{x-u}{x-w}}dx \sim \\
&\sim U_{2w}(u)e^{i\widetilde{B}_1(u)\ln(u-w)}\lim_{\varepsilon\to 0}\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(x)\ln(x-u)}dx \sim \\
&\sim U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)}\lim_{\varepsilon\to 0}\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(u)\ln(x-u)}dx
\end{aligned}
$$

as $t\to+\infty$. After suitable change of the variable the last integral in the relations (86) takes the form

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(u)\ln(x-u)}dx = \\
(87)\qquad &= -e^{itu}t^{i\widetilde{B}_1(u)}e^{\frac{\pi}{2}\widetilde{B}_1(u)}\lim_{\varepsilon\to 0}\int_0^{(b-u)te^{-i\frac{\pi}{2}}} e^{-\theta}e^{((\varepsilon-1)I-i\widetilde{B}_1(u))\ln\theta}d\theta \sim \\
&\sim -e^{itu}t^{i\widetilde{B}_1(u)}e^{\frac{\pi}{2}\widetilde{B}_1(u)}\lim_{\varepsilon\to 0}\mathbf{\Gamma}(\varepsilon I-i\widetilde{B}_1(u)) = \\
&= -\pi i e^{itu}t^{i\widetilde{B}_1(u)}e^{\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))(\sinh(\pi\widetilde{B}_1(u)))^{-1}
\end{aligned}
$$

as $t\to+\infty$. In the last equality in (87) we have used the existence and the form of the limit $\lim\limits_{\varepsilon\to 0}\mathbf{\Gamma}(\varepsilon I-i\widetilde{B}_1(u))$ for the analogue in $\mathbb{C}^m$ of the classical gamma-function (obtained in [9]). Consequently, from (86) and (87) it follows that

$$
\begin{aligned}
(88)\qquad &\lim_{\varepsilon\to 0}\int_u^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}F_w^+(x,u)dx \sim \\
&\sim -\pi i e^{itu}U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)}t^{i\widetilde{B}_1(u)}e^{\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))(\sinh(\pi\widetilde{B}_1(u)))^{-1}
\end{aligned}
$$

as $t\to+\infty$.

Now from (82), (83), (85), (88) and the representation of $Q_w^+(u)$ as

$$Q_w^+(u) = U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)}e^{\pi\widetilde{B}_1(u)}$$

show that

$$
(89)\qquad \lim_{\varepsilon\to 0}\int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}(\widetilde{g}_x{}'^{+}(w)-\widetilde{g}_x{}'^{-}(w))F_w^+(x,u)dx \sim 0
$$

as $t \to +\infty$.

Analogously for the second integral in (79) it can be proved that

$$\lim_{\varepsilon\to 0} \int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} (\widetilde{g}_x{}'^{+}(w) - \widetilde{g}_x{}'^{-}(w)) \widetilde{F}_w^{+}(x,u)dx \sim 0 \tag{90}$$

as $t \to +\infty$ using the corresponding inequalities for the matrix function $T(x) = -i\widetilde{B}_2(x)$.

For the third integral in (79) because of (68), (81) like in (82) and the form of $P_w(x,u)$ we obtain

$$\lim_{\varepsilon\to 0} \int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) P_w(x,u)dx \sim \widetilde{g}_u{}'^{-}(w) \lim_{\varepsilon\to 0} \int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}} P_w(x,u)dx$$

as $t \to +\infty$. But in [9] has been obtained that

$$\lim_{\varepsilon\to 0} \int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}} P_w(x,u)dx \sim$$
$$\sim 2\pi i e^{itu} U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)} t^{i\widetilde{B}_1(u)} e^{-\frac{\pi}{2}\widetilde{B}_1(u)} \mathbf{\Gamma}^{-1}(I + i\widetilde{B}_1(u)).$$

Hence

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) P_w(x,u)dx \sim \\ &\sim 2\pi i e^{itu} \widetilde{g}_u{}'^{-}(w) U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)} t^{i\widetilde{B}_1(u)} e^{-\frac{\pi}{2}\widetilde{B}_1(u)} \mathbf{\Gamma}^{-1}(I + i\widetilde{B}_1(u)) \end{aligned} \tag{91}$$

as $t \to +\infty$. Analogously for the integral $\lim\limits_{\varepsilon\to 0} \int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) \widetilde{P}_w(x,u)dx$ one can find that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) \widetilde{P}_w(x,u)dx = \lim_{\varepsilon\to 0} \int_w^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{g}_x{}'^{-}(w) \widetilde{P}_w(x,u)dx \sim \\ &\sim 2\pi i e^{itu} \widetilde{g}_u{}'^{-}(w) \widetilde{U}_{2w}(u)(u-w)^{-i\widetilde{B}_2(u)} t^{-i\widetilde{B}_2(u)} e^{\frac{\pi}{2}\widetilde{B}_2(u)} \mathbf{\Gamma}^{-1}(I - i\widetilde{B}_2(u)) \end{aligned} \tag{92}$$

as $t \to +\infty$.

For the other integrals in (79) after analogous calculations we obtain the next asymptotics as $t \to +\infty$:

$$\lim_{\varepsilon\to 0} \int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} (\widetilde{g}_x{}'^{+}(w) - \widetilde{g}_x{}'^{-}(w)) \widetilde{F}_w^{+}(x,b) S F_a^{+}(x,u)dx \chi_{(u;b]}(w) \sim 0, \tag{93}$$

$$\lim_{\varepsilon\to 0}\int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\widetilde{g}_x{}'^{-}(w)\widetilde{P}_w(x,b)SF_a^{+}(x,u)dx =$$
$$(94)\qquad = \lim_{\varepsilon\to 0}\int_w^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\widetilde{g}_x{}'^{-}(w)\widetilde{P}_w(x,b)SF_a^{+}(x,u)dx \sim$$
$$\sim \widetilde{g}_u{}'^{-}(w)\lim_{\varepsilon\to 0}\int_w^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\widetilde{P}_w(x,b)SF_a^{+}(x,u)dx \sim 0,$$

$$\lim_{\varepsilon\to 0}\int_a^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\widetilde{g}_x{}'^{-}(w)\widetilde{F}_w^{-}(x,b)SP_a(x,u)dx =$$
$$(95)\qquad = \lim_{\varepsilon\to 0}\int_a^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}}\widetilde{g}_x{}'^{-}(w)\widetilde{F}_w^{-}(x,b)SP_a(x,u)dx \sim$$
$$\sim 2\pi i e^{itu}\widetilde{g}_u{}'^{-}(w)\widetilde{F}_w^{-}(u,b)SU_{2a}(u)(u-a)^{i\widetilde{B}_1(u)}t^{i\widetilde{B}_1(u)}e^{-\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u)).$$

Now the asymptotics (80) in the case when $t\to+\infty$ follow from (89), (90), (91), (92), (93), (94), (95). Analogously we obtain (80) as $t\to-\infty$ and the proof is complete. □

For the simplification of writing let us denote

$$(96)\qquad S_{\gamma\pm}f(\sigma(u)) = Z(t,u)T_{\pm}\widehat{S}_{\gamma\pm}f(\sigma(u)) = Z(t,u)\widetilde{S}_{\gamma\pm}f(\sigma(u)),$$

where

$$\widehat{S}_{\gamma\pm}f(\sigma(u)) = \int_a^u \widetilde{g}_u{}'^{\mp}(w)\overset{\overset{w}{\leftarrow}}{\int_a} e^{\frac{i\widetilde{B}_1(v)}{v-u}dv}dwJ_1 +$$
$$(97)\qquad + \int_a^u \widetilde{g}_u{}'^{\mp}(w)\overset{\overset{w}{\leftarrow}}{\int_a} e^{\frac{-i\widetilde{B}_2(v)}{v-u}dv}dwJ_2 - \int_a^b \widetilde{g}_u{}'^{\mp}(w)\widetilde{F}_w^{\mp}(u,b)dwS$$

when $f\in H_{ac}^{\gamma}\cap\widetilde{H}_0$ and $\widetilde{g}_u{}^{\pm}(w)$ are defined by (70),

$$(98)\qquad \widetilde{S}_{\gamma\pm}f(\sigma(u)) = T_{\pm}\widehat{S}_{\gamma\pm}f(\sigma(u))$$

and $Z(t,u)$, $T_{\pm}$ are defined by (31), (30).

**4. Wave operators and a scattering operator for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$.** In this part we will show that the basic results from the scattering theory for the selfadjoint operators can be extended for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$ with a nondissipative operator $A\in\widetilde{\Omega}_{\mathbb{R}}$ and $A_\gamma = \gamma A + \overline{\gamma}A^*$. The asymptotics (80) of $A_\gamma$, the asymptotics (26) of nondissipative operator $A$, obtained in [9] allow to obtain the wave operators $W_\pm(A_\gamma, A)$, $W_\pm(A, A_\gamma)$ as strong limits and to obtain an explicit representation of the wave operators, the scattering operator and to establish the similarity of $A$ and $A_\gamma$ by the wave operators. We will also essentially use the results in [9] concerning the form of the form of the operators $\widetilde{S}$ and $\widetilde{S}^{-1}$ establishing the similarity of the nondissipative operator $A$ and the operator of a multiplying by an independent variable.

Let the operator $A \in \widetilde{\Omega}_{\mathbb{R}}$ and $A_\gamma = \gamma A + \overline{\gamma} A^*$ with $Re\gamma = 1/2$ be like above stated. From the asymptotics (80) for $A_\gamma$ and the introduced denotations (96), (31), (98) it follows that

$$\|e^{itu} S_{\gamma\pm} f\|_{\mathbf{L}^2} = \|\widetilde{S}_{\gamma\pm} f\|_{\mathbf{L}^2} = \|f\|_{\mathbf{L}^2} \ \ \forall f \in H^\gamma_{ac}$$

i. e. $(\widetilde{S}_{\gamma\pm}{}^* \widetilde{S}_{\gamma\pm} f, f) = (f, f)$. This implies that $\widetilde{S}_{\gamma\pm}{}^* \widetilde{S}_{\gamma\pm} = I$ onto the subspace $H^\gamma_{ac}$. But from the asymptotics (80) we have $\widetilde{S}_{\gamma\pm} : H^\gamma_{ac} \longrightarrow H^\gamma_{ac}$ because $H^\gamma_{ac}$ is an invariant subspace under $A_\gamma$. Then the isometric operators $\widetilde{S}_{\gamma\pm}$ onto $H^\gamma_{ac}$ imply that the range $R(\widetilde{S}_{\gamma\pm}|_{H^\gamma_{ac}})$ of the operator $\widetilde{S}_{\gamma\pm}|_{H^\gamma_{ac}}$ is a closed linear subspace of $H^\gamma_{ac}$. Then $\widetilde{S}_{\gamma\pm}$ are invertible operators onto $R(\widetilde{S}_{\gamma\pm}|_{H^\gamma_{ac}})$. But

$$S_{\gamma\pm} f(\sigma(u)) = Z(t,u) T_\pm \widehat{S}_{\gamma\pm} f(\sigma(u)) \ , \ \ \widetilde{S}_{\gamma\pm} f(\sigma(u)) = T_\pm \widehat{S}_{\gamma\pm} f(\sigma(u))$$

and when $g \in R(\widetilde{S}_{\gamma\pm}|_{H^\gamma_{ac}})$ it follows that

$$S_{\gamma\pm}{}^{-1} g(\sigma(u)) = \widehat{S}^{-1}_{\gamma\pm} T^{-1}_\pm Z^*(t,u) g(\sigma(u)),$$

$$\widetilde{S}_{\gamma\pm}{}^{-1} g(\sigma(u)) = \widehat{S}^{-1}_{\gamma\pm} T^{-1}_\pm g(\sigma(u)),$$

where the operators $T^{-1}_\pm$ have the form

$$T^{-1}_\pm h =$$

$$=h\Pi(\sigma(u))\sigma'(u) \Big( J_1 i \lim_{\varepsilon\to 0} \mathbf{\Gamma}(\varepsilon I + i\widetilde{B}_1(u)) e^{\pm\frac{\pi}{2}\widetilde{B}_1(u)} (u-a)^{-i\widetilde{B}_1(u)} U^*_{2a}(u) J_1 + \\ + J_2(-i) \lim_{\varepsilon\to 0} \mathbf{\Gamma}(\varepsilon I - i\widetilde{B}_2(u)) e^{\mp\frac{\pi}{2}\widetilde{B}_2(u)} (u-a)^{i\widetilde{B}_2(u)} \widetilde{U}^*_{2a}(u) J_2 \Big) \tag{99}$$

for $h \in \mathbb{C}^n$.

Next we will show that for the couples $(A_\gamma, A)$ and $(A, A_\gamma)$ it can be introduced wave operators like in the selfadjoint case [15, 8, 7], the dissipative case [16] and for $(A^*, A)$ with nondissipative operator $A \in \widetilde{\Omega}_{\mathbb{R}}$ [9]. Let us consider the operators

$$W_t(A_\gamma, A) = e^{itA_\gamma} e^{-itA}, \quad W_t(A, A_\gamma) = e^{itA} e^{-itA_\gamma} \ \ (t \in \mathbb{R})$$

in $\mathbf{L}^2(0,l;\mathbb{C}^n)$. The next theorem states the existence and the form of the strong limits $W_\pm(A_\gamma,A)$ and $W_\pm(A,A_\gamma)$ of $W_t(A_\gamma,A)$ and $W_t(A,A_\gamma)$ onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$ and $H^\gamma_{ac}$ correspondingly as $t\to\pm\infty$. The operators $W_\pm(A_\gamma,A)$ and $W_\pm(A,A_\gamma)$ we will call wave operators as in the selfadjoint case and in the dissipative case.

**Theorem 4.** *Let for the model $A\in\widetilde{\Omega}_{\mathbb{R}}$, defined by (1), next conditions hold:*

1) *the function $\alpha\ :\ [0;l]\longrightarrow\mathbb{R}$ satisfies* (i), (ii), (iii)*;*
2) *$Q^*(x)$ is a smooth matrix function on $[0;l]$;*
3) *$B(x)\in C_{\alpha_1}[0;l]$ $(0<\alpha_1\le 1)$;*
4) *the matrix function $Z(t)$ from the representation* (66) *satisfies the condition* (67)*.*

*Then there exist the strong limits*

$$W_\pm(A_\gamma,A)=s-\lim_{t\to\pm\infty}e^{itA_\gamma}e^{-itA}=\widetilde{S}_{\gamma\mp}^{\,*}\widetilde{S}_\mp\quad\text{onto}\quad\mathbf{L}^2(0,l;\mathbb{C}^n),\tag{100}$$

$$W_\pm(A,A_\gamma)=s-\lim_{t\to\pm\infty}e^{itA}e^{-itA_\gamma}=\widetilde{S}_\mp^{-1}\widetilde{S}_{\gamma\mp}\quad\text{onto}\quad H^\gamma_{ac},\tag{101}$$

*where $\widetilde{S}_{\gamma\pm}$ and $\widetilde{S}_\pm$ are defined by* (98) *and* (29) *respectively, $\widetilde{S}_\pm^{-1}$ have the form*

$$\widetilde{S}_\pm^{-1}=\widehat{S}_\pm^{-1}T_\pm^{-1},\tag{102}$$

*$\widehat{S}_\pm^{-1}$ and $T_\pm^{-1}$ are defined by* (38) *and* (99)*.*

Proof. From the form of $W_t(A_\gamma,A)$ and $W_t(A,A_\gamma)$ we obtain

$$\frac{dW_t(A_\gamma,A)}{dt}=-i\overline{\gamma}e^{itA_\gamma}(A-A^*)e^{-itA};\quad\frac{dW_t(A,A_\gamma)}{dt}=i\overline{\gamma}e^{itA}(A-A^*)e^{-itA_\gamma}$$

onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$. Then

$$W_t(A_\gamma,A)=I-i\overline{\gamma}\int_0^t e^{i\tau A_\gamma}(A-A^*)e^{-i\tau A}d\tau,\tag{103}$$

$$W_t(A,A_\gamma)=I+i\overline{\gamma}\int_0^t e^{i\tau A}(A-A^*)e^{-i\tau A_\gamma}d\tau.\tag{104}$$

Then the existence of the limits

$$W_\pm(A_\gamma,A)=s-\lim_{t\to\pm\infty}e^{itA_\gamma}e^{-itA},\quad W_\pm(A,A_\gamma)=s-\lim_{t\to\pm\infty}e^{itA}e^{-itA_\gamma}$$

follows analogously as in the proof of Theorem 1 for the existence of the limits

$$s - \lim_{t \to \pm\infty} e^{itA^*} e^{-itA}$$

using the uniform boundedness of the sets of operators $\{W_t(A, A_\gamma)\}_{t\in\mathbb{R}}$ and $\{W_t(A_\gamma, A)\}_{t\in\mathbb{R}}$ and the representations

$$A = \widetilde{S}_\pm^{-1} D \widetilde{S}_\pm \quad \text{onto} \quad \mathbf{L}^2(0, l; \mathbb{C}^n),$$

$$A_\gamma = \widetilde{S}_{\gamma\pm}{}^* D \widetilde{S}_{\gamma\pm} \quad \text{onto} \quad H_{ac}^\gamma$$

where $\widetilde{S}_\pm$, $\widetilde{S}_\pm^{-1}$, , $\widehat{S}_\pm^{-1}$ and $T_\pm^{-1}$ are defined by (29), (102), (38), and (99) correspondingly and $\widetilde{S}_{\gamma\pm}$ are defined by (98).

Next in order to prove the equality (100) we will show first that $W_\pm(A_\gamma, A) f \in H_{ac}^\gamma$ for all $f \in \mathbf{L}^2(0, l; \mathbb{C}^n)$. On the one hand we have

$$e^{isA_\gamma} W_\pm(A_\gamma, A) = s - \lim_{t\to\pm\infty} W_{s+t}(A_\gamma, A) e^{isA} = W_\pm(A_\gamma, A) e^{isA}. \tag{105}$$

Then

$$e^{-is\lambda} e^{isA_\gamma} W_\pm(A_\gamma, A) = W_\pm(A_\gamma, A) e^{isA} e^{-is\lambda}. \tag{106}$$

Consequently the next equality

$$(A_\gamma - \lambda I)^{-1} W_\pm(A_\gamma, A) = W_\pm(A_\gamma, A)(A - \lambda I)^{-1} \tag{107}$$

follows from (106) by integrating on $s$ from 0 to $+\infty$ when $\operatorname{Im}\lambda < 0$ and from $-\infty$ to 0 when $\operatorname{Im}\lambda > 0$. If $\{E_\gamma(\tau)\}$ is the spectral family of the selfadjoint operator $A_\gamma$, i. e. $(A_\gamma - \lambda I)^{-1} = \int_0^l \frac{dE_\gamma(\tau)}{\tau - \lambda}$, then from the well-known fact from the theory of the selfadjoint operators it follows that $E_\gamma(\tau)$ has the representation

$$E_\gamma(\tau) = (I - U_\gamma(\tau))/2 \tag{108}$$

where

$$U_\gamma(\tau) = s - \lim_{\substack{\delta\to 0 \\ \rho\to+\infty}} \frac{1}{\pi} \int_\delta^\rho ((A_\gamma - (\tau + i\xi)I)^{-1} + (A_\gamma - (\tau - i\xi)I)^{-1}) d\xi$$

for all $\tau$ such that $E_\gamma(\tau)$ is continuous (see, for example, [8], Lemma VI. 5. 6).

On the other hand from the representation (62) for the resolvent of the coupling $A$ it can be obtained by straightforward calculations that

$$((A-\lambda I)^{-1}f(\sigma(u)),f(\sigma(u)))=\int\limits_a^b \frac{1}{\eta-\lambda}(K(\eta)f(\sigma(u)),f(\sigma(u)))d\eta \tag{109}$$

for each $f\in\widetilde{H}_0$ using the assumptions for $\sigma(u)$, $B(x)$, $Q^*(x)$ and Lemma 8 [9]. The function $(K(\eta)f(\sigma(u)),f(\sigma(u)))$ in (109) is integrable on $[a;b]$ as a function of $\eta$. Then

$$(\omega(x)f(\sigma(u)),f(\sigma(u)))=\int\limits_a^x (K(\tau)f(\sigma(u)),f(\sigma(u)))d\tau \tag{110}$$

$(\forall x\in[a;b])$ is an absolutely continuous on $[a;b]$. In other words (109) takes the form

$$((A-\lambda I)^{-1}f(\sigma(u)),f(\sigma(u)))=\int\limits_a^b \frac{d(\omega(\eta)f(\sigma(u)),f(\sigma(u)))}{\eta-\lambda} \tag{111}$$

for each $f\in\widetilde{H}_0$.

Let us consider now

$$U_{\delta\rho}(\tau)=\frac{1}{\pi}\int\limits_\delta^\rho((A-(\tau+i\xi)I)^{-1}+(A-(\tau-i\xi)I)^{-1})d\xi\ \ (\rho>0,\ \delta>0) \tag{112}$$

$(\tau\in[a;b])$. Then for each $f\in\widetilde{H}_0$ from (111) and (112) it follows that

$$\begin{aligned}(U_{\delta\rho}(\tau)f,f)&=\left(\tfrac{1}{\pi}\int\limits_\delta^\rho((A-(\tau+i\xi)I)^{-1}f+(A-(\tau-i\xi)I)^{-1})f)d\xi,f\right)=\\ &=\tfrac{2}{\pi}\int\limits_a^b\left(\int\limits_\delta^\rho\tfrac{\eta-\tau}{(\eta-\tau)^2+\xi^2}d\xi\right)d(\omega(\eta)f,f)=\\ &=\tfrac{2}{\pi}\int\limits_a^b(\arctan\tfrac{\rho}{\eta-\tau}-\arctan\tfrac{\delta}{\eta-\tau})d(\omega(\eta)f,f).\end{aligned} \tag{113}$$

But $(\omega(\eta)f,f)$ is an absolutely continuous function for each $f\in\widetilde{H}_0$ and $|\arctan\frac{\rho}{\eta-\tau}-\arctan\frac{\delta}{\eta-\tau}|\le\pi$. Then (113) together with the Lebesgue convergence theorem implies that

$$(U(\tau)f,f)=\lim_{\substack{\delta\to0\\ \rho\to+\infty}}(U_{\delta\rho}(\tau)f,f)=(\omega(b)f,f)-2(\omega(\tau)f,f).$$

using the equality $(\omega(a)f, f) = 0$ which can be obtained from the form of $\omega(a)$. Hence

$$(\omega(\tau)f, f) = ((\omega(b)f, f) - (U(\tau)f, f))/2 \tag{114}$$

for all $f \in \mathbf{L}^2(0, l; \mathbb{C}^n)$ (because the subspace $\widetilde{H}_0$ is dense in $\mathbf{L}^2(0, l; \mathbb{C}^n)$). From (108) and (114) we have

$$\begin{aligned}
&(E_\gamma(\tau)W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) = \tfrac{1}{2}(W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) - \\
&- \lim_{\substack{\delta\to 0\\ \rho\to+\infty}} \tfrac{1}{2\pi}\int_\delta^\rho (((A_\gamma - (\tau+i\xi)I)^{-1} + (A_\gamma - (\tau - i\xi)I)^{-1})W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f)d\xi = \\
&= \tfrac{1}{2}(W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) - \\
&- \lim_{\substack{\delta\to 0\\ \rho\to+\infty}} \tfrac{1}{2\pi}\int_\delta^\rho (W_\pm(A_\gamma, A)((A - (\tau+i\xi)I)^{-1} + (A - (\tau - i\xi)I)^{-1})f, W_\pm(A_\gamma, A)f)d\xi = \\
&= \tfrac{1}{2}(W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) - \\
&- \lim_{\substack{\delta\to 0\\ \rho\to+\infty}} \tfrac{1}{2\pi}\int_\delta^\rho (((A - (\tau+i\xi)I)^{-1} + (A - (\tau - i\xi)I)^{-1})f, W_\pm^*(A_\gamma, A)W_\pm(A_\gamma, A)f)d\xi = \\
&= \tfrac{1}{2}(W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) - \tfrac{1}{2}(U(\tau)f, W_\pm^*(A_\gamma, A)W_\pm(A_\gamma, A)f) = \\
&= \tfrac{1}{2}(W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) + (\omega(\tau)f, W_\pm^*(A_\gamma, A)W_\pm(A_\gamma, A)f) - \\
&- \tfrac{1}{2}(\omega(b)f, W_\pm^*(A_\gamma, A)W_\pm(A_\gamma, A)f)
\end{aligned}$$

for each $f \in \mathbf{L}^2(0, l; \mathbb{C}^n)$, where we have used (107). The last equalities show that

$$\begin{aligned}
&(E_\gamma(\tau)W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) = \tfrac{1}{2}(W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f) - \\
&- \tfrac{1}{2}(\omega(b)f, W_\pm^*(A_\gamma, A)W_\pm(A_\gamma, A)f) + (\omega(\tau)f, W_\pm^*(A_\gamma, A)W_\pm(A_\gamma, A)f)
\end{aligned} \tag{115}$$

at first for all $\tau$ of a continuity of $E_\gamma(\tau)$ and then for the other $\tau \in \mathbb{R}$, using that $\omega(\tau)$ is absolutely continuous and $E_\gamma(\tau)$ is continuous from the right. Consequently, the equality (115) implies that the function $(E_\gamma(\tau)W_\pm(A_\gamma, A)f, W_\pm(A_\gamma, A)f)$ is an absolutely continuous function of $\tau$ for each $f \in \mathbf{L}^2(0, l; \mathbb{C}^n)$ because (110) is absolutely continuous. In other words $W_\pm(A_\gamma, A)f \in H_{ac}^\gamma$ for all $f \in \mathbf{L}^2(0, l; \mathbb{C}^n)$, i. e.

$$W_\pm(A_\gamma, A) : \mathbf{L}^2(0, l; \mathbb{C}^n) \longrightarrow H_{ac}^\gamma. \tag{116}$$

From the obtained asymptotics (80) and (26) for the curves $e^{itA_\gamma}f$ and $e^{itA}f$ it follows that

$$\lim_{t\to\pm\infty} (W_t(A_\gamma, A)f, g_1) = (\widetilde{S}_{\gamma\mp}^{\ *}\widetilde{S}_\mp f, g_1) \tag{117}$$

for all $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$ and for all $g_1 \in H^\gamma_{ac}$ But $H^\gamma_s$ is a subspace of $\mathbf{L}^2(0,l;\mathbb{C}^n)$ and (116) gives the equality

$$W^*_\pm(A_\gamma, A)g = 0 \quad \text{for all } g \in H^\gamma_s,$$

which implies that $W_\pm(A_\gamma, A)g = 0$ for all $g \in H^\gamma_s$ and

$$W_\pm(A_\gamma, A) = \widetilde{S}^{\,*}_{\gamma\mp}\widetilde{S}_\mp \quad \text{onto } \mathbf{L}^2(0,l;\mathbb{C}^n). \tag{118}$$

Observe that $W_t(A, A_\gamma)W_t(A_\gamma, A)f = f \ \forall f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$
and $W_t(A_\gamma, A)W_t(A, A_\gamma)g = g \ \forall g \in H^\gamma_{ac}$ hence

$$W_\pm(A, A_\gamma)W_\pm(A_\gamma, A)f = f \quad \forall f \in \mathbf{L}^2(0,l;\mathbb{C}^n), \tag{119}$$

$$W_\pm(A_\gamma, A)W_\pm(A, A_\gamma)g = g \quad \forall g \in H^\gamma_{ac}. \tag{120}$$

Then (118), (119) and (120) imply that $W_\pm(A, A_\gamma) = W^{-1}_\pm(A_\gamma, A)$ (defined onto $H^\gamma_{ac}$) and $W_\pm(A, A_\gamma)\widetilde{S}^{\,*}_{\gamma\mp}\widetilde{S}_\mp f = f$ onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$ and consequently $W_\pm(A, A_\gamma)=\widetilde{S}^{-1}_\mp\widetilde{S}_{\gamma\mp}$. The proof is complete. □

Now using the existence and the explicit form of the wave operators we introduce a scattering operator defined by the formula

$$W^{-1}_-(A, A_\gamma)W_+(A, A_\gamma)$$

on the subspace $H^\gamma_{ac}$. Using (100) and (101) the scattering operator takes the form

$$W^{-1}_-(A, A_\gamma)W_+(A, A_\gamma) = \widetilde{S}^*_{\gamma+}\widetilde{S}_+\widetilde{S}^{-1}_-\widetilde{S}_{\gamma-}$$

where $\widetilde{S}_{\gamma\pm}$, $\widetilde{S}_+$ and $\widetilde{S}^{-1}_-$ are defined by (98), (29) and (102).

Finally from the equality (105) it follows that

$$A_\gamma W_\pm(A_\gamma, A) = W_\pm(A_\gamma, A)A.$$

Then the explicitly constructing of the wave operators establishes the similarity of $A$ and $A_\gamma$ and

$$A = W^{-1}_\pm(A_\gamma, A)A_\gamma W_\pm(A_\gamma, A),$$

i. e.

$$A = \widetilde{S}^{-1}_\pm\widetilde{S}_{\gamma\pm}A_\gamma\widetilde{S}^*_{\gamma\pm}\widetilde{S}_\pm$$

onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$, where $\widetilde{S}_{\gamma\pm}$, $\widetilde{S}_\pm$ and $\widetilde{S}^{-1}_\pm$ are defined by (98), (29) and (102).

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