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APPROXIMATION OF LIPSCHITZ MAPPINGS*

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Abstract. We prove that any Lipschitz mapping from a separable Banach space into any Banach space can be approximated by uniformly Gâteaux differentiable Lipschitz mapping.

If X is a separable Banach space and Y is a Banach space with RNP, then any Lipschitz mapping from X to Y can be approximated by a Gâteaux differentiable Lipschitz mapping ([2]; cf. [1], page 155). The aim of this paper is to show that using a different technique of the proof the assumption on the target space having RNP can be dropped and moreover the approximation can be made uniformly Gâteaux differentiable.

Let X, Y be Banach spaces, f a mapping $f : X \to Y$. Let us define the directional derivative of f at $x \in X$ in the direction $h \in X$ as $D_h f(x) = \lim_{t\to 0} \frac{1}{t}(f(x+th) - f(x))$. If for any fixed x the directional derivative exists for all

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$h \in X$ and $D_h f(x)$ is a bounded linear operator in h, we say that f is Gâteaux differentiable at x. If moreover for all fixed h the limit defining $D_h f(x)$ is uniform for $x \in X$ we say that f is uniformly Gâteaux differentiable (UG for short). For any other unexplained term we refer to [1].

Theorem. *Let X be a separable Banach space, Y a Banach space, $f : X \to Y$ be an L-Lipschitz mapping and $\varepsilon > 0$. Then there is a mapping $g : X \to Y$ which is L-Lipschitz, UG and $\|f - g\| < \varepsilon$.*

Proof. We will construct the function g by using the metod of integral convolution in a countable set of directions which was presented in [3].

Let h_i be a dense subset of S_X and $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots$ be such that $\varphi_i \geq 0$, $\varphi_i \in C^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \varphi_i = 1$ and $\operatorname{supp} \varphi_i \in [-\frac{\varepsilon}{2L2^i}, \frac{\varepsilon}{2L2^i}]$.

Define a mapping $g_n : X \to Y$, $n = 1, 2, \dots$ by

$$g_n(x) = \int\limits_{\mathbb{R}^n} f\left(x - \sum_{i=1}^{n} t_i h_i\right) \prod_{i=1}^{n} \varphi_i(t_i) \, d\lambda_n(t),$$

where we integrate in the Bochner sense with respect to the n-dimensional Lebesgue measure.

The mappings g_n are L-Lipschitz:

$$\begin{aligned} \|g_n(x) - g_n(y)\| &\leq \int\limits_{\mathbb{R}^n} \left\| f\left(x - \sum_{i=1}^{n} t_i h_i\right) - f\left(y - \sum_{i=1}^{n} t_i h_i\right) \right\| \prod_{i=1}^{n} \varphi_i(t_i) \, d\lambda_n \\ &\leq L\|x - y\| \int\limits_{\mathbb{R}^n} \prod_{i=1}^{n} \varphi_i(t_i) \, d\lambda_n = L\|x - y\|. \end{aligned}$$

There is a mapping g such that $g_n \to g$ uniformly on X (and hence g is also L-Lipschitz). Indeed, denote by M_m the set $\prod_{i=1}^{m} \left[-\frac{\varepsilon}{2L2^i}, \frac{\varepsilon}{2L2^i}\right] \subset \mathbb{R}^m$. Then using the Fubini theorem and the fact that $\int_{\mathbb{R}} \varphi_i = 1$ for any i we have for $m > n$ and any $x \in X$

$$\begin{aligned}\|g_m(x) - g_n(x)\| &= \left\| \int\limits_{\mathbb{R}^m} \left(f\left(x - \sum_{i=1}^{m} t_i h_i\right) - f\left(x - \sum_{i=1}^{n} t_i h_i\right)\right) \prod_{i=1}^{m} \varphi_i(t_i)\, d\lambda_m \right\| \\ &\le L \int\limits_{\mathbb{R}^m} \left\| \sum_{i=n+1}^{m} t_i h_i \right\| \prod_{i=1}^{m} \varphi_i(t_i)\, d\lambda_m \le L \int\limits_{M_m} \left(\sum_{i=n+1}^{m} |t_i| \right) \prod_{i=1}^{m} \varphi_i(t_i)\; dl_m \\ &\le L \left(\sum_{i=n+1}^{m} \frac{\varepsilon}{2L2^i} \right) \int\limits_{M_m} \prod_{i=1}^{m} \varphi_i(t_i)\, d\lambda_m \le \frac{\varepsilon}{2\cdot 2^n}.\end{aligned}$$

Moreover, $\|f - g\| < \varepsilon$. Pick g_n such that $\|g_n - g\| < \dfrac{\varepsilon}{2}$. Then

$$\begin{aligned}\|f(x) - g(x)\| &\le \|f(x) - g_n(x)\| + \|g_n(x) - g(x)\| \\ &< \int\limits_{\mathbb{R}^n} \left\| f(x) - f\left(x - \sum_{i=1}^{n} t_i h_i\right)\right\| \prod_{i=1}^{n} \varphi_i(t_i)\, d\lambda_n + \frac{\varepsilon}{2} \\ &\le L \int\limits_{M_n} \left(\sum_{i=1}^{n} |t_i| \right) \prod_{i=1}^{n} \varphi_i(t_i)\, d\lambda_n + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Now it remains to show that g is UG.

First we will show that the directional derivative $D_{h_i} g_n(x)$ exists for any $x \in X$ and $i = 1, 2, \ldots, n$.

$$\begin{aligned}D_{h_i} g_n(x) &= \lim_{\tau\to 0} \frac{1}{\tau}(g_n(x + \tau h_i) - g_n(x)) \\ &= \lim_{\tau\to 0} \frac{1}{\tau} \left(\int\limits_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j + \tau h_i\right) \prod_{j=1}^{n} \varphi_j(t_j)\, d\lambda_n \right. \\ &\qquad \left. - \int\limits_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \prod_{j=1}^{n} \varphi_j(t_j)\, d\lambda_n \right) \\ &= \lim_{\tau\to 0} \frac{1}{\tau} \left(\int\limits_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \varphi_i(t_i + \tau) \prod_{\substack{j=1\\ j\ne i}}^{n} \varphi_j(t_j)\, d\lambda_n \right.\end{aligned}$$

$$- \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \prod_{j=1}^{n} \varphi_j(t_j)\, d\lambda_n \Bigg)$$

$$= \lim_{\tau \to 0} \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \frac{\varphi_i(t_i + \tau) - \varphi_i(t_i)}{\tau} \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n$$

$$= \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \varphi_i'(t_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n \tag{1}$$

For the third equality we use the substitution $t_i \to t_i + \tau$. In order to show the last equality, choose $\eta > 0$. Then there is $\delta > 0$ such that $\left|\frac{1}{\tau}(\varphi_i(t_i+\tau) - \varphi_i(t_i)) - \varphi_i'(t_i)\right| < \eta$ for any $|\tau| < \delta$ and $t_i \in \mathbb{R}$. (Use Mean Value Theorem and compactness of the support of φ_i.) Hence, for $|\tau| < \delta$

$$\left\| \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \frac{\varphi_i(t_i + \tau) - \varphi_i(t_i)}{\tau} \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n \right.$$

$$\left. - \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^{n} t_j h_j\right) \varphi_i'(t_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j) \right\|$$

$$\leq \int_M \left\| f\left(x - \sum_{j=1}^{n} t_j h_j\right)\right\| \left| \frac{\varphi_i(t_i + \tau) - \varphi_i(t_i)}{\tau} - \varphi_i'(t_i) \right| \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n$$

$$\leq \eta \int_M \left(\|f(x)\| + \left\| f(x) - f\left(x - \sum_{j=1}^{n} t_j h_j\right)\right\| \right) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n$$

$$\leq \eta \int_M \left(\|f(x)\| + L \sum_{j=1}^{n} |t_j| \right) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n < \eta \left(\|f(x)\| + \frac{\varepsilon}{2} \right) \frac{2\varepsilon}{L 2^i}, \tag{2}$$

where $M = \mathbb{R}^{i-1} \times \left(\left[-\frac{\varepsilon}{2L2^i}, \frac{\varepsilon}{2L2^i} \right] \cup \left[-\frac{\varepsilon}{2L2^i} - \tau, \frac{\varepsilon}{2L2^i} - \tau \right] \right) \times \mathbb{R}^{n-i} \subset \mathbb{R}^n$. We can see that the limit in (1) is uniform with respect to n.

Our next step is to show that $D_{h_i}g_n \to D_{h_i}g$ uniformly for $x \in X$. Due to the Fubini theorem and the fact that $\int_{\mathbb{R}} \varphi_i = 1$ for any i we have for $m > n \geq i$ and any $x \in X$

$$\|D_{h_i}g_m(x) - D_{h_i}g_n(x)\|$$

$$= \left\| \int\limits_{\mathbb{R}^m} \left(f\left(x - \sum_{j=1}^{m} t_j h_j \right) - f\left(x - \sum_{j=1}^{n} t_j h_j \right) \right) \varphi_i'(t_i) \prod_{\substack{j=1 \\ j\neq i}}^{m} \varphi_j(t_j)\, d\lambda_m \right\|$$

$$\leq L \int\limits_{M_m} \sum_{j=n+1}^{m} |t_j|\, |\varphi_i'(t_i)| \prod_{\substack{j=1 \\ j\neq i}}^{m} \varphi_j(t_j)\, d\lambda_m \leq \frac{\varepsilon}{2\cdot 2^n} \int\limits_{\mathbb{R}} |\varphi_i'(t)|\, dt,$$

hence $D_{h_i}g_n$ converges uniformly on X. Now fix $x \in X$. Then

$$D_{h_i}g(x) = \lim_{\tau\to 0} \frac{1}{\tau}\left(g(x+\tau h_i) - g(x)\right) = \lim_{\tau\to 0}\lim_{n\to\infty} \frac{1}{\tau}\left(g_n(x+\tau h_i) - g_n(x)\right)$$

$$= \lim_{n\to\infty}\lim_{\tau\to 0} \frac{1}{\tau}\left(g_n(x+\tau h_i) - g_n(x)\right) = \lim_{n\to\infty} D_{h_i}g_n(x).$$

The limit (1) is uniform with respect to n and so we can interchange the limits above. Therefore $D_{h_i}g_n \to D_{h_i}g$ uniformly for $x \in X$.

Next, the mapping $D_{h_i}g_n$ is L_i-Lipschitz for any $n \geq i$, where $L_i = L \int_{\mathbb{R}} |\varphi_i'(t)|\, dt$:

$$\|D_{h_i}g_n(x) - D_{h_i}g_n(y)\|$$

$$\leq \left\| \int\limits_{\mathbb{R}^n} \left(f\left(x - \sum_{j=1}^{n} t_j h_j \right) - f\left(y - \sum_{j=1}^{n} t_j h_j \right) \right) \varphi_i'(t_i) \prod_{\substack{j=1 \\ j\neq i}}^{n} \varphi_j(t_j)\, d\lambda_n \right\|$$

$$\leq L\|x-y\| \int_{\mathbb{R}} |\varphi_i'(t)|\, dt = L_i\|x-y\|.$$

Thus the mapping $D_{h_i}g$ is L_i-Lipschitz for $i = 1, 2, \ldots$ (because of the uniform convergence). This implies that the limit in the definition of the directional derivative $D_{h_i}g(x)$ is uniform with respect to $x \in X$. Indeed,

$$\left\| \frac{1}{\tau}(g(x+\tau h_i) - g(x)) - D_{h_i}g(x) \right\| = \left\| \frac{1}{\tau} \int_0^\tau D_{h_i}g(x+sh_i)\,ds - D_{h_i}g(x) \right\|$$

$$\leq \frac{L_i}{|\tau|} \int_0^\tau |s|\,ds \leq L_i|\tau|.$$

Finally, the derivative $D_h g(x)$ exists for any $h \in S_X$ and the limit in the definition is uniform with respect to $x \in X$. To see that, choose $\eta > 0$ and h_i, such that $\|h - h_i\| < \dfrac{\eta}{L}$. Then for any $\tau \in \mathbb{R}$, $\left\| \dfrac{1}{\tau}(g(x+\tau h) - g(x)) - \dfrac{1}{\tau}(g(x+\tau h_i) - g(x)) \right\| \leq \dfrac{L}{|\tau|}\|\tau(h - h_i)\| < \eta$. Thus there is $\delta > 0$ such that

$$\left\| \frac{1}{\tau_1}(g(x+\tau_1 h) - g(x)) - \frac{1}{\tau_2}(g(x+\tau_2 h) - g(x)) \right\|$$

$$\leq 2\eta + \left\| \frac{1}{\tau_1}(g(x+\tau_1 h_i) - g(x)) - \frac{1}{\tau_2}(g(x+\tau_2 h_i) - g(x)) \right\| \leq 3\eta$$

$$\text{for } x \in X,\ |\tau_1| < \delta,\ |\tau_2| < \delta.$$

This means that g is UG, provided that for any fixed x the operator $D_h g(x)$ is a bounded linear operator in h.

The fact that $D_{\lambda h}g(x) = \lambda D_h g(x)$ is trivial and the boundedness of the operator follows easily. Pick any $i, j \in \mathbb{N}$. Then

$$D_{h_i+h_j}g(x) = \lim_{\tau\to 0} \frac{1}{\tau}\left(g(x+\tau(h_i+h_j)) - g(x)\right)$$

$$= \lim_{\tau\to 0} \lim_{n\to\infty} \frac{1}{\tau}\left(g_n(x+\tau(h_i+h_j)) - g_n(x)\right)$$

$$= \lim_{n\to\infty} \lim_{\tau\to 0} \frac{1}{\tau}\left(g_n(x+\tau(h_i+h_j)) - g_n(x)\right)$$

$$= \lim_{n\to\infty} \lim_{\tau\to 0} \frac{1}{\tau}\left(\int_{\mathbb{R}^n} f\left(x - \sum_{k=1}^n t_k h_k + \tau(h_i+h_j)\right) \prod_{k=1}^n \varphi_k(t_k)\,d\lambda_n\right.$$

$$\left.-\int_{\mathbb{R}^n} f\left(x-\sum_{k=1}^n t_k h_k\right)\prod_{k=1}^n \varphi_k(t_k)\,d\lambda_n\right)$$

$$=\lim_{n\to\infty}\lim_{\tau\to 0}\int_{\mathbb{R}^n} f\left(x-\sum_{k=1}^n t_k h_k\right)\frac{\varphi_i(t_i+\tau)\varphi_j(t_j+\tau)-\varphi_i(t_i)\varphi_j(t_j)}{\tau}\prod_{\substack{k=1\\k\neq i,j}}^n \varphi_k(t_k)\,d\lambda_n$$

$$=\lim_{n\to\infty}\left(\int_{\mathbb{R}^n} f\left(x-\sum_{k=1}^n t_k h_k\right)\varphi_i'(t_i)\prod_{\substack{k=1\\k\neq i}}^n \varphi_k(t_k)\,d\lambda_n\right.$$

$$\left.+\int_{\mathbb{R}^n} f\left(x-\sum_{k=1}^n t_k h_k\right)\varphi_j'(t_j)\prod_{\substack{k=1\\k\neq j}}^n \varphi_k(t_k)\,d\lambda_n\right)$$

$$=\lim_{n\to\infty}\big(D_{h_i}g_n(x)+D_{h_j}g_n(x)\big)=D_{h_i}g(x)+D_{h_j}g(x).$$

Note that we can show, similarly as in (2), that $\lim\limits_{\tau\to 0}$ is uniform with respect to n. Hence we can interchange the limits. Now, for arbitrary $u, v \in X$ and $\eta > 0$, we have

$$\begin{aligned}\|D_u g(x)-D_v g(x)\| &\le \left\|D_u g(x)-\frac{1}{\tau}(g(x+\tau u)-g(x))\right\|\\ &\quad+\left\|\frac{1}{\tau}(g(x+\tau u)-g(x+\tau v))\right\|\\ &\quad+\left\|D_v g(x)-\frac{1}{\tau}(g(x+\tau v)-g(x))\right\| \le \eta + L\|u-v\|\end{aligned}$$

for τ small enough. Thus $\|D_u g(x)-D_v g(x)\| \le L\|u-v\|$. Choose h_i and h_j such that $\|u-h_i\|<\eta$ and $\|v-h_j\|<\eta$. Then

$$\begin{aligned}&\|D_{u+v}g(x)-D_u g(x)-D_v g(x)\|\\ &\quad\le \|D_{u+v}g(x)-D_{h_i+h_j}g(x)\|+\|D_{h_i}g(x)-D_u g(x)\|+\|D_{h_j}g(x)-D_v g(x)\|\\ &\quad\le L(\|u+v-h_i-h_j\|+\|u-h_i\|+\|v-h_j\|)\le 4L\eta\end{aligned}$$

for an arbitrary $\eta > 0$.

We have shown that the directional derivatives of g form bounded linear operator and hence g is Gâteaux differentiable. Moreover, since the limits defining the directional derivatives are uniform for $x \in X$, the mapping g is uniformly Gâteaux differentiable. $\square$

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